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The Degree of Approximation by Polynomials Increasing to the Right of the Interval

R. K. BEATSON

Department of Mathematics, University of Texas, Austin, Texas 78712

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INTRODUCTION

Jackson Type Theorems are obtained for approximation of $f \in C^k[-1, 1]$ by polynomials $p_n \in \pi_n$ which are increasing on $[1, \infty)$. The estimates obtained depend both on $n^{-k}\omega(f^{(k)}, n^{-1})$ and on the derivatives of f at $x = 1$. For example it is shown that for each $f \in C^2[-1, 1]$ the degree of approximation by polynomials $p_n \in \pi_n$ increasing to the right of $x = 1$, $E_n^*(f)$, satisfies

$$E_n^*(f) \leqslant D_2 n^{-2}\omega(f^{(2)}, n^{-1}) + \max\left(0, \frac{-f'(1)}{n^2}, \frac{-3f''(1)}{n^2(n^2-1)}\right).$$

This estimate of $E_n^*(f)$ is of the best possible order in that the following negative result holds: If $f'(1) < 0$ then for each $\alpha > 0$,

$$\overline{\lim_{n\to\infty}}\, n^{2+\alpha}E_n^*(f) = \infty.$$

The motivation for the present work was the method of proof used in recent studies of uniform rational approximation to reciprocals of entire functions on $[0, \infty)$ (see, e.g., Meinardus and Varga [5]). Indeed that method of proof may be combined with the polynomial preserving one to one correspondence between $C[0, r]$ and $C[-1, 1]$ given by

$$f(x) = g(y) \qquad \text{where} \qquad x \in [-1, 1] \qquad \text{and} \qquad x = (2y - r)/r;$$

and Corollary 1 of this paper; to yield results concerning uniform rational approximation on $[0, \infty)$. Details appear in the preprint Beatson [1].

Results related to those of the present paper appear in Ling, Roulier, and Varga [3].

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THE RESULTS

Notation. Throughout $C_1, C_2, C_3, \ldots$ denote positive constants not depending on n or f, but possibly depending on k.

Define

$$E_n^*(f) = \inf\{\|f - p\| : p \in \pi_n, p'(x) \geqslant 0 \text{ on } [1, +\infty)\}.$$

where the norm $\|\cdot\|$ is the uniform norm on $[-1, 1]$ and π_n is the space of algebraic polynomials of degree not exceeding n.

LEMMA 1. *There exists a constant M such that for each $f \in C[-1, 1]$ and $n = 1, 2, 3, \ldots$ there exists $p_n \in \pi_n$ with*

$$p_n(1) = f(1);$$
$$p_n'(x) \geqslant 0, \forall x \geqslant 1;$$

and

$$\|f - p_n\| \leqslant M\omega(f, n^{-1}).$$

Remark. Hence $E_n^*(f) \leqslant M\omega(f, n^{-1})$.

Proof. Fix f and n. Define f outside $[-1, 1]$ by

$$f(x) = \begin{cases} f(1), & \text{if } x \geqslant 1 \\ f(-1), & \text{if } x \leqslant -1 \end{cases}$$

Let

$$\phi(x) = (2\delta)^{-1} \int_{-\delta}^{\delta} f(x + t)\, dt \text{ with } \delta = n^{-1}.$$

As is well known (see for example Cheney [2, pp. 143–144]), ϕ is continuously differentiable with

$$\|\phi'\| \leqslant n\omega(f, n^{-1}), \quad \omega(\phi', n^{-1}) \leqslant n\omega(f, n^{-1}), \quad \text{and} \quad \|f - \phi\| \leqslant \omega(f, n^{-1}).$$

Using a theorem of Trigub [9], see also Teljakovskii [8][1] and Malozemov [4], there exists a polynomial $q_n \in \pi_n$ with

$$\|\phi - q_n\| \leqslant C_1 n^{-1} \omega(\phi', n^{-1}) \text{ and } \|\phi' - q_n'\| \leqslant C_2 \omega(\phi', n^{-1}).$$

[1] [8] erroneously states the simultaneous approximation theorem as holding for all n. Nontrivial simultaneous approximation to f and its first k derivatives is possible only by algebraic polynomials of degree $n \geqslant k$.

Hence

$$\|f - q_n\| \leqslant C_3\omega(f, n^{-1}) \text{ and } \|q_n'\| \leqslant C_4 n\omega(f, n^{-1}).$$

We perturb q_n in order to obtain an approximation increasing to the right of $x = 1$. Denote by T_m the m-th Chebyshev polynomial of the first kind. It is well known (see e.g. Rogosinski [7], Rivlin [6, pp. 92–93]) that for $n = 0, 1, 2,...$; $r_n \in \pi_n$ and $\|r_n\| \leqslant 1$ implies $|r_n^{(j)}(x)| \leqslant T_n^{(j)}(x)$ for all $x \geqslant 1$, $j = 0, 1,..., n$. The inequality for $j = 0$ shows that if $h_n(x)$ is any indefinite integral of $\|q_n'\| T_{n-1}$ then

$$h_n'(x) + q_n'(x) \geqslant 0, \forall x \geqslant 1.$$

Use the formula

$$I(T_m, x) = \begin{cases} T_1(x) & , m = 0, \\ T_2(x)/4 & , m = 1, \\ \dfrac{T_{m+1}(x)}{2(m+1)} - \dfrac{T_{m-1}(x)}{2(m-1)}, & m \geqslant 2; \end{cases}$$

obtained from the identity $2\cos n\theta \sin\theta = \sin(n+1)\theta - \sin(n-1)\theta$, to specify a particular indefinite integral operator, operating on the T_m, with the desirable property that

$$\|I(T_m)\| \leqslant C_5(m+1)^{-1}, \qquad m = 0, 1, 2,... .$$

Thus

$$y_n(x) = q_n(x) + \|q_n'\| I(T_{n-1}, x),$$

is an algebraic polynomial of degree not exceeding n, increasing to the right of $x = 1$, with

$$\|f - y_n\| \leqslant \|f - q_n\| + \|q_n'\| \|I(T_{n-1})\| \leqslant C_6\omega(f, n^{-1}).$$

Addition of $[f(1) - y_n(1)]$ to y_n produces a polynomial $p_n \in \pi_n$ with: $p_n'(x) \geqslant 0$, $\forall x \geqslant 1$; $p_n(1) = f(1)$; and $\|f - p_n\| \leqslant 2C_6\omega(f, n^{-1})$. This concludes the proof. ∎

THEOREM 1. *For each $k = 1, 2, 3,...$, there exists a constant D_k, such that for each $f \in C^k[-1, 1]$ and $n > k$ there exists a polynomial $p_n \in \pi_n$ with*

$$\|f - p_n\| \leqslant D_k n^{-k}\omega(f^{(k)}, n^{-1});$$

and

$$p_n'(x) \geqslant t'(x), \forall x \geqslant 1,$$

where $t(x)$ is the Taylor polynomial

$$t(x) = \sum_{j=0}^{k} [f^{(j)}(1)(x-1)^j/j!]$$

Proof. Given n ($>k$), let $p_{n,k}^{(k)}$ be the polynomial of degree $n-k$ approximating $f^{(k)}$ whose existence is guaranteed by Lemma 1. Define a polynomial $p_{n,k}$ in π_n by

$$p_{n,k}(x) = \sum_{j=0}^{k-1} [f^{(j)}(1)(x-1)^j/j!] + \int_1^x \int_1^{t_k} \cdots \int_1^{t_2} p_{n,k}^{(k)}(t_1)\, dt_1 \cdots dt_k\ ;$$

where for $k = 1$ the last term is understood to be $\int_1^x p_{n,1}^{(1)}(t_1)\, dt_1$. Then

$$p_{n,k}^{(j)}(1) = f^{(j)}(1), \quad j = 0,..., k;$$

$$p_{n,k}^{(k+1)}(x) \geqslant 0, \qquad \forall x \geqslant 1;$$

and

$$\| f^{(k)} - p_{n,k}^{(k)} \| \leqslant M\omega(f^{(k)}, (n-k)^{-1}) \leqslant C_7\omega(f^{(k)}, n^{-1}).$$

Now consider $(f - p_{n,k})$. This function has

$$(f - p_{n,k})^{(j)}(1) = 0, \quad j = 0,..., k; \quad \text{and} \quad \|(f - p_{n,k})^{(k)}\| \leqslant C_7\omega(f^{(k)}, n^{-1}).$$

By another application of Lemma 1, this time to $[f^{(k-1)} - p_{n,k}^{(k-1)}]$, followed by $k-1$ indefinite integrations we can find a polynomial $p_{n,k-1}$ in π_n such that

$$p_{n,k-1}^{(j)}(1) = 0, \qquad j = 0,..., k-1;$$

$$p_{n,k-1}^{(k)}(x) \geqslant 0, \qquad \forall x \geqslant 1;$$

and

$$\| [f^{(k-1)} - p_{n,k}^{(k-1)}] - p_{n,k-1}^{(k-1)} \| \leqslant C_1 n^{-1}\omega(f^{(k)}, n^{-1}).$$

Continue this process defining for $i = 2,..., k$ in that order, a polynomial $p_{n,k-i}$ of degree not exceeding n such that

$$p_{n,k-i}^{(j)}(1) = 0, \qquad j = 0,..., k-i;$$

$$p_{n,k-i}^{(k-i+1)}(x) \geqslant 0, \qquad \forall x \geqslant 1;$$

$$\left\| \left[f^{(k-i)} - \sum_{j=0}^{i-1} p_{n,k-j}^{(k-i)} \right] - p_{n,k-i}^{(k-i)} \right\| \leqslant C_{7+i} n^{-i}\omega(f^{(k)}, n^{-1}).$$

Then the polynomial

$$p_n = \sum_{j=0}^{k} p_{n,j}(x) \tag{1}$$

belongs to π_n and

$$\| f - p_n \| \leqslant C_{7+k} n^{-k} \omega(f^{(k)}, n^{-1}).$$

It remains to show that the derivative of p_n *satisfies the stated condition to the right of* 1. Recall that

$$p_{n,k}^{(j)}(1) = f^{(j)}(1), j = 0,..., k; \text{ and } p_{n,k}^{(k+1)}(x) \geqslant 0, \qquad \forall x \geqslant 1.$$

Hence

$$[p_{n,k} - t]^{(j)}(1) = 0, \qquad j = 0,..., k;$$

and

$$[p_{n,k} - t]^{(k+1)} (x) = p_{n,k}^{(k+1)} (x) \geqslant 0, \qquad \forall x \geqslant 1;$$

implying

$$p_{n,k}^{(j)}(x) \geqslant t^{(j)}(x), \qquad j = 0, 1,..., k + 1, \qquad \forall x \geqslant 1. \tag{2}$$

Similarly for $i = 0,..., k - 1$,

$$p_{n,i}^{(j)}(1) = 0, \qquad j = 0,..., i; \qquad \text{and} \qquad p_{n,i}^{(i+1)}(x) \geqslant 0; \qquad \forall x \geqslant 1;$$

implies

$$p'_{n,i}(x) \geqslant 0, \qquad \forall x \geqslant 1. \tag{3}$$

(1), (2) and (3) together imply

$$p'_n(x) = \sum_{i=0}^{k} p'_{n,i}(x) \geqslant t'(x), \qquad \forall x \geqslant 1,$$

COROLLARY 1. *Let* D_k *and* $t(x) = t(f, x)$ *be defined as in Theorem* 1. *Given* $f \in C^k[-1, 1]$ *and* $n > k$ *define* $\epsilon_n(f)$ *as the smallest non-negative number such that*

$$(t + \epsilon_n(f) \, T_n)'(x) \geqslant 0, \qquad \forall x \geqslant 1.$$

Then

(a) $E_n^*(f) \leqslant D_k n^{-k}\omega(f^{(k)}, n^{-1}) + \epsilon_n(f)$.

(b) $0 \leqslant \epsilon_n(f) \leqslant \max_{j=1,\dots,k} \max[0, -f^{(j)}(1)/d_{n,j}]$ *where for* $j = 1,\dots, n$,

$$d_{n,j} = | T_n^{(j)}(1)| = \frac{n^2 \cdot (n^2 - 1) \cdots (n^2 - (j-1)^2)}{1 \cdot 3 \cdots (2j-1)}.$$

(c) *If for some* $\theta > 0$, $t'(x) \geqslant 0$ *for all* x *in the interval* $(1, \cosh\theta)$ *then in addition*

$$\epsilon_n(f) \leqslant \frac{k}{2n}\frac{\exp(k\theta)}{\sinh(n\theta)} \| t \| \leqslant M(\theta, f, k)(e^{-\theta})^n, \qquad \forall n > k.$$

Proof of (a). Let $p_n(x)$ be the polynomial approximation to f whose existence is guaranteed by Theorem 1. Then by choice of $\epsilon_n(f)$ the polynomial $p_n(x) + \epsilon_n(f)\, T_n(x)$ provides the estimate (a).

Proof of (b). Define $\delta_n(f) = \max_{j=1,\dots,k} \max[0, -f^{(j)}(1)/d_{n,j}]$. Then for all $n > k$

$$t^{(k+1)}(x) + \delta_n(f)\, T_n^{(k+1)}(x) = \delta_n(f)\, T_n^{(k+1)}(x) \geqslant 0, \qquad \forall x \geqslant 1,$$

and

$$t^{(j)}(1) + \delta_n(f)\, T_n^{(j)}(1) \geqslant 0, \qquad \forall j = 1,\dots, k.$$

It follows that

$$[t + \delta_n(f)\, T_n]'(x) \geqslant 0, \qquad \forall x \geqslant 1,$$

and hence that $\epsilon_n(f) \leqslant \delta_n(f)$.

Proof of (c). For $x > 1$, $m = 1, 2, 3,\dots$, $T_m(x) = \cosh m\phi$ and $T_m'(x) = m \sinh(m\phi)/\sinh\phi$, where ϕ is the positive solution of $x = \cosh\phi$. Hence

$$\frac{T_k'(x)}{T_n'(x)} = \frac{k \sinh(k\phi)}{n \sinh(n\phi)} \leqslant \frac{k \exp(k\phi)}{2n \sinh(n\phi)}, \qquad \forall \phi > 0.$$

Also

$$\frac{d}{d\phi}\left[\frac{\exp(k\phi)}{\sinh(n\phi)}\right] = \frac{\exp(k\phi)[k \sinh(n\phi) - n \cosh(n\phi)]}{[\sinh(n\phi)]^2} < 0,$$

for all $\phi > 0$ and $n > k$, so that

$$\max_{x \geqslant \cosh\theta} \frac{T_k'(x)}{T_n'(x)} \leqslant \frac{k}{2n} \cdot \frac{\exp(k\theta)}{\sinh(n\theta)}, \qquad \forall n > k. \tag{4}$$

(4) and the extremal property of the first derivative of a Chebyshev polynomial (see previous discussion, Rivlin [6, pp. 92–93], or Rogosinski [7]) imply

$$\max_{x \geqslant \cosh\theta} \frac{|t'(x)|}{T_n'(x)} \leqslant \| t \| \cdot \max_{x \geqslant \cosh\theta} \frac{T_k'(x)}{T_n'(x)} \leqslant \| t \| \cdot \frac{k}{2n} \cdot \frac{\exp(k\theta)}{\sinh(n\theta)}. \tag{5}$$

(5) and the hypothesis that $t'(x) \geqslant 0$ for all x in the interval $(1, \cosh\theta)$, imply

$$t'(x) + \| t \| \frac{k}{2n} \frac{\exp(k\theta)}{\sinh(n\theta)} \cdot T_n'(x) \geqslant 0, \qquad x \geqslant 1.$$

i.e.,

$$\epsilon_n(f) \leqslant \| t \| \frac{k}{2n} \frac{\exp(k\theta)}{\sinh(n\theta)}.$$

In the particular case of functions $f \in C^2[-1, 1]$ part (b) of Corollary 1 reduces to the estimate

$$E_n^*(f) \leqslant D_2 n^{-2} \omega(f^{(2)}, n^{-1}) + \max\left(0, \frac{-f'(1)}{n^2}, \frac{-3f''(1)}{n^2(n^2-1)}\right).$$

This estimate of $E_n^*(f)$ is of the best possible order in that the following negative result holds:

If $f'(1) < 0$ then for each $\alpha > 0$, $\overline{\lim}_{n\to\infty} n^{2+\alpha} E_n^*(f) = \infty$.

The negative result is a trivial corollary to the following lemma

LEMMA 2. *Let f be a function defined on $[-1, 1]$, $1 > \alpha > 0$, $C > 0$, and $\{p_n \in \pi_n\}_{n=1}^{\infty}$ be a sequence of polynomials with $\|f - p_n\| \leqslant Cn^{-2-\alpha}$, $n = 1, 2, 3,\ldots$. Then $f \in C^1[-1, 1]$ and $\|f' - p_n'\| \leqslant DCn^{-\alpha}$, $n = 1, 2, 3,\ldots$, where D depends only on α.*

Proof. The proof is via Bernstein's well known argument. Let $d(n) = Cn^{-2-\alpha}$. The Markov inequality and the Weierstrass M test imply the series $\sum_{k=0}^{\infty} (p_{n2^{k+1}}' - p_{n2^k}')$ converges uniformly having norm not exceeding

$$2 \sum_{k=0}^{\infty} [(n2^{k+1})^2 \, d(n2^k)] = n^{-\alpha}\left(8c \sum_{k=0}^{\infty} r^k\right) \text{ with } r = (1/2)^{\alpha}.$$

Hence well known theorems about the uniform convergence of series imply f' exists and that $[f' - p_n'] = \sum_{k=0}^{\infty} (p_{n2^{k+1}}' - p_{n2^k}')$. This completes the proof.

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