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Deformable orthogonal grids: lemniscates

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Abstract

In this paper we describe a technique, based on complex polynomials, for creating plane regions with a hole and propose a new method to produce an orthogonal grid on it. The thickness of the grid can be easily controlled and the sizes of the cells can be automatically estimated. The grid is automatically adapted to the boundary of the region. We offer parameters for the control of the geometric shape of the region, which depend on the roots of the polynomial and its derivative.

Keywords: Deformable grids; Lemniscates; Body fitting meshes

1. Introduction

The classical book of Thompson et al. [5] introduces the problem of constructing a grid by adapting a coordinate system to the given region. This consists essentially in establishing a homeomorphism between the physical region where the grid is to be placed and a simpler region like an annulus or a rectangle. The grid lines are then isoparametric lines. A frequent example is the construction of a grid on a region topologically equivalent to an annulus via polar coordinates.

In this paper we consider the problem of constructing orthogonal grids on regions topologically equivalent to an annulus. The main difference with the classical methods is that no 1:1 parametrization is established; instead a complex polynomial mapping is constructed to approximate the whole region to be gridded. The resulting orthogonal grid can be refined without need of recomputation or minimization processes.

Our method depends on the conformal properties coming from complex variable theory. In particular, orthogonality of grid lines is very easily dealt with within this context. Complex variables techniques have been used fairly widely in grid generation. See [7] for a list of references. The main

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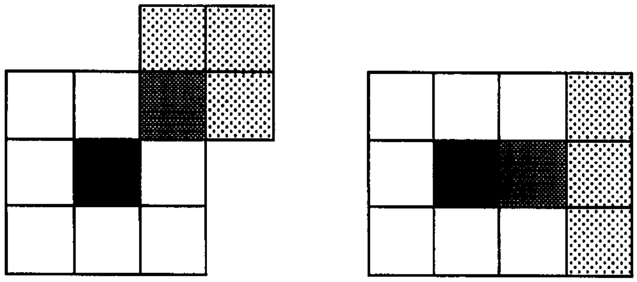

Fig. 1. Tracking directions in Chandler's algorithm.

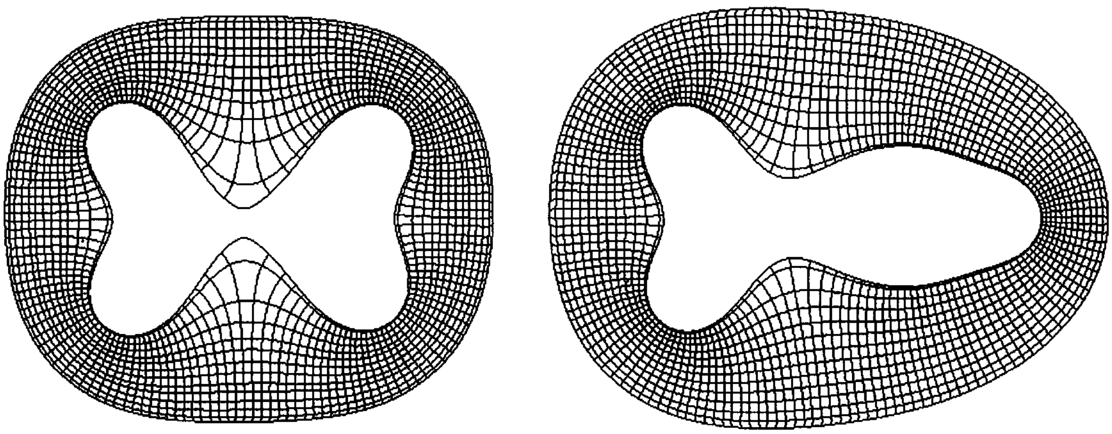

Fig. 2. Interactive region deformation.

novelty of our method is the use of roots and singularities of complex polynomials as interactive control mechanisms for the shape of the physical region to be gridded.

An important consequence of this fact is the following: once the region to be gridded has been approximated, it is easy to construct a grid that interpolates any given boundary or interior points of the region.

2. The grid on a plane region

The display method for the figures in this paper is an adapted version of Chandler's algorithm [2]. Chandler's algorithm is a tracking process to display the points (x, y) that satisfy an equation of the form $h(x, y) = 0$, within a discretization. Usually, it is applied for the purpose of visualization of nonsingular algebraic curves, within the discretization of a computer screen. It starts with a point (x, y) such that $h(x, y) = 0$, then the minimum for $|h(x, y)|$ is chosen among its eight neighbors. This establishes a direction of motion and further evaluations are done among the three points towards which the curve is heading. In Fig. 1, the black square indicates the first point (x, y), the gray the minimum neighbor and the striped squares represent the points where the next evaluations will be performed in order to choose the third point of the curve. The process goes on until it meets a stopping criterion, which depends on the type of grid line being traced: circular or ray.

The technique for creating plane regions with a hole and producing an orthogonal grid on them, is based on level sets given by complex polynomials. By construction, our grids do not present folding problems. The grid is automatically adapted to the boundary of the region and its thickness can be controlled interactively. We also offer parameters for the control of the geometric shape of the region. These are the roots of the defining complex polynomial and the roots of its derivative. Moreover, our graphic interface allows the user to deform the gridded region by pulling the grid along, while preserving its orthogonality, through visually meaningful shape parameters. See Fig. 2.

3. Lemniscates

Given a complex polynomial $f(z)=(z-z_1)\cdots(z-z_n)$ and a positive number ρ, the level curve $|f(z)|=\rho$ is called a lemniscate.

The roots $z_1,\dots,z_n$ are the foci of the lemniscate. If $n=1$, the lemniscate is a circle. If $n=2$ and $z_1\neq z_2$ the topology of the lemniscate depends on ρ and $|z_1-z_2|$. According to ρ, being greater, equal or less than $|z_1-z_2|^2$, the lemniscate $|f(z)|=|(z-z_1)(z-z_2)|=\rho$ consists of a topological circle, a figure eight or two circles, as illustrated in Fig. 3.

Figs. 4–6 illustrate some lemniscates with three foci, so $f'(z)$ has two roots. In Figs. 4 and 5, $f'(z)$ has two simple roots, in Fig. 6 $f'(z)$ has a double root.

Given a polynomial $f(z)$, the main properties of its family of lemniscates, $|f(z)|=\rho$, $\rho>0$ are:

1. For any $\rho>0$, the set $\{z\colon |f(z)|=\rho\}$ is closed and bounded.
2. Given two lemniscates $|f(z)|=\rho_1$ and $|f(z)|=\rho_2$, if $\rho_1<\rho_2$ then $\{|f(z)|<\rho_1\}\subset\{|f(z)|<\rho_2\}$, i.e., the lemniscates corresponding to a given polynomial f are nested.
3. The complex polynomial f, when viewed as a map $w=f(z)$ sends a lemniscate $|f(z)|=\rho$ into a circle centered at the origin.

A lemniscate of the form $|f(z)|=|f(z_0)|$, where z_0 is a root of $f'(z)$, is a singular algebraic curve and will be referred to as a singular lemniscate; otherwise, it will be called regular (see Fig. 7). As Fig. 5 suggests the topology of the lemniscates corresponding to a complex polynomial f is determined by the singular lemniscates: by filling in topological circles. See [6].

We will be interested in regions between lemniscates, because it is very simple to create orthogonal grids on them. In principle, it is possible to approximate an arbitrary region with a hole, by the region between two lemniscates: $|f(z)|=\rho_1$ and ρ_2, for some complex polynomial f. This we will not do here; instead we will show how to deform interactively the region between two regular lemniscates to conform to some prescribed shape or interpolation criteria.

Namely the family of lemniscates of a polynomial f is determined by its roots (and their multiplicities), or equivalently the roots of its derivative and a constant. These are excellent position and shape parameters for the singular lemniscate of the family. Then the region is chosen to be the one between any two consecutive regular lemniscates, i.e., there are no singular lemniscates between them, or equivalently the region between them does not contain any zero of the derivative of f. See Fig. 8.

It is also possible to choose the unbounded region outside the outermost singular lemniscate as presented in Fig. 9.

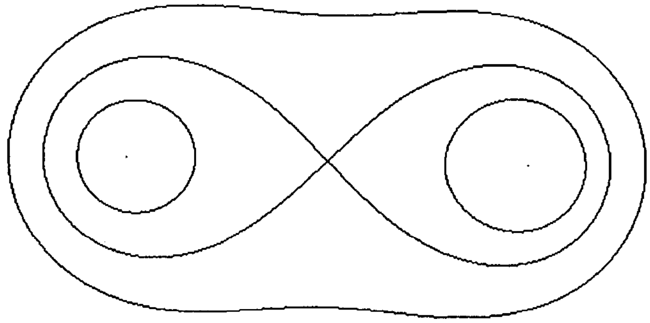

Fig. 3. Lemniscates with two foci.

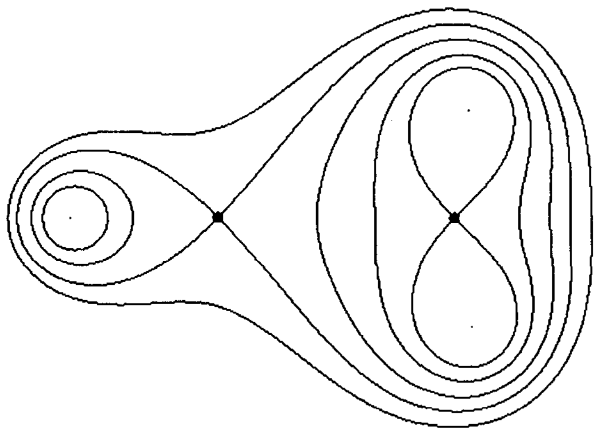

Fig. 4. The roots of $f'(z)=0$ belong to different lemniscates.

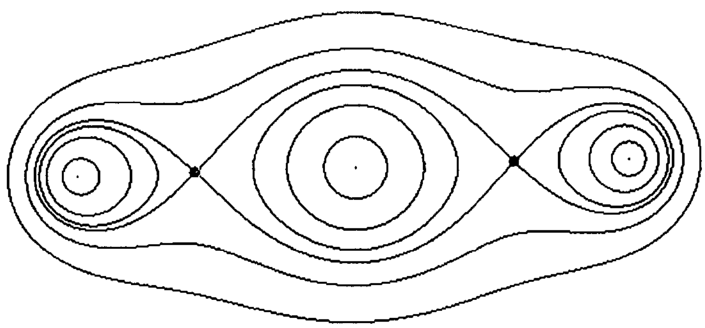

Fig. 5. The roots of $f'(z)=0$ belong to the same lemniscate.

4. Grid construction

Given the physical space, and in order to construct the grid, its external and internal boundaries must be approximated with two lemniscates $\{z\colon\ |f(z)|=\rho_1\}$ and $\{z\colon\ |f(z)|=\rho_2\}$, for some polynomial f and positive numbers ρ_1 and ρ_2. The polynomial f can be controlled via its roots or

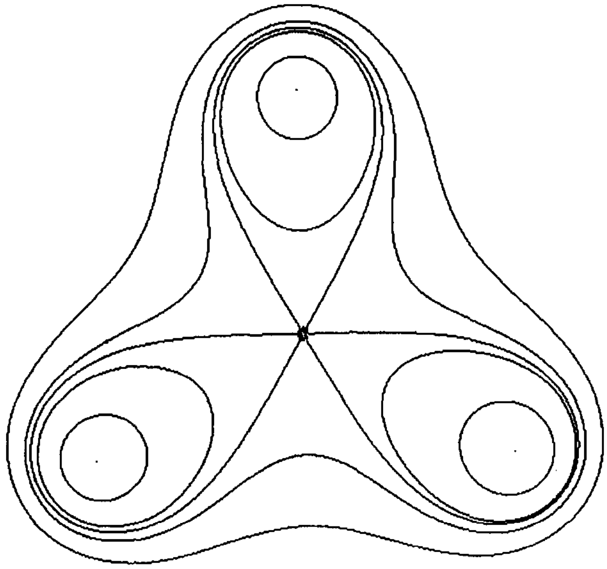

Fig. 6. $f'(z)$ has a double root.

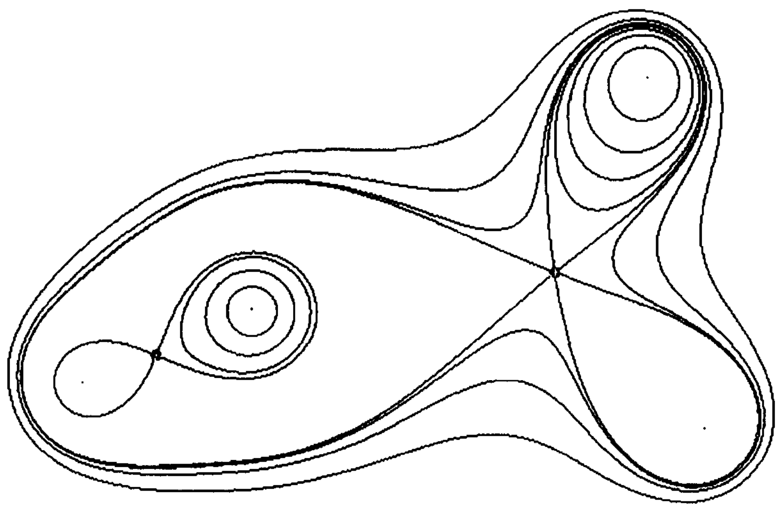

Fig. 7. Singular and regular lemniscates of f.

its singularities (i.e., zeros of f'). In our system these points can be moved interactively. To each polynomial f corresponds a system of singular lemniscates, which moves as the roots of f and the singular points are dragged, or as new roots or singular points are added or removed.

Therefore, a singular system of lemniscates must be constructed, so that the boundaries of the physical region could be modelled by two regular lemniscates lying between two consecutive singular lemniscates. This construction process is interactive.

Once the region to be gridded has been chosen, it is simple to create an adapted orthogonal grid on it.

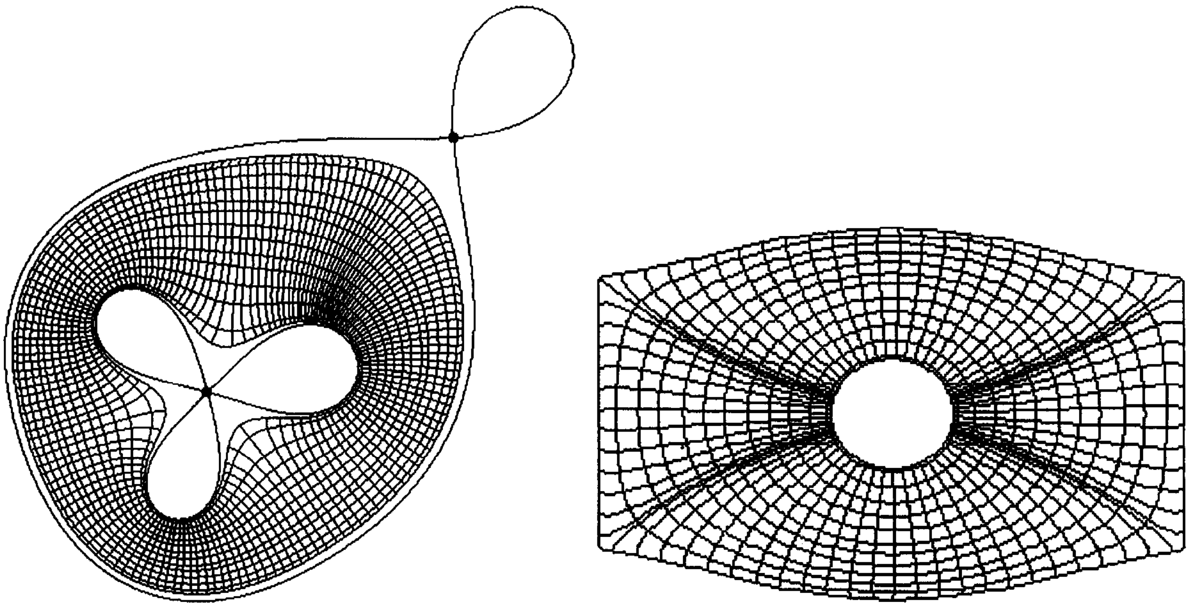

Fig. 8. Region between two consecutive lemniscates.

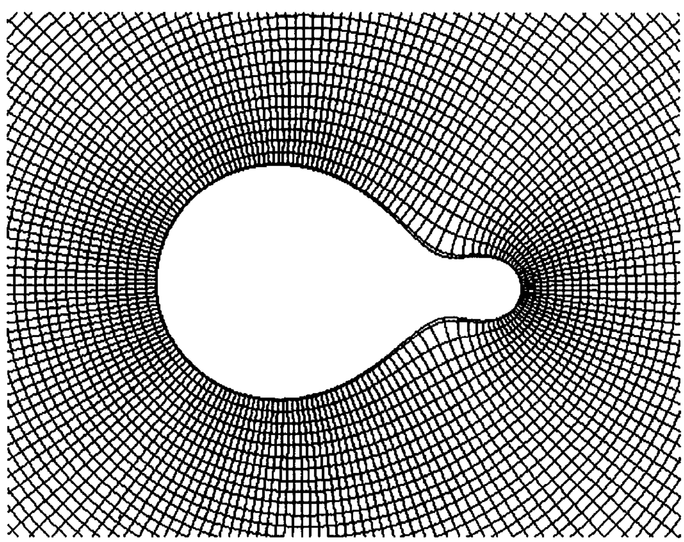

Fig. 9. Unbounded region.

Let f be the generating polynomial, and $|f(z)| = \rho_1$, $|f(z)| = \rho_2$, $\rho_1 < \rho_2$, be two consecutive regular lemniscates. Then, the complex map $w = f(z)$, sends the region $\{z\colon\ \rho_1 < |f(z)| < \rho_2\}$ into the annulus $\{w\colon\ \rho_1 < |w| < \rho_2\}$. The map f is conformal, i.e., it preserves angles, so any standard, not necessarily uniform, grid constructed with concentric circles and radii of the annulus, is transformed into an orthogonal grid in $\{z\colon\ \rho_1 < |f(z)| < \rho_2\}$.

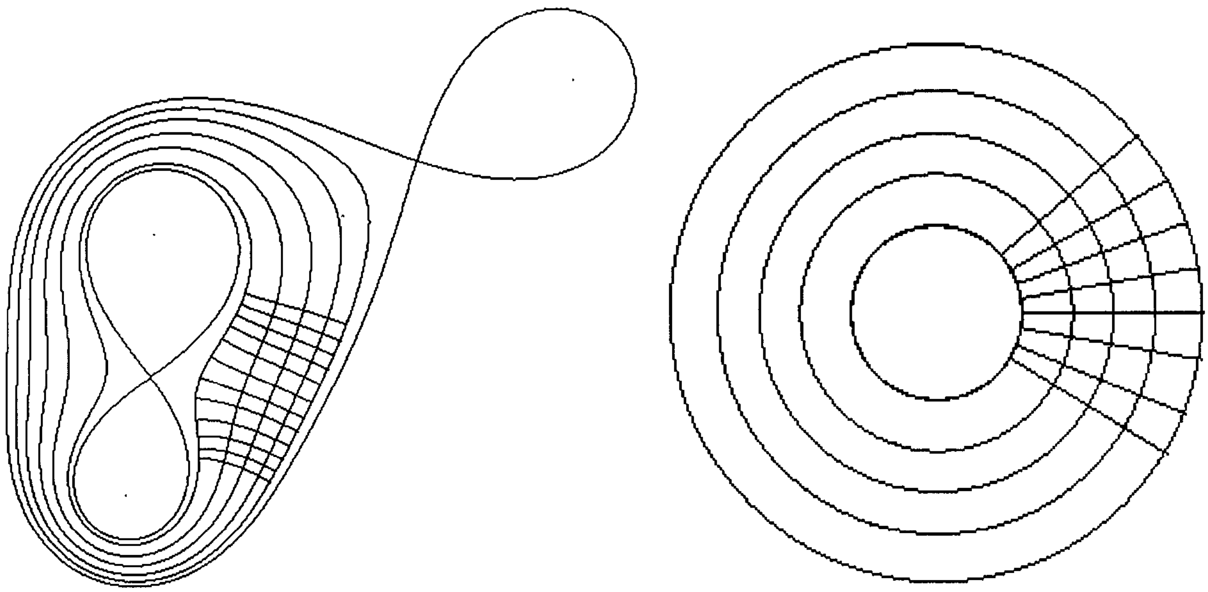

Fig. 10. Construction of the grid.

Note that f is not bijective, so an actual inverse cannot be constructed, but this is not required. The grid lines are just lemniscate components and liftings of straight line segments. See Fig. 10. These can be tracked using Chandler's algorithm.

Our graphic user interface has been designed to favor the creation of a uniform grid in the region, as opposed to mapping a uniform grid from the annulus.

Namely, given a physical domain with a hole, which is bounded by two lemniscates $\{z\colon |f(z)| = \rho_1\}$ and $\{z\colon |f(z)| = \rho_2\}$, with $\rho_1 < \rho_2$, the circular lines of the grid are lemniscates of the form $\{z\colon |f(z)| = \rho\}$, for $\rho_1 < \rho < \rho_2$. The grid rays are liftings (i.e., images under the multivalued inverse of f) of radial segments of the anulus.

To produce a uniform grid that consists of k concentric grid lines (in the sense that they have the same foci) and l grid rays, we mark off l equidistant points on the outer boundary and trace from each a ray. On the shortest ray, k equidistant points are chosen and through each of them the corresponding lemniscate is traced.

Our interface also allows for grid refinement and interpolation, i.e., further grid lines, radial or circular can be readily inserted to augment the density of the originally generated grid. The package is available by request to eyanez@true.net or marco@jade.ciens.ucv.ve.

5. Additional features and future work

Our system actually allows to create orthogonal grids between arbitrary, not necessarily consecutive lemniscates. This of course produces grids which are not structured, i.e. have cells with more than four sides. In fact, grids on regions with several holes can be created. See Fig. 11.

In the context of finite difference schemes, this can be dealt with by introducing special points (one per additional hole), as explained in [5, p. 148].

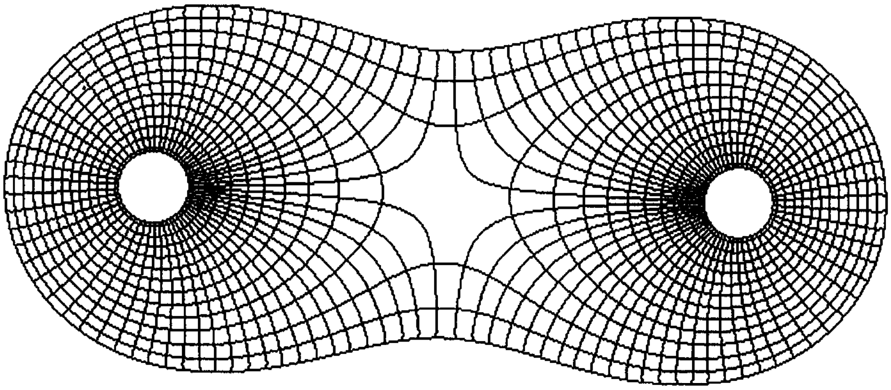

Fig. 11. Orthogonal grid on region with two holes.

A generalization to 3D can be done along the following lines. Given n points in space, a lemniscate surface can be defined as

$$\left\{(x,y,z)\colon \prod_{i=1}^{n}[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2]=\text{constant}\right\}. \tag{1}$$

This is a nested family. Given two such surfaces, and a distribution of points on one of them, one can construct orthogonal trajectories to the surfaces, whose intersections with the intermediate lemniscate surfaces produce an orthogonal grid.

Note added in proof

Marco Paluszny suggested inclusion of the following references: [8, 9]

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