Tensor Products of Matrix Algebras over the Grassmann Algebra

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We develop elementary tools for studying P. I. equivalences between algebras which are tensor products of matrix subalgebras. We then deduce Kemer’s results about the P. I. equivalence of tensor products of matrix subalgebras over the Grassmann algebra. © 1990 Academic Press, Inc.

INTRODUCTION

Let $F$ be a field of characteristic zero, and let $E$ be the infinite dimensional Grassmann (exterior) algebra. Write $E = E_0 \oplus E_1: E_0 = \text{center}(E)$, and if $a, b \in E_1$ then $a_1b_1 = -b_1a_1$. Given $k, l \in \mathbb{N}$, let $n = k + l$ and denote by $M_n(E)$ the algebra of $n \times n$ matrices over $E$; it contains $M_{k,l}(E) \subseteq M_n(E)$, where

$$M_{k,l}(E) = \begin{pmatrix}
E_0 & \cdots & E_0 & E_1 & \cdots & E_1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
E_0 & \cdots & E_0 & E_1 & \cdots & E_1 \\
E_1 & \cdots & E_0 & E_0 & \cdots & E_0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
E_1 & \cdots & E_1 & E_0 & \cdots & E_0
\end{pmatrix}.$$

Both $M_n(E)$ and $M_{k,l}(E)$ are P.I. algebras.

Given two P.I. algebras $A, B$, denote $A \sim B$ if they satisfy the same polynomial identities $\text{Id}(A) = \text{Id}(B)$.

The following three remarkable results are due to Kemer:

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THEOREM 1. \( M_{k,l}(E) \otimes M_{p,q}(E) \cong M_{r,s}(E) \), where \( r = kp + lq \) and \( s = kq + lp \).

THEOREM 2. \( M_{k,l}(E) \otimes E \cong M_{k+1,l}(E) \).

THEOREM 3. \( E \otimes E \cong M_{1,1}(E) \). (Here \( \otimes \) means \( \otimes_{F} \).)

These theorems are consequences of Kemer's structure theory for varieties of associative algebras [2]. An essential tool in that theory is the theorem of Razmyslov and Kemer about the nilpotency of the Jacobson radical in finitely generated P.I. algebras [1, 2, 4].

The main purpose of this paper is to give detailed and direct proofs of the above three theorems, proofs that make no use of the above structure theory nor of the nilpotency theorem. This is done in Section 4 here, while in Sections 1, 2, and 3 we develop the notations and results needed for these proofs. Here, the technique for proving that \( A \sim B \) is, essentially, the following: we construct a linear mapping \( \varphi : A \to B \). Even though \( \varphi \) is neither onto nor one-to-one nor a homomorphism, it almost has these properties:

1. \( \varphi(A) \) is large in \( B \), so \( \varphi(A) \sim B \).
2. \( \varphi \) is a "near-homomorphism" (see Definition 2.4), hence, as in the case of a homomorphism, \( \text{Id}(A) \subseteq \text{Id}(\varphi(A)) \).
3. The kernel of \( \varphi \) is small: if \( f(x) = f(x_1, \ldots, x_n) \) is a polynomial such that \( \varphi \circ f(x) \) is an identity of \( A \), then \( f(x) \) is an identity of \( A \). Thus \( \text{Id}(A) = \text{Id}(\varphi(A)) \). We can therefore conclude that \( A \sim \varphi(A) \sim B \).

The various functions \( \varphi \) in the three proofs are relatives of the "multiplication" function \( \mu : E \otimes E \to E, \mu(x \otimes y) = x \cdot y \).

We now briefly introduce some notations for \( E \):

Let \( V = \text{span}_F\{e_1, e_2, \ldots\} \) be a countable \( F \) space with basis \( \{e_1, e_2, \ldots\} \), then denote by \( E = E(V) \) its Grassmann (exterior) algebra: \( D = \{e_{i_1} \cdots e_{i_l} \mid 1 \leq i_1 < \cdots < i_l, \ l = 0, 1, 2, \ldots\} \) is a basis for \( E : E = \text{span}_F(D) \).

Let \( D(l) = \{e_{i_1} \cdots e_{i_l} \mid 1 \leq i_1 < \cdots < i_l, \ l = 0, 1, 2, \ldots\} \) and \( E^{(n)} = \bigoplus_{l\geq n} \text{span} D(l) \).

Also, let \( E = \bigoplus_{0 \leq l \text{ even}} \text{span} D(l), \ E_{1} = \bigoplus_{1 \leq l \text{ odd}} \text{span} D(l), \ E^{(n)}_{j} = E^{(n)} \cap E_{j}, \ j \in \mathbb{Z}_2 = \{0, 1\} \).

Finally, let \( V_n = \text{span}\{e_1, \ldots, e_n\} \), then denote by \( E(V_n) \) the corresponding Grassmann algebra: \( \dim E(V_n) = 2^n \).

It is shown in Theorem 2.3 that when the \( E^{(n)}_{j} \)'s replace \( E_{j} \)'s (\( j \in \mathbb{Z}_2 \)) in the matrix algebras, the identities remain the same.

Two final remarks: First, the assumption that \( \text{char}(F) = 0 \) is essential here, since in our proofs of P.I. equivalence, we check only the multilinear identities. Second, our tools here enable us to construct new subalgebras...
$A \subseteq M_n(E)$, and we determine P.I. equivalence of such $A$'s with some already known algebras.

1. Matrix-Indexed Systems

Recall that $E = E_0 \oplus E_1$ is $\mathbb{Z}_2$ graded.

1.1. Definition ($\mathbb{Z}_2$ valued sets). Let $I$ be a (finite) set and let $v : I \to \mathbb{Z}_2$; then $(I, v)$ is a $\mathbb{Z}_2$ valued set. Here $I = I_0 \cup I_1$, where $I_g = v^{-1}(g)$, $g \in \mathbb{Z}_2$.

Given a second such set $(I', v')$, $v' : I' \to \mathbb{Z}_2$, we have

$$\tilde{v} - (v, v') : I \times I' \to \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\tilde{v}(i, i') = (v(i), v'(i'))$$

and also

$$v_+ : I \times I' \to \mathbb{Z}_2,$$

$$v_+(i, i') = v(i) + v'(i').$$

1.2. Example. In the algebra $M_{k,I}(E)$, let $I = I_0 \cup I_1$, card$(I_0) = k$, card$(I_1) = l$, with $v : I \to \mathbb{Z}_2$ as above; then $v_+ : I \times I \to \mathbb{Z}_2$ and $M_{k,I}(E) = \{(a_{i,j})_{i,j \in I} | a_{i,j} \in E_{v_+(i,j)} | i, j \in I\}$. Denote $E_{i,j} = E_{v_+(i,j)}$ and note that if $g, h \in \mathbb{Z}_2$ then $E_g E_h \subseteq E_{g+h}$. Thus $E_{l_1,i_1} E_{l_2,i_2} \subseteq E_{l_1 + l_2,i_1}.

To handle the algebras $M_{k,I}(E)$ we introduce

1.3. Definition. (a) Let $K$ be a commutative ring and let $I$ be a (finite) set. The system of $k$-modules

$$\{A_{i,j} | i, j \in I\}$$

is called "matrix indexed" if it satisfies the following condition:

For all $i_1, i_2, i_3 \in I$ there exist a pairing (multiplication) $A_{i_1,i_2} \times A_{i_2,i_3} \to A_{i_1,i_3}$ (i.e., $A_{i_1,i_2} A_{i_2,i_3} \subseteq A_{i_1,i_3}$) which is associative and distributive (i.e., it satisfies the ordinary properties of multiplication): for all $a_{i_1,i_2} \in A_{i_1,i_2}$, etc.,

$$(a_{i_1,i_2} \cdot a_{i_2,i_3}) a_{i_3,i_4} = a_{i_1,i_2}(a_{i_2,i_3} \cdot a_{i_3,i_4}),$$

$$(a_{i_1,i_2} + a'_{i_1,i_2}) a_{i_2,i_3} = a_{i_1,i_2} a_{i_2,i_3} + a'_{i_1,i_2} a_{i_2,i_3}$$

and

$$a_{i_1,i_2} (a_{i_2,i_3} + a'_{i_2,i_3}) = a_{i_1,i_2} a_{i_2,i_3} + a_{i_1,i_2} a'_{i_2,i_3}.$$
(b) Such a system \( \{ A_{i,j} \mid i, j \in I \} \) defines, in a natural way, the matrix algebra \( M_f(A_{i,j} \mid i, j \in I) \): if \( I = \{1, 2, \ldots, n\} \), then

\[
M_f(A_{i,j}) = \left( A_{11}, \ldots, A_{1n} \right) = \left( A_{21}, \ldots, A_{2n} \right) = \cdots = \left( A_{n1}, \ldots, A_{nn} \right).
\]

Formally, \( M_f(A_{i,j}) = \bigoplus_{i,j \in I} A_{i,j} \) as \( k \)-modules, with \( \eta_{i,j} : A_{i,j} \to \bigoplus_{i,j \in I} A_{i,j} \) the natural embeddings. Multiplication in \( M_f(A_{i,j}) \) is now defined to be distributive, and to satisfy

\[
\eta_i(a_{ij}) \eta_p(a_{pq}) = \begin{cases} 0 \in M_f(A_{i,j}) & \text{if } j \neq p \\ \eta_i(a_{ij}a_{pq}) & \text{if } j = p. \end{cases}
\]

Note now that

\[
M_k(E) = M_f(E_{\nu(i,j)}),
\]

where \( I \) is \( \mathbb{Z}_2 \) valued as in Definition 1.1 and Example 1.2.

1.4. Tensor products. Let \( \{ A_{i,j} \mid i, j \in I \} \) and \( \{ B_{p,q} \mid p, q \in P \} \) be two matrix indexed systems of \( K \) modules. Here \( \otimes \) is \( \otimes_K \), and we denote

\[
A_{i,j} \otimes B_{p,q} = D_{(i,p),(j,q)}, \quad (i, p), (j, q) \in I \times P.
\]

By standard arguments for tensor products, the multiplication (or pairing) given by

\[
(a_{i,j} \otimes b_{p,q})(a_{j,k} \otimes b_{q,r}) = a_{i,j}a_{j,k} \otimes b_{p,q}b_{q,r} \in A_{i,k} \otimes B_{p,r}
\]

is well defined, and

\[
\{ D_{a,\beta} \mid a, \beta \in I \times P \}
\]

is a matrix indexed system: it is the tensor product of the above two systems.

1.5. Theorem. Let \( \{ A_{i,j} \mid i, j \in I \} \), \( \{ B_{p,q} \mid p, q \in P \} \) be two matrix indexed systems, with \( \{ D_{a,\beta} \mid a, \beta \in I \times P \} \) as above, their tensor product system. Then we have the \( K \) algebra isomorphism

\[
M_f(A_{i,j}) \otimes_K M_p(B_{p,q}) \cong M_{I \times P}(D_{(i,p),(j,q)}) = M_{I \times P}(A_{i,j} \otimes B_{p,q}).
\]

Proof. As \( K \)-modules

\[
l.h.s. = \left( \bigoplus_{i,j \in I} A_{i,j} \right) \otimes \left( \bigoplus_{p,q \in P} B_{p,q} \right) \cong \bigoplus_{i,j \in I, p,q \in P} (A_{i,j} \otimes B_{p,q}) = r.h.s.
\]
Explicitly, that isomorphism is given by

\[ \eta_{i,j}(a_{i,j}) \otimes \eta_{p,q}(b_{p,q}) \rightarrow \eta_{i+p,j+q}(a_{i,j} \otimes b_{p,q}), \]

and, as is well known, it is also an algebra isomorphism. Q.E.D.

1.6. Definition. Denote \( A \sim B \) if the two P.I. algebras satisfy the same polynomial identities.

1.7. Remark. Let \((I, v), (I', v')\) be two \( \mathbb{Z}_2 \)-valued sets, such that

\[ M_{k,i}(E) = M_{r}(E_{v(i,j)}) \quad \text{and} \quad M_{p,q}(E) = M_{r}(E_{v'_{r}(r,s)}). \]

It follows from Example 1.2 that

\[ M_{k,p + i,q + k + i}(E) \sim M_{r}(E_{v_{r}(i,j) + v'_{r}(r,s)}). \]

By Kemer's theorem (Theorem 1 of the introduction),

\[ M_{k,i}(E) \otimes M_{p,q}(E) \sim M_{k,p + i,q + k + i}(E) \quad (\otimes \text{ is now } \otimes_F, \ F \text{ a field}). \]

By Theorem 1.5, that theorem of Kemer is equivalent to

1.8. Theorem. With notations as in Remark 1.7,

\[ M_{r}(E_{v(i,j) + v'_{r}(r,s)}) \sim M_{r}(E_{v(i,j) - v'_{r}(r,s)}). \]

We shall prove Theorem 1.8 directly, thus providing another proof of Kemer's theorem. That P.I. equivalence will be provided by the map \( \mu^* \).

1.9. Definition. Here \( \otimes \) is \( \otimes_F \). Let \( \mu : E \otimes E \rightarrow E \) be the multiplication \( \mu(x \otimes y) = xy \). Recall the definition of \( E'^{(n)}(g), \ g \in \mathbb{Z}_2 \) (Introduction). Clearly, \( \mu(E_g \otimes F_h) \subseteq F_{g+h} \), where \( g, h, g + h \in \mathbb{Z}_2 \). More precisely, \( \mu(F_0 \otimes E_g) =\mu(E_g \otimes E_0) = E_g \) since \( 1 \in E_0 \), and \( \mu(E_1 \otimes E_1) = E_1^{(2)} \). Denote \( \mu(E_g \otimes E_h) = E_{g+h}^{(1)} \). Given \( \mathbb{Z}_2 \)-valued sets \((I, v), (I', v')\), denote \( E[i, j, r, s] = E_{v(i,j) + v'_{r}(r,s)} \) and \( B_1 = M_{r}(E[i, j, r, s]) \). Also denote \( B = M_{r}(E[v(i,j) + v'_{r}(r,s)]) \) and \( A = M_{r}(E[v(i,j) \otimes E'_{v(r,s)})]. \)

The action of \( \mu \) on the entries of \( A \) induces the map \( \mu^* : A \rightarrow B_1 \subseteq B \) (onto \( B_1 \)). Explicitly, let \( (x_{\alpha, \beta})_{\alpha, \beta} = x \in \mathbb{F} \), then \( \mu^*((x_{\alpha, \beta})) = (\mu(x_{\alpha, \beta})) \).

Obviously, \( \mu^* \) is linear but is Not Even A Ring homomorphism. We call it a "near homomorphism" (see Definition 2.4).

2. Equivalence of P.I. Algebras

The following remark motivates Lemma 2.2.

2.1. Remark. Let \( u_1, \ldots, u_s \in M_{k,i}(E) \); then there exists \( n \in \mathbb{N}, n = n(u_1, \ldots, u_s) \), such that \( u_1, \ldots, u_s \in M_{k,i}(E(V_n)) \). Now let \( u \in F[u_1, \ldots, u_s] \).
and let \( w = e_1 \cdots e_n \), where \( n \leq i_1 \leq i_2 \leq \cdots \leq i_l \). Then \( u = 0 \) if and only if \( wu = 0 \) (trivial).

2.2. **Lemma.** Let \( A \supseteq B \) be two P.I. algebras satisfying the following condition: given any \( u_1, \ldots, u_s \in A \), there exist \( w_1, \ldots, w_s \in \text{center of } A \), such that

1. \( w_i u_i \in B, \ i = 1, \ldots, s \), and
2. For any \( u \in F[u_1, \ldots, u_s] \), \( w_1 \cdots w_s u = 0 \) if and only if \( u = 0 \).

Then \( A \cong B \).

**Proof.** It suffices to show that if \( f(x_1, \ldots, x_s) \) is a multilinear identity of \( B \), then it is also an identity of \( A \). Thus, let \( u_1, \ldots, u_s \in A \), \( u = f(u_1, \ldots, u_s) \), and let \( w_1, \ldots, w_s \in \text{center of } A \), satisfying assumptions (1) and (2). Then

\[
0 = f(w_1 u_1, \ldots, w_s u_s) = w_1 \cdots w_s f(u_1, \ldots, u_s), \text{ hence } f(u_1, \ldots, u_s) = 0. \quad Q.E.D.
\]

As a corollary we obtain

2.3. **Theorem.** Let \( B_1 \subseteq B \) be the algebras in Definition 1.9:

\[
B = M_{I \times I}(E_{v+(i,j)} + v+(r,s)) \quad \text{and} \quad B_1 = M_{I \times I}(E[i, j, r, s])
\]

\( (E[i, j, r, s] \subseteq E_{v+(i,j)} + v+(r,s), \text{ hence } B_1 \subseteq B) \).

Then \( B \cong B_1 \).

**Proof.** Verify Conditions 1 and 2 of Lemma 2.2: given \( u_1, \ldots, u_t \in M_{k,i}(E) \), let \( n \) be an integer such that \( u_1, \ldots, u_t \in M_{k,i}(E(V_n)) \). Recall that \( E = E(V) \), where \( e_1, e_2, \ldots \) is an \( (\infty) \) basis of \( V \) and \( e_1, \ldots, e_n \) is a basis of \( V_n \). Choose \( w_1 = e_n + e_{n+1}, \ w_2 = e_n + e_{n+2}, \ldots \), then (easily) verify conditions (1) and (2). Q.E.D.

2.4. **Definition.** Let \( M(A_{i,j} | i, j \in I) \) and \( M(B_{i,j} | i, j \in I) \) be two matrix algebras as in Definition 1.3(b), and let \( A \subseteq M(A_{i,j}) \) and \( B \subseteq M(B_{i,j}) \) be two corresponding subalgebras. Let \( \varphi: A \to B \) be a linear map. Call \( \varphi \) a "near homomorphism" if there exists a subset \( S \subseteq A \) which satisfies

1. The linear span of \( S \) is \( A \).
2. Given \( \tilde{x}_1, \ldots, \tilde{x}_n \in S \) and \( i, j \in I \), there exists \( \epsilon = \epsilon(i, j, \tilde{x}_1, \ldots, \tilde{x}_n) \neq 0 \) such that for all \( \sigma \in S_n \), the \( (i, j) \) entries satisfy

\[
(\varphi(\tilde{x}_{\sigma(1)} \cdots \tilde{x}_{\sigma(n)}))_{i,j} = \epsilon \cdot (\varphi(\tilde{x}_{\sigma(1)}) \cdots \varphi(\tilde{x}_{\sigma(n)}))_{i,j}
\]

(independent of \( \sigma \), \( S_n \) being the Symmetric group).
2.5. THEOREM. Let \( \varphi: A \to B \) be a near homomorphism of \( A \) onto \( B \) (as in Definition 2.4) with a corresponding subset \( S \subseteq A \). Then every identity of \( A \) is also an identity of \( B \). Assume furthermore that \( \varphi \) is cancelable: for any (multilinear) polynomial \( f(x) = f(x_1, ..., x_n) \), \( \varphi \circ f(x) \) is an identity of \( A \) if and only if \( f(x) \) is.

Then \( A \sim B \).

Proof. Since \( \text{char } F = 0 \), we prove the lemma for multilinear identities: let \( f(x_1, ..., x_n) = \sum_{\sigma \in S_n} \alpha_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)} \) be an identity of \( A \). Since \( \varphi(S) \) spans \( B \), it suffices to show that for any \( i, j \in I \) and \( \bar{x}_1, ..., \bar{x}_n \in S \), \( (f(\bar{x}_1), ..., f(\bar{x}_n)))_{i,j} = 0 \). Let \( \varepsilon = \varepsilon(i, j, \bar{x}_1, ..., \bar{x}_n) \) be as in Definition 2.4; then \( 0 = (f(\bar{x})))_{i,j} = \sum_{\sigma} \alpha_{\sigma}(f(\bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(n)}))_{i,j} = \varepsilon(f(\bar{x}))_{i,j} \), hence \( f(x) \) is an identity of \( B \). Finally, if \( \varphi \) is cancellable, then the same argument shows that every P.I. of \( B \) is a P.I. of \( A \). Q.E.D.

In the remainder of this section we show that the map \( \varphi = \mu^* \) of Definition 1.9 is cancelable (Theorem 2.9).

2.6. Notations. Recall from Definition 1.9

\[
A = M_{I \times I'}(E_{v, (i, j)} \otimes E_{v', (r, s)}),
\]

\[
B_1 = M_{I \times I'}(E[i, j, r, s]), \quad \text{and} \quad \varphi = \mu^*: A \to B_1.
\]

Denote \( V = I \times I' \). Given \( u, v \in V \), let \( e_{u, v} \) be the corresponding matrix unit: \( e_{u, v} \in M_{I \times I'}(E) \) (but, depending on \( u, v \), it is possible that \( e_{u, v} \notin A \) or \( e_{u, v} \notin B_1 \)). With the notation of Definition 1.1, given \( u, v \in V \), write (in \( \mathbb{Z}_2 \times \mathbb{Z}_2 \))

\[
\tilde{v}(u) + \tilde{v}(v) = (g, h), \quad g, h \in \mathbb{Z}_2,
\]

then denote \( E(\tilde{v}(u) + \tilde{v}(v)) = E_g \otimes E_h \). Recall that \( D = D_0 \cup D_1 \) is a basis of \( E \).

Finally, denote

\[
S = \{ (c \otimes d) e_{u, v} \mid u, v \in V \text{ and } c \in D_g, d \in D_h \text{ where } \tilde{v}(u) + \tilde{v}(v) = (g, h) \}.
\]

2.7. Note. In the above, let \( u = (i, r), \ v = (j, s) \in I \times I' \); then \( \tilde{v}(u) + \tilde{v}(v) = (v(i) + v(j), v'(r) + v'(s)) = (v_+(i, j), v'_+(r, s)) = (g, h) \). So, if \( c \in D_g \) and \( d \in D_h \), then \( c \otimes d \in E_{v_+(i, j)} \otimes E_{v'_+(r, s)} \), and these elements \( c \otimes d \) clearly span \( E_{v_+(i, j)} \otimes E_{v'_+(r, s)} \) linearly.

It therefore follows that the set \( S \) in Notations 2.6 spans the above algebra \( A \) linearly.

We shall need the following:

2.8. LEMMA. Let \( a_1, ..., a_n \in D \) (= basis of \( E \)). Then there exist an algebra
homomorphism $\psi : E \rightarrow E$ and $a'_1, \ldots, a'_n \in D$ such that $\psi(a'_i) = a_i$, $1 \leq i \leq n$, and $a'_1 \cdots a'_n \neq 0$.

Proof. Since $\dim E = \infty$, there are $a'_1, \ldots, a'_n \in D$ such that $a'_1 \cdots a'_n \neq 0$ and $a'_i \in D_0$ if and only if $a_i \in D_0$, $1 \leq i \leq n$. Define, first, $\psi(a'_1) = a_1$. Let $F[a'_1, \ldots, a'_n] \subseteq E$ be the subalgebra generated by $a'_1, \ldots, a'_n$. Since $a'_i a'_j = \pm a'_i a'_j$ for all $1 \leq i, j \leq n$, it follows that $H = \{a'_1 \cdots a'_i | 1 \leq i < \cdots < i, \leq n\}$ is a (linear) basis of $F[a'_1, \ldots, a'_n]$. On such basis elements, define now $\psi(a'_1 \cdots a'_i) = a_1 \cdots a_i = \psi(a'_i) \cdots \psi(a'_i)$, then extend—by linearity—to $\psi : F[a'_1, \ldots, a'_n] \rightarrow E$. By the choice of the $a'_i$'s, $\psi$ is an algebra homomorphism. Finally, write $E = F[a'_1, \ldots, a'_n] \oplus E'$ (as vector spaces) and extend $\psi$ trivially to $E' : \psi(E') = 0$.

2.9. Theorem. The map $\mu^* : A \rightarrow B$, of Definition 1.9 is cancelable (see Theorem 2.5): let $f(x) = f(x_1, \ldots, x_n)$ be a multilinear polynomial. Then $\mu^* \circ f(x)$ is an identity of $A$ if and only if $f(x)$ is an identity of $A$.

Proof. Assume that $\mu^* \circ f(x)$ is an identity and show that $f(x)$ is also an identity of $A$. Let $S \subseteq A$ be as in Notations 2.6 and let $\bar{x}_i = (c_i \otimes d_i) e_{u,v} \in S$, $1 \leq i \leq n$. Since $S$ spans $A$, it suffices to show that $f(x) = f(x_1, \ldots, x_n) = 0$.

Case 1. Assume $c_1 \cdots c_n \cdot d_1 \cdots d_n \neq 0$. Calculate the entries of the $\langle c_1 \cdots c_n \otimes d_1 \cdots d_n \rangle$ entry of $f(\bar{x})$ (i.e., the coefficient of $e_{u,v} \in f(\bar{x})$) is $x_{u,v} * c_1 \cdots c_n \otimes d_1 \cdots d_n$, for some $x_{u,v} \in F$. Thus, $x_{u,v} * c_1 \cdots c_n \cdot d_1 \cdots d_n$ is the $u, v$ entry in $0 = \mu^* \circ f(\bar{x})$. Since $c_1 \cdots c_n \cdot d_1 \cdots d_n \neq 0$, $x_{u,v} = 0$, hence $f(x) = 0$.

Case 2. Any $c_1, \ldots, c_n, d_1, \ldots, d_n$. As in Lemma 2.8, let $\psi : E \rightarrow E$ be an algebra homomorphism with $c_1', \ldots, c_n', d_1', \ldots, d_n' \in D$ such that $\psi(c'_i) = c_i$, $\psi(d'_i) = d_i$, $c'_i \in D_0$ if and only if $c_i \in D_0$, similarly for $d'_i$ and $d_i$, and $c'_1 \cdots c'_n \cdot d'_1 \cdots d'_n \neq 0$. Denote $\bar{x}'_i = (c'_i \otimes d'_i) e_{u,v}$, then $\bar{x}'_i \in S$, $1 \leq i \leq n$. Extend $\psi$ to the algebra homomorphism $\psi^* : A \rightarrow A$ via the action of $\psi \otimes \psi$ on the entries:

$$\psi^*(c' \otimes d') e_{u,v} = (\psi(c') \otimes \psi(d')) e_{u,v}.$$  

Thus $\psi^*(\bar{x}'_i) = \bar{x}'_i$, $1 \leq i \leq n$, hence

$$0 = \psi^*(0) = \psi^*(f(\bar{x}')) = f(\psi^*(\bar{x}')) = f(\bar{x}).$$

3. $\mu^*$ is a Near Homomorphism

We know from Definition 1.9 that $\mu^* : A \rightarrow B$ is a linear mapping. Here we show that $\mu^*$ is a near homomorphism (Definition 2.4) with the subset
$S \subseteq A$ as in Notations 2.6. We shall see that here $\varepsilon = \pm 1$ (Definition 3.3, Theorem 3.10).

We fix the following.

3.1. **Notations.** Fix $\bar{x}_i = (c_i \otimes d_i) e_{u_i,v_i} \in S$, $1 \leq i \leq n$ (as in the proof of Theorem 2.9), then denote $e_i = \prod_{t=1}^{n} e_{u_i(t),v_i(t)}$, $\bar{x}_\sigma = \prod_{t=1}^{n} \bar{x}_{\sigma(t)}$, where $\sigma \in S_n$. If $e_\sigma = 0$ then $\bar{x}_\sigma = 0$. Assume $e_\sigma \neq 0$ for some $\sigma \in S$. For simplicity, assume $e_1 \neq 0$. Hence $v_t = u_{t+1}$, $1 \leq t \leq n-1$. Denote also $v_n = u_{n+1}$ and $(u) = (u_1, \ldots, u_{n+1})$. Call $\sigma \in S_n$ "$(u_1, \ldots, u_{n+1})$ permissible" if $e_\sigma \neq 0$: $u_{\sigma(t+1)} = u_{\sigma(t)+1}$, $1 \leq t \leq n-1$, then denote for such $\sigma$

$$(u)\sigma = (u_{\sigma(1)}, \ldots, u_{\sigma(n)}, u_{\sigma(n)+1}).$$

Note that if $\eta \in S_n$ is $((u)\sigma)$ permissible, then $((u)\sigma)\eta = (u)((\sigma)\eta)$ and $\sigma\eta$ is $u$ permissible.

3.2. **Remark.** Let $\bar{x}_i \in S$ as in Notations 3.1; then clearly $d_1 c_2 = \varepsilon c_2 d_1$, $\varepsilon = \pm 1$, and hence $\mu^*(\bar{x}_1 \cdots \bar{x}_n) = \varepsilon^{\mu^*(\bar{x}_1)} \mu^*(\bar{x}_2)$. If both sides here are $\neq 0$, $\varepsilon$ is unique: it depends on the parities of $d_1$ and $c_2$, hence $\varepsilon$ is determined by $u_1, u_2, u_3$. Similarly for any product of $\bar{x}_i$'s, and in particular for the product $\bar{x}_1 \cdots \bar{x}_n$: here $\varepsilon = \pm 1$ is determined by $\mu = (u_1, \ldots, u_{n+1})$.

We formally introduce

3.3. **DEFINITION.** (a) Let $\bar{x}_1, \ldots, \bar{x}_n \in S$ and $(u) = (u_1, \ldots, u_{n+1})$ as in Notations 3.1, then define $\varepsilon((u)) = \varepsilon(u_1, \ldots, u_{n+1})$ via

$$\mu^*(\bar{x}_1 \cdots \bar{x}_n) = \varepsilon(u_1, \ldots, u_{n+1}) \cdot \mu^*(\bar{x}_1) \cdots \mu^*(\bar{x}_n).$$

(b) If $\sigma \in S_n$ is $(u)$ permissible, then $\varepsilon((u)\sigma)$ is given by

$$\mu^*(\bar{x}_{\sigma(1)} \cdots \bar{x}_{\sigma(n)}) = \varepsilon((u)\sigma) \mu^*(\bar{x}_{\sigma(1)}) \cdots \mu^*(\bar{x}_{\sigma(n)}).$$

3.4. **Remark.** Recall from Notations 2.6 that for $g, h \in \mathbb{Z}_2$, $E(g, h) = \text{def} E_g \otimes E_h$. Also, $\bar{x}_i = (c_i \otimes d_i) e_{u_i,v_i}$, $c_i \otimes d_i \in E(\bar{v}(u_i) + \bar{v}(v_i))$, $1 \leq i \leq n$. Since here $v_t = u_{t+1}$, $1 \leq t \leq n$ (and $z + z = 0$ in $\mathbb{Z}_2 \times \mathbb{Z}_2$), it follows that for any $1 \leq a < b \leq n$,

$$\prod_{t=a}^{b} (c_i \otimes d_i) e_{u_t,v_{t+1}} \in E(\bar{v}(u_a) + \bar{v}(u_{b+1})).$$

3.5. **LEMMA.** Let $\varepsilon(u)$ as in Definition 3.3 and let $1 \leq a \leq n-1$, then

$$\varepsilon(u_1, \ldots, u_{n+1}) = \varepsilon(u_1, \ldots, u_{a+1}) \varepsilon(u_{a+1}, \ldots, u_{n+1}) \varepsilon(u_1, u_{a+1}, u_{n+1}).$$

**Proof.** Let $\bar{x}_1, \ldots, \bar{x}_n$ as in Notations 3.1. By Remark 3.4,

$$\prod_{t=1}^{a} \bar{x}_t = (c' \otimes d') e_{u_1, u_{a+1}}, \quad c' \otimes d' \in E(\bar{v}(u_1) + \bar{v}(u_{a+1}))$$
and
\[ \prod_{t=1}^{n} \tilde{x}_t = (c^n \otimes d^n) e_{u_0, u_{a+1}} e_{u_{a+1}, u_{a+1}}, \quad c^n \otimes d^n \in E(\tilde{v}(u_{a+1}) + \tilde{v}(u_{a+1})). \]

The proof now follows by computing and comparing the actions of \( \mu \) on both sides of the equation:
\[ \prod_{t=1}^{n} \tilde{x}_t = \left( \prod_{t=1}^{n} \tilde{x}_t \right) \cdot \left( \prod_{t=a+1}^{n} \tilde{x}_t \right). \]
Q.E.D.

3.6. **Graph Theory.** Let \( G = (V(G), E(G)) \) be a graph, \( e_t \in E(G) \) (directed edges), \( 1 \leq t \leq n \), with origin \( u_t \) and terminus \( v_t \); write \( e_t = u_t e_t v_t \). Then \( w = e_1 \cdots e_n = (u_1 e_1 v_1) \cdots (u_n e_n v_n) \) is a \((u_1, v_n)\) walk if and only if \( v_t = u_{t+1}, 1 \leq t \leq n - 1 \). Now, \( \sigma \in S_n \) is a \((u_1, v_n)\) w-permutation if \( (w)\sigma = e_{\sigma(1)} \cdots e_{\sigma(n)} \) is also a \((u_1, v_n)\) walk [5]. The main result in [5] is that if \( \sigma \in S_n \) is a \((u_1, v_n)\) w-premutation, then \( \sigma \) is a successive product of very simple walk-permutations (called \(" \text{block-transpositions}\"\)). We shall state that result below (Theorem 3.8) in matrix language, but we first establish the connection, due to R.G. Swan [6], between graphs and matrices.

3.7. **Swan’s Correspondence.** The rows and columns of the matrix algebra are indexed by the set of vertices \( V = V(G) \). Given \( u, v \in V \), the unit matrix \( e_{u,v} \) corresponds to the (directed) edge \( u \rightarrow v \), and for a sequence \( w, w = (u_1 e_1 v_1) \cdots (u_n e_n v_n) \leftrightarrow e_{u_1, v_1} \cdots e_{u_n, v_n} \). Thus \( \prod_{t=1}^{n} e_{u_t, v_t} \neq 0 \) if and only if \( w \) is a \((u_1, v_n)\) walk.

Recall from [5] that \( \eta \in S_n \) is a block transposition if we can write \( x_1 \cdots x_n = ABCDE \) and \( x_{\eta(1)} \cdots x_{\eta(n)} = ADCBE \) \((B, D \neq 1)\). We can now translate:

3.8. **Theorem [5].** Let \( e_{u_t, v_t} = e_{u_{t-1}, u_{t+1}} \), \( 1 \leq t \leq n \), as in Notations 3.1, so \( \prod_{t=1}^{n} e_{u_t, u_{t-1}} = e_{u_1, u_{a+1}} \). Let \( \sigma \in S_n \) such that (also) \( e_{\sigma} = e_{u_1, u_{a+1}} \). Then there exist block-transpositions \( \eta^{(1)}, \ldots, \eta^{(m)} \in S_n \) such that if we write \( \eta^{(1)} \cdots \eta^{(m)} = \sigma^{(l)}, 1 \leq l \leq m \), then

1. \( \sigma = \sigma^{(m)} \)
2. For each \( 1 \leq l \leq m \),
\[ e_{\sigma^{(l)}} = e_{u_1, u_{a+1}}. \]

We can now prove the following crucial

3.9. **Lemma.** Let \( \tilde{x}_t, 1 \leq t \leq n \) be as in Notations 3.1 with corresponding
(y) = (u_1, ..., u_{n+1}), let σ ∈ S_n be (y) permissible such that u_1 = u_{σ(1)}, u_{n+1} = u_{σ(n)+1}, and let ε be as in Definition 3.3. Then ε(y) = ε((y)σ).

Proof. It clearly follows from Notations 3.1 and Theorem 3.8 that it suffices to prove Lemma 3.9 in the case where σ is a (y) permissible block transposition.

We assume that x_1 ⋅ ⋅ ⋅ x_n = ABCDE and x_{σ(1)} ⋅ ⋅ ⋅ x_{σ(n)} = ADCBE. Moreover, u_{σ(1)} = u_1 and u_{σ(n)+1} = u_{n+1}. It follows that we can assume, w.l.o.g., that A = E = 1, x_1 ⋅ ⋅ ⋅ x_n = BCD, and x_{σ(1)} ⋅ ⋅ ⋅ x_{σ(n)} = DCB. These blocks clearly correspond to the sequences B → (u_1, ..., u_{a+1}), C → (u_{a+1}, ..., u_{b+1}), and D → (u_{b+1}, ..., u_{n+1}). Since σ is (y) permissible,

u_1 = u_{b+1} \quad \text{and} \quad u_{a+1} = u_{n+1}.

(*)

Moreover,

\[
\begin{array}{ccc}
D & \xrightarrow{\sigma} & C \\
\downarrow & & \downarrow \\
\underbrace{u_{b+1}, \ldots, u_{n+1}, u_{a+1}, u_{a+2}, \ldots, u_{b+1}, u_2, \ldots, u_1} & & \underbrace{u_{a+1}}
\end{array}
\]

Now apply Lemma 3.5 (twice): ε(y) = ε_1 ε_2 ε_3 ε_4 ε_5, where ε_1 = ε(u_1, ..., u_{a+1}), ε_2 = ε(u_{a+1}, ..., u_{b+1}), ε_3 = ε(u_{b+1}, ..., u_{n+1}), ε_4 = ε(u_1, u_{a+1}, u_{b+1}) and ε_5 = ε(u_1, u_{b+1}, u_{n+1}). It follows from (*) above, by a similar calculation, that ε((y)σ) = δ_5 ε_2 ε_1 δ_4 δ_5, where δ_4 = ε(u_{b+1}, u_{n+1}, u_{b+1}) and δ_5 = ε(u_{b+1}, u_{b+1}, u_{n+1}). Finally, by (*) above, ε_4 = δ_4 and ε_5 = δ_5, and the proof is complete. Q.E.D.

We can now prove

3.10. Theorem. The map μ*: A → B_1 of Definition 1.9 is a near homomorphism (see Definition 2.4).

Proof. Let S ⊆ A be as in Notations 2.6, Note 2.7, with \( \bar{x}_i = (c_i \otimes d_i) e_{u,v} \varepsilon S, 1 ≤ i ≤ n \). Let σ ∈ S_n, u, v ∈ V, and denote

\[
\text{l.h.s.}(σ) = \left( \mu^* \left( \prod_{i=1}^n \bar{x}_{σ(i)} \right) \right)_{u,v}
\]

\[
\text{r.h.s.}(σ) = \left( \prod_{i=1}^n \mu^* \left( \bar{x}_{σ(i)} \right) \right)_{u,v}.
\]

As in Definition 3.3, we know that (l.h.s.)(σ) = ε(σ, u, v, x_1, ..., x_n) * (r.h.s.)(σ). We show that ε is in fact independent of σ. We may assume that for some σ ∈ S_n, (l.h.s.)(σ), (r.h.s.)(σ) ≠ 0. Without loss of generality, we
may assume that (l.h.s.)(1), (r.h.s.)(1) \neq 0. Thus \( v_t = u_{t+1}, 1 \leq t \leq n \) (hence \((y) = (u_1, \ldots, u_{n+1})\) is given). Write \((l.h.s.)(1) = e \ast (r.h.s.)(1)\) \((e = \pm 1)\). We now show that for that \(e\), \((l.h.s.)(\sigma) = e(r.h.s.)(\sigma)\) for any \(\sigma \in S_n\).

Clearly, \((l.h.s.)(\sigma) = 0\) if and only if \((r.h.s.)(\sigma) = 0\), hence assume both \(\neq 0\). It follows that \(\sigma\) is \((u)\) permissible and \(u_1 = u_{\sigma(1)} = u u_{\sigma(n+1)} = u_{n+1} = v\).

By Lemma 3.9 we now have

\[
\mu^* \left( \prod_{i=1}^n \tilde{x}_{\sigma(i)} \right) = \varepsilon(u) \prod_{i=1}^n \mu^*(\tilde{x}_{\sigma(i)})
\]

(where \(\varepsilon = \varepsilon(u) = \varepsilon((u)\sigma)\)), hence the same is true for the \((u, v)\) entries.

Q.E.D.

4. Applications: Kemer's Result

4.1. THEOREM (Kemer). Let \(k, l, p, q \in \mathbb{N}\) and let \(\alpha = kp + lq, \beta = kq + lp\). Then the algebras \(A_1 = M_{k,l}(E) \otimes M_{p,q}(E)\) and \(B = M_{\alpha,\beta}(E)\) satisfy the same polynomial identities: \(A_1 \sim B\) (here \(\oplus\) is \(\otimes\)).

Proof. The algebra \(A_1\) is isomorphic, by Theorem 1.5, to the algebra \(A\) of Definition 1.9, while the algebra \(B\) of Definition 1.9 is the same algebra as \(B = M_{\alpha,\beta}(E)\) here (Remark 1.7). As in Definition 1.9, let \(B_1 = M_{1 \times 1}(E[i,j,r,s])\). By Theorem 2.3, \(B \sim B_1\). By Theorems 3.10, 2.9, and 2.5, \(A_1 \sim B_1\).

Q.E.D.

As a second application we now prove Kemer's second theorem:

4.2. THEOREM (Kemer). \(M_{k,l}(E) \otimes E \sim M_{k+1,l}(E)\).

To prove Theorem 4.2 we first introduce

4.3. DEFINITIONS. (a) Let \(a = a_0 + a_1 \in E, a_0 \in E_0, a_1 \in E_1\), and define \(g_0(a_0 + a_1) = (a_0, a_1)\) and \(g_1(a_0 + a_1) = (a_1, a_0)\). Then \(g_1, g_0: E \to M_2(E)\).

Denote \(g_i(E) = \Omega_i, i = 1, 2, \Omega_0 = \Omega_0 \oplus \Omega_0 \subset M_2(E)\) (in fact, \(\Omega_0 \subset M_{1 \times 1}(E)\)).

Note that \(\Omega_0\) is an algebra, \(\Omega_0 \cong E\) (and \(\Omega_1\) is an \(\Omega_0\) module).

Clearly, \(\Omega = \{(x, y) | x, y \in E\}\) is an algebra.

(b) Define \(f: \Omega \to E\) as follows: \(f(x, y) = x + y\). Trivially, \(f\) is an algebra homomorphism. Also, \(f \circ g_0 = f \circ g_1 = 1_E\), hence \(f|_{\Omega_0}\) and \(f|_{\Omega_1}\) are one-to-one.

(c) Define the algebra \(\mathcal{U}\) as follows: let \((I, \nu)\) be a \(\mathbb{Z}_2\) valued set (with \(|I_0| = k, |I_1| = l\)), then define \(\mathcal{U} \subseteq M_{2(k+l)}(E)\) by \(\mathcal{U} = \{g_{\nu,\nu}(x_{i,j}) | i, j \in I, x_{i,j} \in E\}\). In fact, let \(I' = I \times \mathbb{Z}_2\) with the obvious \(\mathbb{Z}_2\) valuation: \(\nu'(i, z) = \nu(i) + z \in \mathbb{Z}_2\), then \(\mathcal{U} \subseteq M_{I'}(E_{\nu'(i,z)} | r, s \in I')\).
4.4. Lemma. With the notations and definitions of Definition 4.3, \( \mathfrak{A} \cong M_f(E) \).

Proof. Define \( f^* : \mathfrak{A} \to M_f(E) \) via
\[
f^*(g_{r,(i,j)}(x_{i,j})) = (f \circ g_{r,(i,j)}(x_{i,j})) = (x_{i,j}) \quad (i, j \in I).
\]
Now \( f^* \) is an algebra homomorphism since \( f \) was such; \( f^* \) is obviously onto; it is one-to-one since both \( f \mid \Omega_0, \ f \mid \Omega_1 \) were such. Q.E.D.

4.5. Remark. Let \( E' = E \setminus F \); let \( \mathfrak{A} \) be as in Definition 4.3(c); and let \( \mathfrak{A} \supseteq \mathfrak{A}' = \{ \ldots \ \mid x_{i,j} \in E' \} \). By an argument similar to that of Theorem 2.3, \( \mathfrak{A} \sim \mathfrak{A}' \). Thus, for any \( \mathfrak{A}' \subseteq \mathfrak{A}'' \subseteq \mathfrak{A}, \mathfrak{A}'' \sim \mathfrak{A} \).

4.6. The Proof of Theorem 4.2. Let
\[
M_{k,l}(E) = M_f(E_{v+(i,j)},) \quad M_{1,1}(E) = M_{Z_2}(E_{z_1 + z_2} | z_1, z_2 \in Z_2),
\]
and denote \( \mathfrak{A}_1 = M_f(E_{v+(i,j)}) \otimes E \).

The embedding \( g_0 : E \hookrightarrow M_{1,1}(E) \) induces the embedding
\[
\tilde{g}_0 : \mathfrak{A}_1 \hookrightarrow M_f(E_{v+(i,j)}) \otimes M_{Z_2}(E_{z_1 + z_2}) \equiv M_{I \times Z_2}(E_{v+(i,j)} \otimes Z_{z_1 + z_2}) = \Omega.
\]
Denote \( \mathfrak{A} = \tilde{g}_0(\mathfrak{A}_1) \subseteq \Omega \) and apply \( \mu^* \):
\[
\mu^* : \Omega \to M_{I \times Z_2}(E_{v+(i,j)} + z_1 + z_2).
\]
Let \( \mu^*(\mathfrak{A}) = \mu^*(\tilde{g}_0(\mathfrak{A}_1)) = \mathfrak{A}'' \), and let \( \mathfrak{A}, \mathfrak{A}' \) be as in Definition 4.3(c) and in Remark 4.5; then it is easy to check that \( \mathfrak{A} \subseteq \mathfrak{A}'' \subseteq \mathfrak{A} \). By Remark 4.5, \( \mathfrak{A}'' \sim \mathfrak{A} \sim \mathfrak{A}' \).

The same proof that \( \mu^* \) was a cancelable near homomorphism on \( \Omega \) applies now to the restriction of \( \mu^* \) to the subalgebra \( \mathfrak{A} \subseteq \Omega \), with the corresponding spanning subset \( S \cap \mathfrak{A} \). Hence, by Theorem 2.5,
\[
\mathfrak{A} \sim \mu^*(\mathfrak{A}) = \mathfrak{A}'' \sim \mathfrak{A}.
\]
Also, since \( g_0 \) is an embedding, \( \mathfrak{A}_1 \cong \mathfrak{A} \), hence \( \mathfrak{A}_1 \sim \mathfrak{A} \).

Finally, by Lemma 4.4, \( \mathfrak{A} \cong M_f(E) = M_{k+l}(E) \) (hence \( \sim \)) and the proof follows. Q.E.D.

We now turn to the proof of Kemer's third theorem.

4.7. Theorem (Kemer). \( E \otimes E \sim M_{1,1}(E) \ (\otimes \ is \ \otimes_f) \).

The proof is given in several steps. Let \( E^{\text{op}} \) denote the opposite algebra of \( E \). Multiplication in \( E^{\text{op}} \) is denoted by \( * : a, b \in E^{\text{op}}, a * b = ba \) (in \( E \)). It is easy to show that \( E^{\text{op}} \sim E \) (in fact, \( E^{\text{op}} \cong E \)), hence \([3] \) \( E \otimes E \sim E \otimes E^{\text{op}} \).
4.8. **Definition.** (a) Let \( 1 = (1 \ 0), \ i = (0 \ 1), \ j = (0 \ -1), \ k = (i \ 0); \ i j = k = -ji; \) also, \( i^2 = j^2 = 1, \ j^2 = -1. \)

(b) Define \( g: E \to M_{1,1}(E), \ h: E^{op} \to M_{1,1}(E) \) as follows: if \( a \in E, \) then \( a = a_0 + a_1, \ a_0 \in E_0, \ a_1 \in E_1, \) etc. Now

\[
g(a) = g(a_0 + a_1) = a_0 1 + a_1 i
\]

\[
h(b) = h(b_0 + b_1) = b_0 1 + b_1 j.
\]

4.9. **Properties of \( g, h. \)** (1) \( g \) is an algebra isomorphism of \( E \) into \( M_{1,1}(E). \)

(2) \( h \) is an algebra isomorphism of \( E^{op} \) into \( M_{1,1}(E). \)

(3) \( g(a) h(b) = h(b) g(a) \) for all \( a, b. \)

The proofs are trivial. Introduce now

4.10. **Definition.** With \( g, h \) as in Definition 4.8, define \( \varphi: E \otimes E^{op} \to M_{1,1}(E) \) via \( \varphi(a \otimes b) = g(a) \cdot h(b). \)

4.11. **Properties.** (1) The above \( \varphi \) is an algebra homomorphism, and

\[
\varphi(E \otimes E^{op}) = \begin{pmatrix} E_0 & E_1 \\ E_1 & E_0 \end{pmatrix}
\]

**Proof.** (1) is trivial.

(2) If \( a_1 \in E_1, \) then \( \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix} \in \varphi(E \otimes E^{op}) \) since \( \frac{1}{2} [\varphi(a_1 \otimes 1) + \varphi(1 \otimes a_1)] = \begin{pmatrix} 0 & 0 \\ a_1 & 0 \end{pmatrix}. \) Similarly for \( \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix} \). Thus \( \begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix} \), and similarly \( \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, \) both are in \( \varphi(E \otimes E^{op}), \) and the proof of (2) follows.

Q.E.D.

Thus, if we denote \( R = E \otimes E^{op}, \) then, as in Theorem 2.3, \( \varphi(R) \sim M_{1,1}(E). \) Therefore, the proof of Theorem 4.7 is complete if we show that \( R \sim \varphi(R). \) Since \( \varphi \) is a homomorphism, clearly \( \text{Id}(R) \subseteq \text{Id}(\varphi(R)). \) The proof of Theorem 4.7 is therefore complete once we prove

4.12. **Lemma.** With the above notations, \( \text{Id}(R) \supseteq \text{Id}(\varphi(R)). \)

**Proof.** W.l.o.g. we prove it for the multilinear polynomials: let \( f(x) = \sum_{\sigma \in S_n} x_{\sigma(1)} \cdots x_{\sigma(n)} \in \text{Id}(\varphi(R)). \) To show that \( f(x) \in \text{Id}(R) \) it clearly suffices to show that for any basis elements \( a_1, \ldots, a_n, b_1, \ldots, b_n \in D \) \( (x_i \to x_i, a_1 \otimes b_1), \) \( f(a_\otimes b) = f(a_1 \otimes b_1, \ldots, a_n \otimes b_n) = 0. \) Calculate \( f(a_\otimes b) = \sum_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)} \otimes b_{\sigma(1)} \cdots * b_{\sigma(n)}. \) Denote \( d(a_i) = d, \in \mathbb{Z}_2 \) if \( a_i \in E_d, \) and define the sign function \( \varepsilon: S_n \times \mathbb{Z}_2^n \to \pm 1 \) via \( a_{\sigma(1)} \cdots a_{\sigma(n)} = \varepsilon(\sigma; d(a_1), \ldots, d(a_n)) a_\otimes b. \) Trivially, \( b_{\sigma(1)} \cdots * b_{\sigma(n)} = \varepsilon(\sigma; d(b_1), \ldots, d(b_n)) b_1 \cdots * b_n. \)
Thus \( f(a \otimes b) = \beta \cdot (a_1 \cdots a_n) \otimes (b_1 \ast \cdots \ast b_n) \cdots \) (***), where \( \beta = \sum a \varepsilon(\sigma; d(a)) \cdot \varepsilon(\sigma; d(b)) \). We need to show that \( \beta = 0 \). Since \( \beta \) depends only on \( d(a), d(b) \), we can choose the \( a \)'s and \( b \)'s such that \( a_1 \cdots a_n b_1 \cdots b_n \neq 0 \) in \( E \). Now apply \( \varphi \) to both sides of (***): \( \varphi(f(a \otimes b)) = 0 \) since \( f \in \text{Id}(\varphi(a)) \). On the other hand, \( \varphi(a_1 \cdots a_n \otimes b_1 \ast \cdots \ast b_n) = a_1 \cdots a_n \cdot b_1 \cdots b_n \cdot i^r \ast j^s \) for some \( r, s \), so
\[
0 = \beta \cdot a_1 \cdots a_n \cdot b_1 \cdots b_n \cdot i^r \ast j^s.
\]
Since \( i^r j^s \) is a nonzero matrix (with entries 0, ±1) and \( a_1 \cdots a_n \cdot b_1 \cdots b_n \neq 0 \), hence \( \beta = 0 \). The proof of Theorem 4.7 is complete.

Q.E.D.

4.13. FURTHER CONSTRUCTIONS. The functions \( g, h \) of Subsection 4.9 and \( \mu^* \) (of Definition 1.9) can be used to construct new algebras which are P.I. equivalent to known algebras of matrices over the Grassmann algebra. We demonstrate this in the following

EXAMPLE. Apply \( g \) and \( h \) (several times). For example, deduce the imbedding
\[
g \otimes h: E \otimes E^{\text{op}} \to g(E) \otimes h(E^{\text{op}}) \subseteq M_{1,1}(E) \otimes M_{1,1}(E).
\]
Apply now \( \mu^*: M_{1,1}(E) \otimes M_{1,1}(E) \to M_{2,2}(E) \). As in Section 3, it follows that
\[
g(E) \otimes h(E^{\text{op}}) \sim \mu^*(g(E) \otimes h(E^{\text{op}})).
\]
Denote
\[
\mathcal{S} = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| A = a_0 \mathbf{1} + a_1 i, B = b_0 \mathbf{1} + b_0 j; a_0 \in E_0, b_0 \in E_0^{(2)}, a_1, b_1 \in E_1 \right\}.
\]
It is easy to verify that \( \mu^*(g(E) \otimes h(E^{\text{op}})) = \mathcal{S} \). Since \( g(E) \otimes h(E^{\text{op}}) \sim E \otimes E^{\text{op}} \sim M_{1,1}(E) \), it follows that \( \mathcal{S} \sim M_{1,1}(E) \).

REFERENCES