Jónsson Extensions of One-Dimensional Semilocal Domains

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1. INTRODUCTION

Let \( R \) be a ring (always commutative with unity), and let \( T \) be a ring of which \( R \) is a subring (always with the same unity). In [GH2], \( T \) is called a "Jónsson \( \omega_0 \)-generated extension" or "J-extension" of \( R \) if and only if it is not itself finitely generated as an \( R \)-algebra but every ring between \( R \) and \( T \) not equal to \( T \) is a finitely generated \( R \)-algebra. The purpose of the present paper is to characterize all the J-extensions of a one-dimensional semilocal (Noetherian) domain within its field of fractions in terms of ideals in its completion and to provide some examples of J-extensions of one-dimensional local domains.

As usual, we abbreviate "rank-one discrete valuation domain" to DVR. Example 2.22 of [GH2] is an example of a pair of DVRs \( V \) and \( W \) for which \( W \) is a J-extension of \( V \). The argument there relied heavily on the fact that the characteristic was 2. (The construction, for arbitrary nonzero characteristic, is repeated in Example 3.5 below.) In that case, of course, \( W \) is not contained in the field of fractions of \( V \), but for any element \( y \) of \( W \) not in \( V \), \( W \) is a J-extension of \( V[y] \) within its field of fractions. Moreover, the rings between \( V \) and \( W \) are linearly ordered by inclusion. The investigation reported in the present paper was motivated by the question

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of whether repeating the construction in other nonzero characteristics would also yield DVRs $V$ and $W$ in which $W$ is a $J$-extension of any $V[y]$, though the intermediate rings are no longer linearly ordered by inclusion. We found instead that $W$ is not a $J$-extension of such a $V[y]$, but it contains a (unique) $J$-extension $T$, and the rings between $V[y]$ and $T$ are linearly ordered.

Section 2 deals with the general case of a one-dimensional semilocal domain $R$. By [GH2, Theorem 2.4], any $J$-extension of $R$ is integral over $R$, so any $J$-extension within the field of fractions $K$ of $R$ is contained in the integral closure $R'$ of $R$. Thus there can be a $J$-extension of $R$ in $K$ only if $R'$ is not finitely generated (as an $R$-module or, equivalently, as an $R$-algebra). On the other hand, the $R$-module $K/R$ is Artinian (by [V, Proposition 2*] or [SV, Theorem 3.12]), since the socle—the sum of the modules $(R:_RM)/R$ as $M$ varies over the maximal ideals of $R$—is finitely generated and essential. So, if $R'$ is not finitely generated, then we can find minimal elements among the submodules $T/R$ of $R'/R$ for which $T$ is a ring not finitely generated over $R$; and such a $T$ is a $J$-extension of $R$. Thus, the results of this section apply to the one-dimensional local domains with non-finitely generated integral closure treated in [Ak, p. 332; Be; C, p. 38; L, Sect. 2; Sc, p. 445; Z, p. 24].

For a one-dimensional semilocal domain $R$ and any domain $J$-extension $T$ of $R$, it follows from [GH2, Theorem 2.1, Corollary 2.2] that the maximal ideals of $T$ are the extensions of the maximal ideals of $R$, and that for each maximal ideal $M$ of $R$ the residue fields $R/M$ and $T/MT$ are isomorphic. By Cohen's Theorem 8 [Mu, p. 212, Lemma; ZS, Vol. II, Chap. VIII, Sect. 3, p. 259, Theorem 7], the map on completions (with respect to the Jacobson radicals) of these rings, $R^* \to T^*$, is surjective. We shall occasionally use this fact for the case in which $T$ is not contained in the field of fractions $K$ of $R$. But, more importantly, for the case of $T$ within $K$; this fact is the basis for the main result of the paper, Theorem 2.1. There it is shown that the kernel of the map on completions is a nilpotent ideal $I$ in $R^*$ that is minimal among those that are "pure height zero"—i.e., those ideals whose associated primes are precisely the minimal primes of $R^*$, or equivalently those nilpotent ideals contracted from the total quotient ring $R^\sim$ of $R^*$. Moreover, we can recover the $J$-extension $T$ from such a minimal ideal $I$, in that $T$ is the intersection of $R^*/I$ with the copy of $K$ within $R^\sim/IR^\sim$.

The other results in Section 2 are consequences of this characterization of $J$-extensions of $R$ in $K$ with few added hypotheses. Among these results: If $S$ is an extension of $R$ within $R'$ and $T$ is a $J$-extension of $R$ within $K$ but not contained in $S$, then the compositum $T[S]$ is a $J$-extension of $S$; and, if $T$ is a $J$-extension of $R$ in $K$, then the multiplicity of $T$ must be strictly smaller than that of $R$. 
Section 3 provides some results on $J$-extensions of more specific one-dimensional semilocal domains. It is shown there that, if a one-dimensional local domain is Gorenstein and its integral closure is also local (i.e., a DVR), then it has at most one $J$-extension within its field of fractions; and if in addition its multiplicity is 2, then the rings between it and its integral closure are linearly ordered. Examples show that the hypotheses are necessary. It is also shown that Example 2.22 of [GH2] is essentially unique: If a DVR has a domain $J$-extension, then that $J$-extension is also a DVR, the characteristic is 2, and the fields of fractions of these domains constitute a purely inseparable field extension of degree 2.

In the short Section 4 we consider three natural questions about $J$-extensions, giving partial answers to each.

Again, all the rings we consider are commutative with unity, modules are unitary, and subrings share the unity. The words "local" and "semilocal" implicitly include "Noetherian." The symbol $<\,$ between sets means proper inclusion. For a ring $R$, we will write $R'$ for its integral closure (within its total quotient ring), $\text{jac}(R)$ for its Jacobson radical, $\text{nil}(R)$ for its nilradical, and $\text{soc}(R)$ for its socle (the ideal generated by the elements whose annihilators are maximal ideals). For subsets $A$ and $B$ of $R$, $A : KB$ or $A : B$ means the set of elements $x$ of the total quotient ring $K$ of $R$ for which $xb$ is in $A$ for every element $b$ of $B$. (But if $L/K$ is a finite algebraic field extension, then $(L : K)$ means the degree of $L$ over $K$.) If $R$ is semilocal, we write $\text{mult}_R(I, A)$ for the multiplicity of the $R$-module $A$ with respect to the ideal $I$ (sometimes suppressing $R$, or $A$ if it is $R$, or $I$ if it is $\text{jac}(R)$, by the usual abuse of notation) and $R^*$ for its completion in the $\text{jac}(R)$-adic topology. And if $R$ is local, we write $\text{max}(R)$ for its maximal ideal (so that $\text{max}(R) = \text{jac}(R)$).

Let $R$ denote a one-dimensional semilocal domain, and let $T$ be a domain containing $R$ so that the fields of fractions of $T$ and $R$ constitute a finite algebraic field extension. Then by the well-known Krull–Akizuki theorem [$N$, (33.2); $Kp$, Theorem 93; $Bo$, Chap. VII, Sect. 2.5, Proposition 5], $T$ is also one-dimensional and (Noetherian) semilocal. We will often make implicit use of these facts below, for instance, in speaking of the completion of such a $T$.

We shall also make occasional use of the following: Let $R$ be a one-dimensional semilocal domain with quotient field $K$, and let $T$ be a one-dimensional semilocal ring extension of $R$ in which the nonzero elements of $\text{jac}(R)$ are nonzerodivisors in $T$ and elements of $\text{jac}(T)$. Then the total quotient ring of $T$ contains a copy of $K$, so the intersection $T \cap K$ is meaningful. Assume that this intersection is $R$. Now the topology on $R$ is determined, not only by the powers of $\text{jac}(R)$, but also by the powers of $yR$ for any nonzero element $y$ of $\text{jac}(R)$ (since the powers of $\text{jac}(R)$ and the powers of $yR$ are cofinal); and a similar statement holds for $T$ provided the
element is a nonzerodivisor. Let \( y \) be any nonzero element of \( \text{jac}(R) \); then since \( T \cap K = R \), we have \( y^n T \cap R = y^n R \) for each positive integer \( n \). Thus, the \( \text{jac}(R) \)-adic topology on \( R \) is the same as the subspace topology that \( R \) inherits from \( T \); i.e., \( R \) is a subspace of \( T \). Hence, \( R^* \to T^* \) is injective. If \( R \) is dense in \( T \), then this map on completions is an isomorphism.

As a final preliminary remark, we note that Theorem 2.1 and its consequences can be extended to \( J \)-extensions within the total quotient ring of a one-dimensional semilocal reduced ring. To see this, first use [Kp, p. 122, Exercise 151] to isolate the maximal ideals of height zero; then we may assume that all maximal ideals of such a ring \( R \) have height one. The 1–1 correspondence between the minimal primes of \( R^* \) and the maximal ideals of \( R' \) (described for domains in more detail below, and basic to the proof of Theorem 2.1) exists in this more general context. For, the total quotient ring of \( R \) is a finite direct sum of fields; let \( S \) be the result of adjoining to \( R \) the indecomposable (i.e., minimal) idempotents in this direct sum. Then \( R = S \) and the total quotient rings of \( R^* \) and \( S^* \) are equal; since \( S \) is a direct sum of one-dimensional semilocal domains, the 1–1 correspondence follows from the domain case. Rather than state our main results in this generality, however, we felt it to be more natural to focus on the domain case.

2. The Main Result

Let \( R \) be a one-dimensional semilocal domain with field of fractions \( K \), and let \( T \) be a ring between \( R \) and \( R' \). Since \( T^* \) is also one-dimensional, adjoining to \( T^* \) the inverse of any nonzerodivisor in \( \text{jac}(T^*) \) will yield the total quotient ring \( T^* \) of \( T^* \). In particular, we could reach \( T^* \) by adjoining to \( T^* \) the inverse of any nonzero element of \( \text{jac}(R) \), so \( T^* \) is also \( T^*[K] \).

If the map \( R^* \to T^* \) is surjective, then \( T \) cannot be finitely generated over \( R \) unless \( T = R \). (For, if \( R^* \to T^* \) is surjective, then \( R/\text{jac}(R) = T/\text{jac}(T) \) and \( \text{jac}(T) = \text{jac}(R) T \), so that \( T = R + \text{jac}(R) T \); if \( T \) were finitely generated, then the NAK lemma [Mu, p. 11, (1.M)] would imply \( T = R \).) But for any \( T \) between \( R \) and \( R' \), the map \( R^* \to T^* \) is always surjective. (If \( T \) is not finitely generated over \( R \), then we can replace \( R \) with a \( J \)-extension inside \( T \), for we saw above that the map from \( R^* \) to the completion of a \( J \)-extension is surjective, and hence so is the map on their total quotient rings. It will follow from Corollary 2.2 below that we can form only a finite sequence of \( J \)-extensions within \( T \), so we may assume that \( T \) is finitely generated over \( R \). But then \( T^* = R^*[T] \), and since \( T[K] = K \) we see that \( R^* = T^* \).)

By [N, (17.7)], \( R' \) is the direct sum of the completions \( V_1, \ldots, V_n \) of the localizations of \( R' \) at its maximal ideals \( M_1, \ldots, M_n \); these localizations are
DVRs and hence so are the $V_i$'s. The minimal primes $P_1, \ldots, P_n$ of $R^*$ are the kernels of the compositions of the map on completions $R^* \rightarrow R'^*$ with the projections onto the $V_i$'s [N, p. 122, Exercise 1; Kt, Corollary 5]; and passing to the total quotient ring $R$ makes these minimal primes maximal as well. So the maximal ideals $M_1, \ldots, M_n$ of $R$ are in 1-1 correspondence with the maximal ideals of $R'$. We shall continue to use this notation below.

If $R$ has a $J$-extension within $K$, then, as we saw above, $R'$ is not finitely generated over $R$, and so by [N, (32.2)] nil($R^*$) and hence also nil($R$) are nontrivial. We now show that the $J$-extensions of $R$ within $K$ are in 1-1 correspondence with the minimal nonzero nilpotent ideals of $R$.

2.1. Theorem. Let $R$ be a one-dimensional semilocal domain with field of fractions $K$. Then the $J$-extensions of $R$ within $K$ are in 1-1 correspondence with the minimal nonzero nilpotent ideals in the total quotient ring $R$ of $R^*$.

For such a $J$-extension $T$, the corresponding ideal is the kernel of the surjective map $R \rightarrow T$. For such an ideal $I$, the corresponding $J$-extension is the intersection of the copy of $K$ in $R/I$ with $R/(R\cap I)$.

Proof: Let $I$ be a minimal nonzero nilpotent ideal in $R$, and set $T = K \cap (R/(R\cap I))$, where $K$ and $R/(R\cap I)$ are regarded as subrings of $R/I$. Then since $T$ is a subspace of the complete ring $R/(R\cap I)$ (see the introduction), dense because it contains the image of $R$, it follows that $R/(R\cap I) = T$; and so the Noetherian ring $R^*$ is not isomorphic to $T^*$. Since $R < T$ (because their completions are different) and $R^* \rightarrow T^*$ is surjective, $T$ cannot be finitely generated over $R$. So $T$ contains a $J$-extension $S$ of $R$. The map $R^* \rightarrow S^*$ is surjective, and its kernel is the contraction to $R^*$ of the kernel $I$ of $R \rightarrow S$. Now $I$ is contained in $I$ since the map $R \rightarrow T$ factors through $S$; and since $R = K \cap R^*$, while $S = K \cap S^* = K \cap (R^*/(R^*\cap J))$, $I$ must be nonzero. Thus, $J = I$, and $T = S$ is a $J$-extension of $R$.

Conversely, suppose $T$ is a $J$-extension of $R$ within $K$. Then, since $R^* \rightarrow T$ is surjective with kernel extending to $\ker(R \rightarrow T)$, $T = R/(R\cap I)$ and $T = K \cap (R/(R\cap I))$ for some nonzero ideal $I$ in $R$. Since the minimal primes of $T^*$ are in 1-1 correspondence with the maximal ideals of $T' = R'$ and hence with the minimal primes of $R^*$, $I \cap R^*$ is in every minimal prime of $R^*$, and so $I$ is nilpotent. It remains to show that $I$ is minimal among the nonzero ideals; so suppose $J$ is an ideal properly between $0$ and $I$, and set $S = K \cap (R/(R\cap J))$. Then $R/(R\cap J) = S$, so $S$ is not finitely generated over $R$, and restricting to the elements in $K$ the maps $R^* \rightarrow R^*/(R\cap J) \rightarrow R^*/(R\cap I)$ shows that $S$ lies properly between $R$ and $T$, contradicting the fact that $T$ is a $J$-extension of $R$. 

This theorem directs our attention to the minimal nonzero nilpotent ideals of \( R^- \), or equivalently to their contractions to \( R^* \). Such a contraction has primary decomposition of the form \( Q_1 \cap \cdots \cap Q_n \), where \( Q_i \) is primary for \( P_i \) for each \( i \) (in the notation of the opening paragraphs in this section), and each of the \( Q_i \)'s except one is the \( P_i \)-primary component of zero (i.e., the kernel of the map from \( R^* \) to the localization at \( P_i \)). Moreover, the \( Q_i \) that is not the \( P_i \)-primary component of zero becomes, in the localization of \( R^* \) at \( P_i \) (or, equivalently, of \( R^- \) at \( M_i^- \)), a simple ideal in the socle, i.e., a one-dimensional vector space over the field \( R^*/M_i^- \).

For a \( J \)-extension of \( R \), with its corresponding ideal in \( R^* \), it is natural to regard the minimal prime \( P_i \) in \( R^* \) and the maximal ideal \( M_i^- \) in \( R^- \) at which the ideal does not localize to zero, as well as the corresponding maximal ideal \( M'_i \) of \( R' \), as "associated" to the \( J \)-extension.

For each integer \( r > 1 \), [FR, Proposition 3.1] gives a one-dimensional local domain (of residue characteristic zero) having infinitely many \( J \)-extensions within its field of fractions (in fact, a "projective space of dimension \( r - 1 \)" of them).

2.2. Corollary. Let \( R \) be a one-dimensional semilocal domain, and let \( T \) be a \( J \)-extension of \( R \) within its field of fractions. Then mult(\( T \)) = mult(\( R \)) - mult(\( R^*/P \)) for some minimal prime \( P \) of \( R^* \).

Proof. Since \( R \) and \( R^* \) have the same Hilbert–Samuel polynomial (for, length\(_R(R/jac(R))^n\) = length\(_{R^*}(R^*/jac(R^*))^n\) for each positive integer \( n \)), they have the same multiplicity; and the same is true of \( T \) and \( T^* \). Moreover, \( T^* - R^*/I \) for the corresponding ideal \( I \) of \( R^* \) (already contracted from \( R^- \)). Now the length of a \( T^* \)-module is the same as its length as an \( R^* \)-module, so the multiplicity of \( T^* \) is the same as the multiplicity of \( R^*/I \) as an \( R^* \)-module. Using [N, (23.5)] to compute mult\(_{R^*}(R^*)\) and mult\(_{R^*}(R^*/I)\), we find that the terms in the sums are equal except at one minimal prime \( P \) of \( R^* \), namely the one associated to \( T \) in the sense above. For that \( P \), the length of \( (R^*/I)_P \) as a module over \( R^*_P \) is one less than the length of \( R^*_P \). Since mult\(_{R^*}(jac(R^*) + P)/P\) = mult\(_{R^*}(jac(R^*/P))\), we get the stated formula. 

Thus, starting with a one-dimensional semilocal domain \( R \), if we find a \( J \)-extension \( R_1 \) of \( R \) in the field of fractions \( K \), and then a \( J \)-extension \( R_2 \) of \( R_1 \) in \( K \), and so on, then the multiplicities of these rings strictly decrease. Since the multiplicity is always a positive integer, mult(\( R \)) is a strict upper bound on the integer \( n \) for which we can find a ring \( R_n \). In particular, for any \( T \) between \( R \) and \( R' \), we can take a finite sequence of \( J \)-extensions inside \( T \) so that \( T \) is a finitely generated extension of the last. The last ring in such a chain of \( J \)-extensions is uniquely determined, because it is \( K \cap (R^*/(R^* \cap J)) \), where \( J \) is the kernel of the map \( R^- \to T^- \); and the
length of any such chain is the length of the ideal \( J \). In particular, we can only reach \( T \) by a chain of \( J \)-extensions starting from \( R \) if the map \( R^* \to T^* \) is surjective.

2.3. Proposition. Let \( R \) be a one-dimensional semilocal domain, and let \( S \) and \( T \) be rings between \( R \) and \( R' \) with \( T \) a \( J \)-extension of \( R \). If \( T \) is not contained in \( S \), then the compositum \( T[S] \) is a \( J \)-extension of \( S \).

Proof. Starting with \( R \), take a sequence of \( J \)-extensions within \( S \) until \( S \) is a finitely generated extension of the last, \( D \). Then \( D^* \) has the form \( R^*/J \) for some nilpotent ideal \( J \) contracted from \( R^* \) (though \( JR^* \) is not necessarily minimal). Since \( T \) is not contained in \( S \) and hence not in \( D \), the map \( R \to D \) does not factor through \( T \), and so the ideal \( I \) of \( R^* \) corresponding to \( T \) is not contained in \( J \). Thus, \((I+J)R^* \cap R^*/J \) is an ideal of \( D^* \) corresponding to a \( J \)-extension \( E \) of \( D \); and \( E \) contains \( T \) because the map on completions \( R^* \to E^* \) factors through \( R^*/I = T^* \).

Assume \( T[D] \) is a finitely generated extension of \( D \). Then there is a nonzero element that multiplies \( T[D] \) into \( D \), and that element could be chosen from \( R \). That element multiplies \( T \) into \( T \cap D \), which is properly contained in \( T \) and hence a finite extension of \( R \), and we can find a nonzero element of \( R \) that multiplies this intersection into \( R \). Thus, we can build a nonzero element \( y \) of \( R \) multiplying \( T \) into \( R \). It follows that \( T \) is contained in the \( R \)-module generated by \( 1/y \), so that \( T \) is finitely generated over \( R \). This contradicts the fact that \( T \) is a \( J \)-extension of \( R \). Thus \( T[D] \) is not finitely generated over \( D \); but it is contained in the \( J \)-extension \( E \), so \( T[D] = E \).

Now \( T[S] \) is not finitely generated over \( S \) (for, if it were, then the product of its nonzero conductor into \( S \) with the conductor of \( S \) into \( D \) would give a nonzero conductor of \( T[D] \) into \( D \), contradicting the fact that that extension is not finitely generated). But any ring \( A \) between \( S \) and \( T[S] \) not equal to \( T[S] \) must meet \( T[D] \) in a proper subring, which has a nonzero conductor into \( D \). The nonzero conductor of \( S \) into \( D \) also conducts \( A \) into \( A \cap T[D] \), so we get a nonzero conductor of \( A \) into \( D \) and hence into \( S \). Thus \( A \) is finitely generated over \( S \), and we conclude that \( T[S] \) is a \( J \)-extension of \( S \).

2.4. Corollary. Let \( R \) be a one-dimensional semilocal domain with field of fractions \( K \), and let \( S \) be a finite integral extension of \( R \) in \( K \). Then the set of \( J \)-extensions of \( R \) in its field of fractions \( K \) is in 1-1 correspondence with the set of \( J \)-extensions of \( S \) in \( K \) (with a \( J \)-extension \( T \) of \( R \) corresponding to \( T[S] \)).

Proof. In view of Proposition 2.3, we need only show that any \( J \)-extension of \( S \) contains only one \( J \)-extension of \( R \). The next proposition proves
2.5. **PROPOSITION.** Let \( V \) be a one-dimensional Noetherian domain and \( W \) be a \( J \)-extension of \( V \) that is also a domain. If \( R \) is a subring of \( V \) with the same field of fractions such that \( V \) is a finite integral extension of \( R \), then \( R + (R : V)W \) is the unique \( J \)-extension of \( R \) contained in \( W \).

**Proof.** Let \( C \) denote the (nonzero) conductor \( R : V \). By [GH2, Theorem 2.1], \( W = V + CW \), so \( W \) is finitely generated over \( T = R + CW \); thus \( T \) cannot be finitely generated over \( R \). But for any ring \( S \) between \( R \) and \( W \) that is not finitely generated over \( R \), \( S[V] \) is not finitely generated over \( V \), so \( S[V] = W \). Since \( CW \) is contained in \( S \), \( S \) contains \( T \). 1

2.6. **COROLLARY.** Let \( R \) be a one-dimensional semilocal domain, and let \( T \) be a ring between \( R \) and \( R' \). If \( T \) is not finitely generated over \( R \), then \( T \) is a \( J \)-extension of some ring containing \( R \).

**Proof.** Take a sequence of \( J \)-extensions \( R = R_0 < R_1 < \cdots < R_n \) in \( T \) so that \( T \) is a finitely generated extension of \( R_n \), and let \( S \) be an extension of \( R \) generated by a finite set of generators of \( T \) over \( R_n \). Then the rings \( R_n[S] \) form a sequence of \( J \)-extensions ending in \( T \), so \( T \) is a \( J \)-extension of \( R_n[S] \). 1

The last result of the section is a partial justification for the restriction to \( J \)-extensions within the field of fractions, in that it shows that domain \( J \)-extensions not contained in the field of fractions are rare.

2.7. **PROPOSITION.** Let \( R \) be a one-dimensional semilocal domain with field of fractions \( K \), and let \( T \) be a \( J \)-extension of \( R \) that is a domain. If \( T \) is not contained in \( K \), then \( T \cap K = R \), \( R' \) is a DVR finitely generated over \( R \), and the field of fractions of \( T \) is an algebraic, purely inseparable extension of \( K \) of degree 2. In particular, \( R \) is local and the characteristic of these domains is 2.

**Proof.** Let \( T \cap K = S \), and assume \( R < S \). The map on completions \( f: R^* \to T^* \) factors through the completion \( S^* \) of \( S \); say \( f = hg \), where \( g: R^* \to S^* \) and \( h: S^* \to T^* \). As noted in the introduction, \( h \) is injective. Also, since \( S \) is finitely generated over \( R \), \( S^* = R^*[S] \); the inclusion of \( R \) into \( S \) is injective but not surjective, so \( g \) is also injective but not surjective. But this is a contradiction, since \( hg = f \) is surjective; so \( T \cap K = R \).

Now we are free to give a new meaning to the letter \( S \): By [GH2, Proposition 1.7], the field of fractions \( L \) of \( T \) is a finite algebraic field extension of \( K \). Take an element \( s_1 \) of \( T \) not in \( K \); replace \( s_1 \) with a multiple...
by a nonzero element of $R$, if necessary, so that the monic minimal polynomial of $s_1$ over $K$ has coefficients in $R$. If $L = K(s_1)$, then set $S = R[s_1]$. Otherwise take an element $s_2$ of $T$ not in $K(s_1)$, and multiply it, if necessary, by a nonzero element of $R$ so that its monic minimal polynomial over $K(s_1)$ has coefficients in $R[s_1]$. If $L = K(s_1, s_2)$, then set $S = R[s_1, s_2]$; otherwise continue. After at most $(L : K)$ repetitions, $S$ gains its new definition.

Then $S$ is a finite free $R$-module of rank $(L : K)$ and $T$ is a $J$-extension of $S$ within its field of fractions. Since $\text{jac}(S)$ and $\text{jac}(T)$ still contain $\text{jac}(R)$, the total quotient rings $S^\sim$ and $T^\sim$ of $S^*$ and $T^*$ are still obtained by adjoining $K$ to $S^*$ and $T^*$. Thus the map $R^\sim \to S^\sim$ is injective, the map $S^\sim \to T^\sim$ is surjective with kernel of length one, and the composition $R^\sim \to T^\sim$ is an isomorphism. Since the residue fields of $S^\sim$ are caught between those of $R^\sim$ and $T^\sim$, which are equal, we see that the length of an $S^\sim$-module is the same as its length as an $R^\sim$-module. Since $S^\sim$ is a free $R^\sim$-module of rank $(L : K)$, we get

$$\text{length}(R^\sim) = \text{length}(T^\sim) = \text{length}(S^\sim) - 1 = (\text{length}(R^\sim) \times (L : K)) - 1.$$ 

It follows that $\text{length}(R^\sim) = 1$ and $(L : K) = 2$. The fact that $R^\sim$ has length 1 means that it is a field, so $R^*$ is a domain. Thus, $R'$ is finitely generated over $R$ [N, (32.2)], and it has only one maximal ideal [N, p. 122, Exercise 1] and so it is a DVR.

Finally, since $T[R']$ is a $J$-extension of $R'$ by Proposition 2.3, the integral closure of $R'$ in $L$ is not finitely generated over $R'$. Thus, the field extension $L/K$ is not separable [N, (10.16); ZS, Vol I, p. 265, Corollary 1; Bo, Chap. V, Sect. 1.6, Corollary 1 to Proposition 18, p. 318], so it must be purely inseparable.

In view of Corollary 2.18 of [GH2], it follows that the only Noetherian domain $R$ that may admit a domain $J$-extension not contained in the field of fractions of $R$ is a one-dimensional local domain of characteristic 2. This answers a question raised in Remark 2.23 of [GH2].

3. Special Results and Examples

In this section we explore the extension of Example 2.22 of [GH2] to nonzero characteristics greater than 2. As mentioned, some of the more interesting and desirable properties of this example are the uniqueness of $J$-extensions and the linear ordering of intermediate rings with respect to
inclusion. The next two results below "explain" why that example had these properties.

3.1. PROPOSITION. Let $R$ be a one-dimensional local domain for which $R'$ is also local (i.e., a DVR). If $\text{mult}(R) = 2$, then the rings between $R$ and $R'$ are linearly ordered by inclusion. If, in addition, $R'$ is not finitely generated over $R$, then $R'$ is the unique $J$-extension of $R$ in its field of fractions.

Proof. By [Sa, Chap. 3, p. 49, Theorem 1.11], every ideal in $R$ has at most two generators. In particular, this is true of $\text{max}(R) = M$. By [Kp, p. 163, Exercise 1], $R$ is a Gorenstein ring; i.e., every fractional ideal is its own double inverse [Kp, p. 167, Theorem 222]. Since $M$ is not principal and hence not invertible, $M^{-1}M$ is not equal to $R$, so it is equal to $M$; i.e., $M^{-1} = M : \kappa M$, which is a ring. Now for any ring $S$ between $R$ and $R'$ that is finitely generated over $R$ but not equal to it, $S$ is also a fractional ideal of $R$, and $S^{-1} = R : S$ is contained in $M$, so $S = (R : S)^{-1}$ contains $M^{-1}$. Thus, $M^{-1}$ is contained in every ring properly containing $R$. It follows from [ZS, Vol. II, p. 297, Theorem 241] that $\text{mult}(M^{-1}) \leq 2$. If it is equal to 2, then we can repeat the above argument. If it is equal to 1, then the result from [Sa] or [N, (40.6)] shows that $R'$ is a DVR and so equals $R'$.

If $R'$ is finitely generated over $R$, we need only repeat the argument a finite number of times. If not, we repeat it a countably infinite number of times, reaching a $J$-extension of $R$, and hence so are the localizations of $R'$ at its minimal primes $P_i$, by [Kp, p. 164, Exercise 12]. For each $i = 1, \ldots, n$, there is only one minimal nonzero ideal in the localization of $R'$ at $P_i$ (and it is the whole localization, not part of the socle, when the localization is a field). So there are at most $n$ possible primary decompositions of a minimal nonzero "pure height zero" ideal in $R'$. By Theorem 2.1 and the discussion following it, there are at most $n$ $J$-extensions of $R$ in $K$. 

The next result shows that, under weaker hypotheses, we lose the linear ordering of the intermediate rings, but not (yet) the uniqueness of the $J$-extension, provided $R'$ is also local.

3.2. PROPOSITION. If the one-dimensional local domain $R$ is Gorenstein, then the number of $J$-extensions of $R$ within its field of fractions is at most the number of maximal ideals in $R'$.

Proof. Since $R$ is Gorenstein, so is $R^*$ (for, if $r$ is a nonzero element of $R$, then $R/rR = R^*/rR^*$, and the Gorenstein property can be tested on these factor rings); and hence so are the localizations of $R^*$ at its minimal primes $P_i$, by [Kp, p. 164, Exercise 12]. For each $i = 1, \ldots, n$, there is only one minimal nonzero ideal in the localization of $R^*$ at $P_i$ (and it is the whole localization, not part of the socle, when the localization is a field). So there are at most $n$ possible primary decompositions of a minimal nonzero "pure height zero" ideal in $R^*$. By Theorem 2.1 and the discussion following it, there are at most $n$ $J$-extensions of $R$ in $K$. 

The $R$ in Example 3.8 below has two $J$-extensions in its field of fractions, one for each of the maximal ideals in $R'$.

Suppose the one-dimensional local domain $R$ has a unique $J$-extension $T$ within its field of fractions $K$. This means that the zero-dimensional ring $R^\sim$ has one-dimensional socle (i.e., its localization at the maximal ideal $M^\sim$ associated with $T$ is Gorenstein, and its localizations at other maximal ideals are fields). Of course, $T$ may have no $J$-extension at all in $K$, because $R'$ may be finitely generated over it. And when $T$ has a $J$-extension in $K$, it need not be unique; for, $T^\sim = R^\sim / \text{soc}(R^\sim)$, and $\text{soc}(T^\sim)$ need not be one-dimensional. Indeed, suppose that $R^\sim$ is local. Then the inverse image of $\text{soc}(T^\sim)$ in $R^\sim$ is the annihilator of the square of $\text{max}(R^\sim)$, and by Matlis duality [Mi, Corollary 4.33 in the self-injective ring $R'$] we have:

$$\text{length}((0 : \text{max}(R^\sim)^2)/(0 : \text{max}(R^\sim))) = \text{length}(\text{max}(R^\sim)/\text{max}(R^\sim)^2).$$

The right side of this equation is the minimum number of generators required by $\text{max}(R^\sim)$. Thus, $T$ has a unique $J$-extension in $K$ if and only if $\text{max}(R^\sim)$ is principal (or, equivalently, the ideals of $R^\sim$ are linearly ordered) and $\text{max}(R^\sim)^2$ is nonzero. Example 3.6 below shows that these equivalent conditions need not hold.

For ease of reference, we isolate the important features of Example 2.22 of [GH2] and of Example 3.5 in the following discussion.

3.3. General Example. Let $R$ be a one-dimensional Gorenstein local domain for which $R'$ is also local and not finitely generated over $R$ and for which $R'/\text{max}(R') = R/\text{max}(R)$. Then $R'$ is a DVR; denote the associated valuation on the field of fractions $K$ by $w$. By Proposition 3.2, $R'$ contains a unique $J$-extension $T$ of $R$. We claim that $T$ is the union of the fractional ideals $Z^n$, where $I_n$ is the set of elements $r$ of $R$ for which $w(t) \geq n$.

Moreover, (1) the fractional ideals $I_n^{-1}$ are themselves rings, and (2) $T/R$ is an "almost finitely generated" $R$ module (i.e., all its proper submodules are finitely generated; see [Ar, GH1, HL, W]).

As a first step toward seeing all this, let $J_1 > J_2 > \cdots$ be a strictly descending chain of ideals of $R$. The intersection of the $J_n$'s is zero, because any proper homomorphic image of $R$ is an Artinian ring; we claim that the intersection of their extensions to $R'$ is also zero. For this, take a chain of $J$-extensions starting from $R$ and ending in a ring $S$ over which $R'$ is finitely generated. Then it suffices to show that the extensions of the $J_n$'s to $S$ intersect in zero, because the nonzero conductor of $R'$ into $S$ multiplies $J_n$ into $J_nS$. Now $S = K \cap (R^\ast/P)$, where $P$ is the unique minimal prime of
so we need only show that the intersection of the ideals \( J_n R^* + P \) in \( R^* \) is \( P \). Assume by way of contradiction that this intersection properly contains \( P \); then it is primary for \( \max(R^*) \), so it contains a nonzero element \( y \) of \( \max(R) \) (by the \( 1-1 \) correspondence between \( \max(R) \)-primary ideals and the \( \max(R^*) \)-primary ideals). Let \( m \) denote the index of nilpotence of \( P \). For each positive integer \( n \), we can write \( y = z_n + t_n \), where \( z_n \) is in \( J_n R^* \) and \( t_n \) is in \( P \); and hence \( z_n = y - t_n \). Multiplying by the sum of the terms \( y^{m-j-1} t_n^j \) for \( j = 0, \ldots, m - 1 \), shows that \( y^m \) is in \( J_n R^* \) as well as \( R \), and so is in their intersection \( J_n \). This contradiction to the fact that the \( J_n \)'s intersect in zero proves the claim.

The preceding paragraph shows that, for any strictly descending chain of ideals \( J_1 > J_2 > \cdots \) in \( R \) and for any positive integer \( n \), we can find a positive integer \( m \) for which \( J_m < I_n \). In particular, take a chain of rings \( R < T_1 < T_2 < \cdots \) contained in the \( J \)-extension \( T \); each \( T_m \) is a fractional ideal of \( R \), so we can set \( J_m = T_m^{-1} \) (i.e., \( T_m = R : J_m \) — recall that, since \( R \) is Gorenstein, all its fractional ideals are their own double inverses). Then we have \( R < I_n^{-1} < J_m^{-1} < T_m < T \). Since \( T \) is contained in the DVR \( R' \), any element of \( I_n^{-1} \) has nonnegative \( w \)-value; so we get that \( I_n^{-1} = I_n : R I_n \), a ring. Since the intersection of the ideals \( I_n \) is zero, the union of their inverses is not finitely generated over \( R \), so it is \( T \). And finally, if \( R < F_1 < F_2 < \cdots \) is a strictly ascending chain of \( R \)-submodules of \( T \), then the ideals \( F_m^{-1} \) strictly descend, so for each \( n \) there is an \( m \) for which \( F_m^{-1} < I_n \), and hence \( I_n^{-1} < F_m \). It follows that the union of the \( F_m \)'s is \( T \), so that \( T/R \) is an almost finitely generated \( R \)-module.

Now suppose in addition that \( \max(R') = \max(R) R' \); in other words, \( R \) has an element \( x \) of \( w \)-value 1. Using the fact that the residue fields of \( R \) and \( R' \) are equal, it is easy to “adjust” the rest of a list of generators of \( \max(R) \) so that they all have \( w \)-value at least as large as any given positive integer \( n \); that is, so that they are in \( I_n \). (The “adjusting” proceeds as follows: If \( w(y) = t \), say, then \( y/x^t \) is an element of \( R' \), but there is an element \( z \) of \( R \) for which \( w((y/x^t) - z) > 0 \). The element \( y - x^t \) of \( R \) has value greater than \( t \), and this element and \( x \) generate the same ideal as \( y \) and \( x \).) Thus, for each positive integer \( n \), \( \max(R/I_n) \) is principal, generated by the image of \( x \); so that the ideals of \( R/I_n \) are linearly ordered by inclusion. Thus the fractional ideals between \( R \) and \( I_n^{-1} \) are also linearly ordered. It follows that all the \( R \)-submodules of \( T \), including the rings between \( R \) and \( T \), are linearly ordered by inclusion.

The additional assumption that \( \max(R') = \max(R) R' \), together with Corollary 2.2, gives a bit more information: The \( P \) in that corollary must be the unique minimal prime of \( R* \); \( R*/P \) is the image of \( R^* \) in \( R'^* \), and the hypotheses that \( \max(R') = \max(R) R' \) and \( R'/\max(R') = R/\max(R) \) yield that \( R* \to R'^* \) is surjective. Since \( R*/P = R^* \) is a DVR, it has multiplicity 1, so we conclude that \( \mult(T) = \mult(R) - 1. \)
The argument in the last paragraph can be extended to yield the following statement: Let $T$ be a $J$-extension of the one-dimensional semilocal domain $R$ and $M'$ be the ideal of $R'$ associated with $T$. Then $\text{mult}(T) = \text{mult}(R) - 1$ if and only if the contraction $M$ of $M'$ to $R$ satisfies $MR_M' = M'R'_M$ and $R/M = R'/M'$.

For the $R$ and $T$ of Example 3.7, the intermediate rings are not linearly ordered; so the hypothesis in General Example 3.3 that $\text{max}(R') = \text{max}(R)R'$ cannot be deleted.

3.4. Remark. The second paragraph of General Example 3.3 is an argument valid in arbitrary characteristic for a fact that we discovered in nonzero characteristic by a simpler argument. We include the pivotal fact of that argument here, in the hope that others may also find it useful: Let $R$ be a one-dimensional local domain with nonzero characteristic for which $R'$ also is local. Then there is a positive integer $q$ such that, for all elements $y$ in $\text{max}(R')$, $y^q$ is in $R$. (Question: Is this true without the hypothesis of nonzero characteristic?) To see this, take a finite extension $S$ of $R$ so that a finite sequence of $J$-extensions in $R'$ starting from $S$ ends in $R'$, as in the proof of Corollary 2.6. Let $a$ be a power of the characteristic at least as large as the index of nilpotence of the kernel of the surjective map $S^* \rightarrow R'^*$. Then each element $y$ of $R'$ has a preimage $z$ in $S^*$, and the difference $y - z$ is a meaningful element of $S^*$. The $a$th power $y^a - z^a$ of this element is zero, so we see that $y^a$ is in both $S^*$ and the field of fractions of $S$, so it is in $S$. Now some power, say the $b$th, of $\text{max}(S)$ is in the conductor of $S$ into $R$, and the integer $q = ab$ has the desired property. (Note the close relationship of this fact to [Be, Lemma (2.1), p. 134].)

To see how the assertion of the second paragraph of General Example 3.3 follows from this fact, let $J_1 > J_2 > \cdots$ be a strictly descending chain of ideals, and assume by way of contradiction that $J_nR' = J_{n+1}R' = \cdots$. Let $x$ be a nonzero element of $R$ with $w$-value greater than the minimum $w$-value of the elements of $J_n$. Then for each $m \geq n$, there is an element $y_m$ of $J_m$ for which $x/y_m$ is in $\text{max}(R')$. Since $x^q/y_m^q$ is in $R$, we have $x^q$ is in $J_m$, so the intersection of the $J_m$'s is nonzero, the desired contradiction.

3.5. Example. Let $k$ be a field of characteristic $p > 0$, and let $x$ be an indeterminate over $k$. Pick an element $y$ of the formal power series ring $k[[x]]$ that is transcendental over $k(x)$ and has zero constant term and nonzero linear term as a power series (i.e., its order is 1). Let $V$ and $W$ denote the DVRs that are the intersections of the DVR $k[[x]]$ with the fields $k(x, y^p)$ and $k(x, y)$. The $p$th power of every element of $W$ is in $V$, so $W$ is integral over $V$; but $W$ is not finitely generated over $V$ (see below). The residue fields of both $V$ and $W$, and hence of all their intermediate
rings, are $k$, and the maximal ideal of $R = V[y]$ is generated by $x$ and $y$, so $R$ is Gorenstein by [Kp, p. 163, Exercise 1] and $\text{max}(R) = V$. By General Example 3.3, $R$ has a unique $J$-extension $T$ in $W$, and the rings between $R$ and $T$ are linearly ordered by inclusion. Because $R$ is a free $V$-module of rank $p$ and $V/xV = k$, $R/xR$ is a free $k$-module of that rank. Since $y^p/x^p$ is in $V$, it is easy to see that, for $n \geq p$, $(x, y)^n = x^{n-p}(x, y)^p$, and hence for $n > p$, $(x, y)^n : x = (x, y)^{n-p} - (x, y)^p = (x, y)^{n-1}$. Thus $x$ is a superficial element of $\text{max}(R)$, and hence [Sa, p. 6, Remark (2)] $\text{mult}(R) = \text{mult}(R/xR) = \text{length}(R/xR) = p$. Thus, $\text{mult}(T) = p - 1$; and since $T$ inherits the important hypotheses from $R$ (i.e., maximal ideal generated by two elements and extending to the maximal ideal of $W$, residue field equal to that of $W$), we can form a chain of exactly $p - 1$ $J$-extensions starting with $T$ and ending with $W$.

It may be useful to describe some explicit computations: Write $y = \sum a_i x^i$, where the $a_i$'s are in $k$, and define $z_n = y - (a_1 x + a_2 x^2 + \cdots + a_n x^n)$. Then $\text{max}(R)$ is generated by $x$ and $z_n$, and so $I_n$ is generated by $x^n$ and $z_n$. (Proceed by induction: $I_1 = \text{max}(W) \cap R = \text{max}(R)$. Suppose $r$ is in $I_n$, and write $r = bx^{n-1} + cz_{n-1} = (b + ca_n x)x^{n-1} + cz_n$, where $b$ and $c$ are in $R$. Then $\omega(b + ca_n x) = \omega(r - cz_n) - n + 1 > 0$, so $b + ca_n x$ is in $\text{max}(R) = (x, z_n)$, and we can write $r$ in terms of $x^n$ and $z_n$.) Set $t_n = z_n^{n-1}/x^n$. Then $t_n$ is in $I_{n-1}$; we claim it is not in $I_{n-1}$. The list $1, y, \ldots, y^{p-1}$ is a free $V$-basis for $R$ and a $k(x, y^p)$-basis for $k(x, y)$. In the expression for $x^{n-1}(z_n^{n-1}/x^n)$ with respect to this basis, the coefficient of $y^{p-1}$ is $1/x$, not in $V$, so this element is not in $R$. This proves the claim. Thus, $t_n$ is an element of the ring $I_{n-1}$ not in $I_{n-1}$. Since $\text{length}(I_{n-1}/I_{n-1}) = \text{length}(I_{n-1}/I_n) = 1$, it follows that $I_{n-1} = I_{n-1}$. Moreover, $t_{n-1} - xt_n$ is an element of $R$, so we conclude that $I_{n-1} = R[t_n]$. Thus $T = R[t_1, t_2, \ldots]$.

It is not hard to check that $t_n^2$ is in $R$ and then that a $V$-basis for $R[t_n]$ is $1, y, \ldots, y^{p-2}, t_n$. Suppose $p > 2$ and write $y/x$ in terms of this $k(x, y^p)$-basis for $k(x, y)$. It follows that $y/x$ is an element of $W$ not in $T$.

3.6. Example. This is an example of a one-dimensional local domain $R$ having a unique $J$-extension $T$ in its field of fractions, but for which $T$ has many $J$-extensions: For $k$ and $x$ as in Example 3.5, take two elements $y$ and $z$ of $k[[x]]$ algebraically independent over $k(x)$, and let $V$ and $W$ be the intersections of $k[[x]]$ with the fields $k(x, y^p, z^p)$ and $k(x, y, z)$. Then $R = V[y, z]$ shares many of the properties of the $R$ in that example—in particular, it is still Gorenstein because it has the form $V(Y, Z)/(Y^p - y^p, Z^p - z^p)$, a regular ring modulo a regular sequence—so it has a unique $J$-extension $T$ in $W$. But since $V^*$ contains copies of the power series $y$ and $z$, the maximal ideal of $R^*$ requires two generators (the $y$ in $R$ minus the copy of $y$ in $V^*$, and similarly for $z$). Thus, $R^*$ is not a
principal ideal ring, and the socle of $T^* = R^*/\text{soc}(R^*)$ is not one-dimensional. By Theorem 2.1, there are many $J$-extensions of $T$ in $W$.

3.7. **Example.** This is an example of a one-dimensional local domain $R$ with a $J$-extension $T$ within its field of fractions $K$ for which the rings between $R$ and $T$ are not linearly ordered (and $T$ is the unique $J$-extension of $R$ in $K$): In Example 3.5, let the characteristic be 2, so that $W$ is a $J$-extension of $V$. Write $V = k + M$ and $W = k + N$ where $M = \max(V)$ and $N = \max(W)$. Then by Proposition 2.5, the unique $J$-extension of $k + M^2$ in $W$ is $T = k + N^2$. Now $T$ is also a $J$-extension of $R = (k + M^*)[y']$. The rings $(k + M^2)[xy]$ and $(k + M^2)[y^2]$ lie between $R$ and $T$, and neither is contained in the other. To see this, note that, if an element $z$ of $W$ is not in $k(x, y^2)$, then $1, z$ is a $k(x, y^2)$-basis for $k(x, y)$; so an element of $W$ is in $(k + M^2)[z]$ if and only its coefficients in this basis are in $k + M^2$. Thus, for instance, since $w(y^2/x) = 1$, $y^2/x$ is in $M$ but not $M^2$, and so $y^3 = (y^2/x)(xy)$ is not in $(k + M^2)[xy]$. But since $y^5/xy = (y^2/x)(y^2)$, $y^5$ is in $(k + M^2)[xy]$.

3.8. **Example.** This is an example of a one-dimensional local Gorenstein domain with more than one $J$-extension within its field of fractions: We make the following modifications to Example 3.5. Let $k$ be as in that example and $x, y$ and $z$ be indeterminates. Pick a power series $u$ in $k[[z]]$ with zero constant term and nonzero linear term such that $u$ is transcendental over $k(z)$. Consider the two isomorphisms $k(x, y) \rightarrow k(z, u)$, both sending $y$ to $u$, but one sending $x$ to $z$ and the other sending $x - 1$ to $z$. The inverse images of the DVR $k(z, u^p) \cap k[[z]]$ under these isomorphisms are DVRs $V_1$ and $V_2$ of the form $V_i = k + M_i$, where $M_i = \max(V_i)$; we have $xV_1 = M_1$, $(x - 1)V_2 = M_2$, and $y^pV_i = M_i^p$.

Set $M = M_1 \cap M_2$ and $S = k + M$. For any $t$ in $V_1 \cap V_2$, there are elements $a$ and $b$ of $k$ for which $t - a$ is in $M_1$ and $t - b$ is in $M_2$; so $(t - a)(t - b)$ is in $M$, and we see that $S' = V_1 \cap V_2$. Moreover, the fact that $x(t - b) - (x - 1)(t - a)$ is in $M$ shows that $t$ is in $S + Sx$, so $S' = S + Sx$. Now $M = x(x - 1)S'$, so $x(x - 1)$ and $x^2(x - 1)$ generate $M$ as an ideal in $S$. By [Kp, p. 163, Exercise 1], $S$ is Gorenstein.

Set $R = S[y] = S[Y]/(Y^p - y^p)$. By [Kp, p. 164, Exercise 13, 14], $R$ is also Gorenstein. Now $R^* = R^*[1/\lambda(x - 1)] = (S^*[Y]/(Y - y)^p)[1/\lambda(x - 1)]$ (since $S^*$ contains a copy of $y$) $= S^*[Y]/(Y - y)^p$. Since $S'$ is finitely generated over $S$, $S^* = S'^*$, the direct sum of the fields of fractions of $V^*_1$ and $V^*_2$. So $\text{soc}(R^*)$ has two summands, and hence $R$ has two $J$-extensions in its field of fractions.

3.9. **Remark.** Since the basic construction of this section has been the
"two DVRs" of Example 3.5, let us add to Proposition 2.7 the comment that if \( R \) is assumed to be a DVR, then, since \( T \) is also a one-dimensional Noetherian domain with principal maximal ideal, it is also a DVR.

4. THREE QUESTIONS

Let \( R \) be a ring and \( x \) be an indeterminate. Recall that \( R(x) \) denotes the localization of the polynomial ring \( R[x] \) at the multiplicative set of polynomials whose coefficients generate the ring \( R \) as an ideal.

(Q1) If a ring \( T \) is a \( J \)-extension of a ring \( R \), must \( T(x) \) be a \( J \)-extension of \( R(x) \)?

If this statement were true, it would facilitate working with multiplicities of \( J \)-extensions, because passing to \( R(x) \) is the standard way of assuring infinite residue field for working with superficial elements. We know the answer only in the case considered in most of the present paper; it is proved in the proposition below. In preparation for this proof, we note the following:

Let \( R \) be a one-dimensional semilocal domain and \( x \) be an indeterminate. Then \( R(x)^* \) is the completion of the faithfully flat extension \( R(x) \) of \( R \), so it is also faithfully flat over \( R \). Thus the powers of \( \text{jac}(R) \) are contracted from \( R(x)^* \), so that \( R \) is a subspace of \( R(x)^* \). Since the latter ring is complete, we can regard \( R^* \) as the closure of \( R \) in \( R(x)^* \)—indeed, we can regard \( R(x)^* \) as the completion of \( R^*(x) \).

4.1. PROPOSITION. If \( R \) is a one-dimensional semilocal domain, \( T \) is a \( J \)-extension of \( R \) that is itself a domain, and \( x \) is an indeterminate, then \( T(x) \) is a \( J \)-extension of \( R(x) \).

Proof. Suppose first that \( T \) is not contained in the field of fractions \( K \) of \( R \). By Proposition 2.7, the integral closures \( R' \) and \( T' \) are DVRs, finitely generated over \( R \) and \( T \) respectively, and the field of fractions \( L \) of \( T \) has degree 2 over \( K \). Now \( R'(x) \) and \( T'(x) \) are also DVRs; for example, the valuation on \( K(x) \) associated with \( R'(x) \) is the "inf" valuation, assigning to a nonzero polynomial the infimum of the \( R' \)-values of its nonzero coefficients. We know that \( T'(x) \) contains a \( J \)-extension of \( R'(x) \), and, by Remark 3.9 and the fact that there are no fields between \( K(x) \) and \( L(x) \), that \( J \)-extension is \( T'(x) \). Since \( R'(x) \) and \( T'(x) \) are finitely generated over \( R(x) \) and \( T(x) \) respectively, it is not hard to argue using conductors that \( T(x) \) is a \( J \)-extension of \( R(x)[f] \) for any element \( f \) of \( T(x) \) not in \( R(x) \); and so \( T(x) \) is a \( J \)-extension of \( R(x) \).

Suppose on the other hand that \( T \) is contained in \( K \); we want to show that \( T(x) \) has the form described in Theorem 2.1. By Cohen's Theorem 8,
the map on completions $R(x)^* \to T(x)^*$ is surjective; so to complete the proof, it suffices to show that the kernel of the corresponding map on total quotient rings $R(x)^- \to T(x)^-$ is a minimal nonzero ideal. (Since this kernel surely contains the extension of the kernel of the map $R^- \to T^-$, it will follow that it is nilpotent.) Equivalently, it suffices to show that the length of the Artinian ring $T(x)^-$ is exactly one less than that of $R(x)^-$. This will follow if we show that length($R^-$) = length($R(x)^-$), for the same proof will also show that length($T^-$) = length($T(x)^-$).

The length of an Artinian ring is computed by finding the length of a chain of ideals whose factors are isomorphic to factors of the given ring by prime (i.e., maximal) ideals. Since every prime ideal in $R(x)^-$ lies over a prime in $R^-$, it suffices to prove that the extensions to $R(x)^-$ of primes in $R^-$ are prime in $R(x)^-$ (cf. [N, (19.1)]). Now $R^-$ and $R(x)^-$ are the results of adjoining to $R^*$ and $R(x)^*$ respectively the inverse of a nonzero element of $\text{jac}(R)$, so we must show that, for a minimal prime $P$ in $R^*$, $PR(x)^*$ is prime. The last assertion is equivalent to the assertion that $R(x)^*/PR(x)^* = (R^*(x)/PR^*(x))^* = ((R^*/P)(x))^*$ is a domain. Now by [N, p. 122, Exercise 1], the integral closure $(R^*/P)'$ of the complete local domain $R^*/P$ is local and finitely generated over $R^*/P$, so $(R^*/P)(x)' = (R^*/P)'(x)$ is also local and finitely generated over $(R^*/P)(x)$. By the same exercise, $((R^*/P)(x))^*$ is a domain, and the proof is complete.

In trying to extend results on $J$-extensions of domains to $J$-extensions of rings with zero-divisors, it is natural to ask:

(Q2) If the Noetherian ring $T$ is a $J$-extension of the Noetherian ring $R$, is there a unique minimal prime $P$ of $T$ for which $T/P$ is a $J$-extension of $R/(P \cap R)$?

In view of [GH2, Corollary 2.7], the hypotheses of (Q2) imply that $T/\text{nil}(T)$ is also a $J$-extension of $R/\text{nil}(R)$, so we may assume that $R$ and $T$ are reduced. If we add another hypothesis, we get an affirmative answer:

4.2. PROPOSITION. Let $R$ be a Noetherian reduced ring and $T$ be a $J$-extension of $R$ within its total quotient ring. Then there is a unique minimal prime ideal $P$ of $T$ for which $T/P$ is a $J$-extension of $R/(P \cap R)$.

Proof. Note first that $T$ is Noetherian by [GH2, Corollary 2.8]. The total quotient ring $K$ of $R$ is a finite direct sum of domains, let $R^\wedge$ and $T^\wedge$ be the results of adjoining to $R$ and $T$ respectively the indecomposable idempotents $e_1, \ldots, e_n$ in this direct sum. We claim that $T^\wedge$ is a $J$-extension of $R^\wedge$. To see this, suppose $S$ is a ring containing $R^\wedge$ and properly contained in $T^\wedge$. Then $S \cap T$ is properly contained in $T$, so it is finitely generated over $R$, and hence there is a nonzerodivisor in $R$ multiplying $S \cap T$ into $R$; thus $S \cap T$ is finitely generated as an $R$-module. Similarly,
there is a nonzerodivisor in $R$ multiplying $e_1, \ldots, e_n$ into $T$ and hence multiplying $S$ into $S \cap T$, so $S$ is a finitely generated $(S \cap T)$-module. Thus $S$ is finitely generated over $R$, and the claim follows.

Now $R^\langle$ and $T^\langle$ are themselves direct sums of the domains $Re_i$ and $Te_i$. It follows easily that, for exactly one such $e_i$ (say $e_1$, without loss of generality), $Te_1$ is a $J$-extension of $Re_1$, and $Te_i = Re_i$ for $i > 1$. Then since the minimal primes of $T$ have the form $P_i = (1 - e_i)K \cap T$ for $i = 1, \ldots, n$, we conclude that $P_1$ is the unique minimal prime for which $T/P_1$ is a $J$-extension of $R/(P_1 \cap R)$.

Finally:

(Q3) If a ring $T$ is a $J$-extension of a ring $R$, when is $T/R$ an almost finitely generated $R$-module?

We have seen that this holds under the hypotheses of General Example 3.3; it may hold when $R$ is a general one-dimensional semilocal domain. On the other hand, if $R$ and $T$ are fields, then it does not hold, for then $T/R$ is an infinite-dimensional vector space over $R$. An example of two fields for which the larger is a $J$-extension of the smaller is given in [GH3].

REFERENCES

[HL] W. Heinzer and D. Lantz, Artinian modules and modules of which all proper submodules are finitely generated, J. Algebra 95 (1985), 201–216.


