A reductive technique for enumerating non-isomorphic planar maps

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Abstract

A general method for the counting of unrooted planar maps is proposed. It reduces the problem to the counting of rooted maps of several classes of three kinds: planar, projective and 'circular'. The latter are reduced further (for the set of all maps) to certain generalized rooted quadrangular dissections of the disc. Their counting in a 'closed' form remains so far an open problem.

The method is based upon an exhaustive classification of the periodic homeomorphisms of the geometrical sphere, including orientation-reversing ones, into five types. A general formula of enumerating under orthogonal actions of a group is also derived.

1. Introduction

1.1. In [10, 11] we developed a general technique for the exact counting of unrooted planar maps up to sense-preserving sphere homeomorphisms. It reduces the problem to that for rooted planar maps of the same and several auxiliary classes which depend heavily on the class under enumeration. This method turned out rather effective for a large number of particular classes of maps. On the other hand, Wormald [19, 20] developed an algorithmic method for enumerating planar maps up to arbitrary symmetries including orientation-reversing ones. However, it leads to much more complicated ('unclosed') formulae, even when restricted to orientation-preserving symmetries.

The aim of the present paper is to extend our reductive approach (in a slightly improved and generalized form) to sense-reversing homeomorphisms. We achieve it, though, with considerable complications: the reduction leads not only to rooted planar maps but also to two different, more difficult, generalized kinds of rooted maps — projective and circular. For the latter type, an effective enumerative technique has not been developed yet and hardly can turn out as simple as in the case of rooted planar maps. A bijection between circular maps and certain generalized dissections of the disc (cf. [2]) is also established.
We think that in many cases this natural approach reveals the inherent difficulties of the problem and will lead to the simplest possible formulae. We hope that such formulae will not only be found afterwards, but will turn out rather convenient, for a large number of important classes of maps (such a closed form of results is typical for the map enumeration in general, but seems to have not found a sufficiently convincing explanation yet). The results of the present paper may serve to stimulate further activity in the vast contemporary theory of the counting of rooted maps.

The method is based upon a strict classification of planar map symmetries into five types. Then, quotient maps of planar maps with respect to their automorphisms of all the types are introduced. It is this notion that distinguishes our approach from the approaches developed in [2, 19, 20]. We think it would be worth looking for a combined approach especially to overcome obstacles which arise for circular maps.

The main result (Theorem 5.3) is a general uniform formula; when applied to a particular class of planar maps, it needs to be supplemented with ad hoc means for studying and enumerating the rooted quotient maps of the classes that arise. In this paper, only one important, and, evidently, simplest class — all planar maps (our primary goal) — is considered and is reduced as far as we have been able to.

It should be noted that four different presentations of maps are used here: topological (as they are), geometric, combinatorial in terms of cell incidences, and, finally, combinatorial–algebraic in terms of permutations acting on doubly oriented edges. One auxiliary general result concerning the enumeration under two orthogonal actions of a group is also derived.

1.2. By a map we mean a finite cell dissection of a closed topological surface, i.e., a dissection of it to open 0-, 1- and 2-dimensional cells called vertices, edges, and faces, respectively. The vertices and edges of a map form a connected pseudo-graph. If one cell belongs to the boundary of another cell, they are called incident. In general, an edge is incident to two different vertices (its ends) and to two different faces (its sides). If both ends coincide, an edge is called a loop. If both sides coincide, an edge is called an isthmus. Two edges may have a pair of ends in common (and a pair of sides, as well). Maps on the sphere are called planar. From now on, we assume that the set of edges is not empty. Other details can be found in [5,9,11,19,20].

Any edge may be doubly oriented along and across it, i.e., we can select a direction to one of its ends and a direction to one of its sides. Each edge, including a loop or an isthmus, has four possible orientations. Such a doubly oriented edge will be called a flag (various authors use also different terms such as blade or arrow).

It is well known (cf. [4,5,9]) that maps allow a precise combinatorial description as triples of permutations acting on the flags. Let $X$ be an abstract set that corresponds bijectively to the set of flags, $|X| = 4n$, where $n, n \geq 1$, is the number of edges. The simplest model has the form of a triple $(\phi, \theta, \psi)$ of fixed-point-free involutions on $X$ which generate a transitive group and satisfy the conditions that $\phi \theta = \theta \phi$ and this product is also a fixed-point-free involution. In the map interpretation, $\phi$ interchanges both ends of any edge (to be more precise, both flags of the same edge oriented to the
same side), \( \theta \) interchanges both sides of any edge and \( \psi \) interchanges both sides of any corner (a corner is described by the pair of flags that lie on both its sides and are oriented to its vertex and inside it). They can be depicted as follows:

\[
\phi: |�|, \quad \theta: −|−|, \quad \psi: ^−^-
\]

In particular, the following assertion is valid.

**1.3. Lemma.** For \( n \geq 2 \) up to trivial homeomorphisms preserving all cells, each map automorphism is defined uniquely by its action on the flags. This action is regular, i.e., it consists of cycles of an equal length.

As a matter of fact, any map automorphism is represented (up to trivial homeomorphisms) as a permutation on \( X \) that commutes with \( \phi, \theta \) and \( \psi \) (an element of their centralizer). Therefore, the last assertion in Lemma 1.3 follows from the transitivity of the permutation group \( \langle \phi, \theta, \psi \rangle \) generated by \( \phi, \theta \) and \( \psi \). For \( n = 1 \) this presentation is not faithful. For example, the map with \( \phi = (12) (34), \theta = (13) (24) \) and \( \psi = (14) (23) \) (it represents a map on the projective plane consisting of one vertex, one face and one ‘twisted’ loop), has only trivial symmetries though \( \phi, \theta \) and \( \psi \) commute with each other.

2. **Enumeration under orthogonal actions**

Here we derive a simple auxiliary algebraic result that seems to present a certain interest of its own. It is an enumerative corollary of a property that generalizes Lemma 1.3. It is described in terms of generalized ‘rooted’ objects. Some elementary facts from group theory are used.

2.1. Let \( G \) be a finite group that is endowed with actions on two finite disjoint sets \( X \) and \( Y \). In other words, we are given two homeomorphisms of \( G \) into the symmetric groups \( \Sigma(X) \) and \( \Sigma(Y) \). We assume also that the action on \( X \) is faithful, that is, the corresponding homeomorphism is injective.

In what follows we adopt the next *orthogonality condition*:

\[
|\text{Fix}_gx||\text{Fix}_gy| = 0 \quad \text{for all } g \in G, \, g \neq 1,
\]

where \( \text{Fix}_gx \) denotes the set of all elements of \( X \) kept fixed by \( g \) and \( |X| \) means the number of elements in \( X \).

2.2. **Lemma.** The induced componentwise action of \( G \) on the set \( X \times Y \) is semi-regular, i.e., it consists of regular permutations only. It is faithful on every non-empty invariant subset.

This is evident from the orthogonality condition and the faithfulness of \( G \) on \( X \).
2.3. The main theorem of enumerative combinatorics, well known as the (or Burnside’s Cauchy–Frobenius) lemma, asserts that

$$|\text{Orb}(G, Y)| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}_yg|,$$

where $\text{Orb}(G, Y)$ denotes the set of orbits of $G$ in its action on $Y$.

2.4. The normalizer $N(g) = N_G(\langle g \rangle)$ in $G$ of the cyclic group $\langle g \rangle$ generated by $g \in G$ is obviously invariant on $\text{Fix}_rg$.

**Lemma.**

$$|\text{Fix}_rg| = \frac{|N(g)| |\text{Orb}(N(g), X \times \text{Fix}_rg)|}{|X|}.$$

**Proof.** According to Lemma 2.2 each orbit of $N(g)$ on $X \times \text{Fix}_rg$ contains $|N(g)|$ elements. Therefore (this is valid for empty $\text{Fix}_Xg$ as well),

$$|X| |\text{Fix}_Xg| = |N(g)| |\text{Orb}(N(g), X \times \text{Fix}_rg)|.$$

2.5. Let $Z_X = Z_X(G)$ denote the set of cyclic subgroups of $G$ that act *semi-regularly* on $X$ and $C_X = C_X(G)$ denote the set of conjugacy classes (in $G$) of groups in $Z_X$. For every $c \in C_X$, we select an element $c^o \in G$ such that $\langle c^o \rangle = c$. From the orthogonality condition 2.1

$$\sum_{g \in G} |\text{Fix}_yg| = \sum_{g \text{ is regular on } X} |\text{Fix}_yg| = \sum_{c \in C_X} \sum_{\langle g \rangle = z} |\text{Fix}_yg|,$n

$$= \sum_{c \in C_X} |c| \phi(c)|\text{Fix}_yc^o|,$n

where $\phi(c)$ denotes the number of elements $g \in G$ such that $\langle g \rangle = \langle c^o \rangle$. So, $\phi(c)$ is the Euler totient function of the order of $c^o$. It is well known that $|c| = |G|/|N(c)|$ where $N(c)$ denotes $N(c^o)$. Then according to Lemma 2.4, enumerative formula 2.3 may be written as follows.

**Corollary.**

$$|\text{Orb}(G, Y)| = \frac{1}{|X|} \sum_{c \in C_X} \phi(c) |\text{Orb}(N(c), X \times \text{Fix}_yc^o)|.$$

2.6. In combinatorial terms elements of $X$ are often called points or ‘figures’, elements of $Y$ — (labelled) configurations, elements of $X \times Y$ — rooted configurations, elements of $\text{Orb}(G, Y)$ — non-isomorphic (unlabelled) configurations (with respect to $G$) and elements of $\text{Orb}(G, X \times Y)$ — non-isomorphic rooted configurations. Elements of $Y$ are usually certain objects constructed on $X$ such as graphs, tuples of permutations, etc. Then the root means simply a point selected or marked on such an object.
More generally, under \( c \)-rooted (labelled) configurations we mean (cf. [19]) pairs \(((s), y)\) where \((s)\) is a cycle of \( c^° \) in its action on \( X \) and \( y \) is an element in \( \text{Fix}_{c^°} \).

In order to define non-isomorphic \( c \)-rooted configurations we endow \((s)\) with the structure of the cyclic permutation group \( \langle s \rangle \) generated by \((s)\). The action of \( G \) on \( X \) induces its setwise action on the subsets of \( X \). Furthermore, for any \( s \subseteq X \) any element \( g \in G \) transforms a cyclic structure \((s)\) to \( \langle g(s) \rangle \) and its cyclic group structure \( <s> \) to \( <g(s)> \). It is clear that in this action the set

\[
\Pi_c(X) = \{ \langle s \rangle \}_{(s)} \text{ is a cycle of } c
\]

is invariant with respect to \( N(c) \). Moreover, \( N(c) \) is its setwise stabilizer in \( G \). Orbits in \( \text{Orb}(N(c), \Pi_c(X) \times \text{Fix}_{c^°}) \) are called non-isomorphic \( c \)-rooted configurations. These become rooted configurations when \( c = 1 \).

There is a bijection between non-isomorphic \( c \)-rooted configurations and orbits in \( \text{Orb}(N(c), X \times \text{Fix}_{c^°}) \). Indeed, if there is an element \( g \in N(c) \) such that \( g(x_1) = x_2 \) for \( x_1 \in s_1, x_2 \in s_2 \) where \( \langle s_1 \rangle, \langle s_2 \rangle \in \Pi_c(X), \) then \( g\langle s_1 \rangle = \langle s_2 \rangle \). Conversely, if \( g\langle s_1 \rangle = \langle s_2 \rangle \), then the elements of \( s_1 \) and \( s_2 \) belong to a common orbit of \( N(c) \) since \( <c> \), a subgroup of \( N(c) \), acts transitively on every \( s_i \). Together with the previous assertions, this yields the desired general result.

**Proposition.** Under the orthogonality condition 2.1, the number of non-isomorphic configurations is expressed by the formula

\[
|\text{Orb}(G, Y)| = \frac{1}{|X|} \sum_{c \in C_X(G)} \phi(c) R_Y(c),
\]

where \( R_Y(c) \) denotes the number of non-isomorphic (with respect to \( G \)) \( c \)-rooted configurations.

2.7. For the problem under consideration, \( Y = \mathcal{M}(n) \) is a certain set of maps with \( n \) edges \((n \geq 2)\), \( X \) is the set of (labelled) flags. We will assume thereafter the validity of the closure (or closedness) condition [11] which ensures that any procedure of relabelling the flags of a map belonging to \( \mathcal{M}(n) \) leaves it in \( \mathcal{M}(n) \). In other words, \( \mathcal{M}(n) \) is invariant with respect to (the induced action of) the symmetric group \( \mathcal{G}(X) = \mathcal{S}_X \) as \( G \). We are interested in the number of unlabelled maps in \( \mathcal{M}(n) \), that is, \( |\text{Orb}(\mathcal{G}(X), \mathcal{M}(n))| \).

In accordance with Section 2.6, a map is defined to be rooted if some of its flag is distinguished as the root [17]. Each class \( c \in C_X(\mathcal{G}(X)) \) is determined by a unique parameter — the order \( L = L(c) \) of its groups. Therefore, we will write \( L \) instead of \( c \) everywhere. \( L^° \) will denote a selected regular permutation of order \( L \) (for definiteness, \( L^° = (123 \ldots L)(L + 1 \ldots) \)). In terms of these, Proposition 2.6 gets the following form.
Theorem.

\[ M(n) = \frac{1}{4n} \sum_{L > 1, L \leq 4n} \varphi(L) M(L)(n), \quad n \geq 2, \]

where \( M(n) \) is the number of non-isomorphic (unlabelled) maps in \( \mathcal{M}(n) \), \( M(L)(n) \) is the number of non-isomorphic \( L \)-rooted maps corresponding to \( \mathcal{M}(n) \).

Indeed, according to Lemma 1.3, the action of \( S_{4n} \) on \( X \) is orthogonal to its action on \( \mathcal{M}(n) \).

We recall that an unlabelled \( L \)-rooted map is a map which has a selected cyclically ordered \( L \)-element set of root-flags (a cycle of \( L^o \)), is invariant with respect to the permutation \( L^o \) and is considered up to symmetries preserving the group generated by the selected cycle. In other words, all flags, other than root-flags, are indistinguishable, the root-flags are endowed with the structure of a cyclic group (generated by a cyclic permutation) and are otherwise indistinguishable one from another too, and, finally, \( L^o \) leaves the map invariant. This definition generalizes naturally, the notion of a rooted, i.e. 1-rooted map. For the number of rooted maps we will write \( M'(n) \) instead of \( M(1)(n) \).

2.8. If the surface is orientable and we are given a fixed orientation on it which must be preserved by the symmetries, then there is a similar well-known map model (cf. [18]) that describes maps in terms of along-oriented edges called darts. Since the analog of Lemma 1.3 is also valid, we obtain likewise

Theorem (cf. [11]).

\[ M^+(n) = \frac{1}{2n} \sum_{L > 1, L \leq 2n} \varphi(L) M^+(n), \quad n \geq 2, \]

where \( M^+(n) \) (and \( M^+(1)(n) \)) is the number of non-isomorphic \( L \)-rooted, resp.) maps in \( \mathcal{M}(n) \) considered up to orientation-preserving homeomorphisms.

Notice that \( M^{(+, 1)}(n) = M^{(1)}(n) = M'(n) \).

The dart model of maps on orientable surfaces may be described in terms of flags in the form of pairs of permutations \((\alpha, \sigma)\) where \( \alpha = \theta \phi \) is a fixed-point-free involution and \( \sigma = \theta \psi \) consists of cycles that correspond to rotations around the vertices.

2.9. Remark. It seems possible to replace the symmetric groups \( S_{4n} \) and \( S_{2n} \), used in the previous results, with wreath products of \( S_n \) (permutations of abstract edges) by the dihedral group \( D_2 \) and by the cyclic group \( C_2 \), respectively (the symmetry groups of an edge). In the case of orientable surfaces, it seems also possible to restrict the values of \( L \) to \( L \leq 2n \) in Theorem 2.7 and to \( L \leq n \) in Theorem 2.8 (see Section 5).
3. Classification of planar map automorphisms

3.1. In the case of the sphere $S$ we will use, in addition, geometric presentations of maps. According to a result of Mani [13], any planar 3-connected (polyhedral) graph may be represented on the geometrical sphere in such a way that all its automorphisms are induced by symmetries (moves) of the sphere. This theorem is transferred easily to arbitrary planar maps by means of the standard construction of their triangulations. Therefore, for our purposes, the following assertion is important.

Lemma (Coxeter [6, Theorem 7.4.1]; cf. Eilenberg [7]). Every non-trivial periodic symmetry of the sphere is a rotation, a reflection or the product of a (commuting) pair which consists of a reflection and a rotation around the axis perpendicular to the reflection plane.

3.2. We need a more thorough (for the third case) classification which we found in literature only implicitly. There exist five types (classes) of non-trivial periodic sphere symmetries that will be denoted throughout by mnemonic Greek letters.

$I$ — rotations. Every $\rho \in I$ is defined by (i) its order $l$, $l \geq 2$, (ii) the pair of poles fixed by it and lying on the rotation axis and (iii) the rotation angle $\frac{2\pi d}{l}$ where $1 \leq d < l$, $(d, l) = 1$ (hereafter $(d, l)$ will denote the g.c.d. of the numbers $d$ and $l$).

$O$ — reflections. Every $\omega \in O$ has order 2 and the set of points fixed by it forms the equatorial circle lying on the rotation plane.

$X$ — the central (antipodal) inversion. This class contains a unique symmetry $x$. It is of order 2, has no fixed points and coincide with the product of any reflection by the rotation of order 2 around the perpendicular axis.

$\Theta$ — reflections combined with the corresponding rotations of even orders greater than 2. Every $\lambda \in \Theta$ is defined by (i) its order $l$, $l \geq 4$, $l$ is even, (ii) the pair of poles interchanged by it and (iii) the rotation angle $\frac{2\pi d}{l}$ where $1 \leq d < l$, $(d, l) = 1$. All points, except for the poles, have order $l$.

$\Phi$ — reflections combined with the corresponding rotations of odd orders greater than 1. Every $\alpha \in \Phi$ is defined by (i) its order $2l$, $l \geq 3$, $l$ is odd, (ii) the pair of poles interchanged by it and (iii) the rotation angle $\frac{2\pi d}{l}$ where $1 \leq d < l$, $(d, l) = 1$. Equatorial points have order $l$, the other points, except for the poles, have order $2l$.

In the cases of $\Theta$ and $\Phi$, the equator is defined by the reflection plane which is the plane that passes through the sphere centre and is perpendicular to the rotation axis. Each point passes first into its mirror image and then revolves. However, in the case of $\Theta$, equatorial points do not differ, in essence, from the other non-axial points.

$I$-symmetries are orientation-preserving, the other types are orientation-reversing.

Let us denote $K = \{I, O, X, \Theta, \Phi\}$. Sometimes, we will refine these symmetry classes with various parameters. In particular, the rotation order will be written as a superscript in parentheses, for example, $I^{(3)}$, $\Theta^{(3)}$. Clearly $O^{(2)} = O$, $X^{(2)} = X$, $\Theta^{(0)} = \emptyset$ for an odd $l$, $\Phi^{(0)} = \emptyset$ for an even $l$. 
3.3. We apply the above classification to planar maps. The cells containing the poles are called \textit{axial} and those intersected with the reflection plane are called \textit{equatorial}. It will be important to distinguish \textit{axial edges}. Their number will be written as a subscript. Clearly, in the case of $I^{(0)}$ a map may contain one or two axial edges if only, $l = 2$. In addition, axial edges are possible in the case of $\Theta^{(4)}$. Indeed, if $\alpha = \rho \omega \in \Theta \cup \Phi$ (where $\rho \in I$, $\omega \in O$) then $\alpha^2 = \rho^2$ is a rotation symmetry around the same axis. Therefore, if the axis intersects an edge then $\rho^2$ must be of order 2 and both its axial cells are edges. Thus, we have subclasses $I_1^{(2)}$, $I_2^{(2)}$ and $\Theta_2^{(4)}$ while the other classes have no axial edge (in [11] instead of the first two symbols we used Greek letters, $T$ and $H$, respectively, and refined them further depending on properties of both axial cells).

A simple example of type $\Theta_2^{(4)}$ is the symmetry of the tetrahedron which moves its vertices cyclically: $(1234)$. Its rotation axis intersects two edges, namely, (1,3) and (2,4).

The next property differentiates $\Theta$ and $\Phi$ as classes of \textit{map} symmetries. If $\alpha = \rho \omega \in \Phi^{(0)}$ is a map automorphism then, clearly, $\alpha^2 = \omega$ and thus, both $\rho$ and $\omega$ are also automorphisms of the map. On the other hand, in the case of $\Theta$ (and $X$ as well) a map may not be invariant with respect to the corresponding reflection and rotation taken separately (the 'non-automorphical' case) as in the example for the tetrahedron given above.

For a better understanding of map symmetries, we describe briefly possible cycle structures of their induced action on the set of all $2n + 2$ cells ($l^m$ means $m$ cycles of length $l$):

- $(1^2 l^m)$ for $I^{(0)}$ where $lm = 2n (l | n$ for $l > 2)$;
- $(1^{2k} 2^m)$ for $O$ where $k + m = n + 1$, $k \geq 1$;
- $(2^* + 1)$ for $X$ where $n$ is even;
- $(2^* l^m)$ for $\Theta^{(0)}$ where $l \geq 4$ is even, $lm = 2n$, $n$ is even ($l | n$ for $l > 4$);
- $(2^1 2^2 2^m)$ for $\Phi^{(0)}$ where $l \geq 3$ is odd, $l(k + m) = n$, $k \geq 1$.

If $L$ denotes the automorphism order, then $L = l \ (l = 2$ for $O$ and $X$) in all cases, except for $\Phi^{(0)}$ when $L = 2l$.

It is worth noting also that, in general, a map symmetry is \textit{not} defined by its induced action on the set of cells. For example, the map-chain has a rotation and a reflection both of which act identically on the set of cells (it has also another reflection that leaves all the cells fixed, i.e., acts as the trivial automorphism).

3.4. \textbf{Proposition.} Each non-trivial automorphism of a planar map belongs to a unique class of $K$. It is defined by the pair of axial cells and the rotation angle in the cases of $I$, $\Theta$ and $\Phi$ or by the set of equatorial cells in the case of $O$.

This follows directly from Mani’s theorem, the above classification and the simple fact that the rotation angle is an invariant of automorphisms, not depending on the way in which the map is represented geometrically.
3.5. The equatorial circle of a map automorphism of types \( O \) or \( \Phi \) is partitioned by cells into parts of four kinds: vertices, edges, segments intersecting with faces and points intersecting with edges. This simple fact will be used later.

4. Reduction to quotient maps

The central construction of the present paper is the quotient map of a planar map by a symmetry, which turns out a certain generalized map.

4.1. A planar map is called punctured if two of its cells, other than edges, are distinguished as axial. A projective map (that is, a map on the projective plane) is called punctured if one of its vertices or faces is distinguished as axial. Any pendant (i.e., of valency 1) axial vertex of a punctured map may be declared as singular. An edge with a singular end is also called singular (half-edge). It is considered as having only 2 flags oriented to both sides and only to the non-singular end (hereafter we exclude the planar map with a sole edge both of whose ends are singular).

As we pointed out in Section 3.5, the equator of a map symmetry may be considered as a polygon with two types of points (polygon vertices) and two types of links (polygon sides). Its points of intersections with edges of the map are called singular. Links of intersections with faces of the map are called quasi-edges, as opposed to the other map edges called ordinary. Moreover, such polygons satisfy the following condition:

(C1) both links incident to a singular point are quasi-edges (they may coincide in the case of a 1-gon, i.e. a loop).

Such a polygon will be called generalized.

**Definition** (cf. [4, 16]). A circular map means a finite cell dissection of a closed disc that induces a generalized polygon on the boundary and possesses the following properties:

(C2) each face is incident to at most one boundary quasi-edge;
(C3) each boundary vertex is incident to at least one ordinary edge;
(C4) each singular boundary vertex is incident to exactly one internal (i.e., not boundary) edge;
(C5) each ordinary edge is incident to at most one singular vertex.

In circular maps, we consider a quasi-edge as belonging to the face incident to it. Such a face will be called boundary, as opposed to the other faces called internal. Thus, a quasi-edge has no flags (i.e., no ‘usual’ sides and ends). On the contrary, an ordinary boundary edge has only one side and two flags. An ordinary internal edge, incident to a singular vertex, is also called singular. It has one end and two flags.
Internal non-singular edges are called normal. An internal edge may connect two boundary vertices.

A circular map is called punctured if one of its internal vertices or faces is distinguished as axial. Again, a pendant axial vertex may be declared as singular (see, however, property C5).

Planar, projective and circular maps together, punctured or not as described above, are called generalized maps. The number of flags in a generalized map is always even. The number of edges in such a map is an integer or a half-integer: it is equal to the number of normal edges plus half of the number of singular internal edges and boundary edges (quasi-edges are not taken into account at all!).

4.2. The results described below are merely straightforward consequences of well-known facts from the theory of Riemann surfaces and covering spaces (cf. [21, 14, 9]; we could also use the so-called 'orbifolds', i.e., quotient manifolds with singularities, after Thurston, cf. [15]).

**Definition.** A quotient map (an $\Omega$-quotient map) $B = A/\omega$ a planar map $A$ with respect to an automorphism $\omega, \omega \in \Omega, \Omega \in K$, means the quotient space $S/\langle \omega \rangle$ (the orbit space) and its induced cell dissection that forms a generalized map. It has

(i) images of axial vertices and faces of $A$ (if any) as the distinguished axial cells;
(ii) images of points where edges of $A$ intersect with the rotation axis or with the equator as the singular vertices;
(iii) images of segments where faces of $A$ intersect with the equator as the quasi-edges.

Note that singular vertices and quasi-edges of $B$ are the only cells that correspond to no cells of $A$. They are necessary for obtaining a cell dissection of the quotient space. If $A$ has a root (independent of $\omega$) it is transformed to the root of $B$.

4.3. **Lemma.** The quotient map of a planar map with respect to a non-trivial automorphism is of one of the following forms:

$I$-quotient maps are punctured planar maps, which may contain singular axial vertices in the case of $I^{(2)}$;

$O$-quotient maps are circular maps;

$X$-quotient maps are projective maps;

$\Theta$-quotient maps are punctured projective maps, which may contain a singular axial vertex in the case of $\Theta^{(4)}$;

$\Phi$-quotient maps are punctured circular maps.

This is easily verified, including properties C1–C5 for $O$ and $\Phi$.

It is convenient to build quotient maps geometrically with the help of the Mani theorem. The $I^{(0)}$-quotient map is obtained by cutting out a sphere sector with angle
2π/l bearing upon the poles and then by gluing its boundary half-circles [11]. The O-quotient map is half of the original map that lies above the equator and is endowed with the appropriate singular vertices and quasi-edges on the boundary. The standard 2-fold covering of the projective plane by the sphere induces the X-quotient map. The case of Φ(0), l odd, is reduced to I(0) and O because \( A/\alpha \cong (A/\rho)/\omega \) where \( \alpha = \rho \omega \in \Phi(0), \rho \in I(0), \omega \in O \). Similarly, the case of Θ(0), l even, is reduced to I(0/2) and X: \( A/\alpha \cong (A/\alpha^2)/\chi \) where \( \alpha \in \Theta(0), \alpha^2 \in I(0/2), \chi \in X \). In fact, the quotient group \( \langle \alpha \rangle/\langle \alpha^2 \rangle \) acts, clearly, on the punctured planar map \( A/\alpha^2 \) as \( \chi \).

For example, the above-mentioned \( \Theta^{(4)} \)-automorphism (1234) of the tetrahedron generates a punctured projective map that contains one vertex, one face, one edge, one singular edge and one singular axial vertex. Thus, it is merely the non-shrinkable twisted loop (the middle line of the Moebius band) with one vertex to which a singular edge is attached. It has \( 4 + 2 = 6 \) flags (and one edge and a half) as required.

4.4. Lemma 4.3 can be reversed (cf. [11]). Let \( \Omega(r) \in K \) and \( B \) be an arbitrary generalized map of the form that corresponds to \( I(0) \) according to Lemma 4.3.

**Lemma** (on lifting). \( B \) is the \( \Omega(r) \)-quotient map of a uniquely defined planar map \( A \) with respect to a given automorphism \( \alpha \) of type \( \Omega(r) \) with the corresponding rotation axis and/or reflection plane. There is a bijection between flags of \( B \) and cycles of \( \alpha \) on the flags of \( A \).

Proof follows from the general theory (cf. [21]). For \( I \) it was given (together with a construction of \( A \)) in [11]. For \( O \) it is evident from geometrical considerations (Definition 4.1 with properties \( C1-C5 \) does ensure that gluing two mirror copies of \( B \) results in a cell dissection of the sphere after erasing the singular vertices and quasi-edges). For \( X \) the assertion is well known in the theory of covering spaces [14]. Finally, the cases of \( \Theta \) and \( \Phi \) are reduced (as shown above) to \( I \) and \( X \) and to \( I \) and \( O \), respectively.

The necessity to establish a bijection between cycles of \( \alpha \) and flags of \( B \) have just predetermined our way of specifying flags on singular and boundary edges and quasi-edges.

4.5. Let \( \alpha \) be an \( L \)-regular permutation on the set of \( 4n \) flags, \( L \geq 2, L \mid 2n \), and \( \Omega(r) \in K \) where \( l = L \) for \( \Omega \neq \Phi \) and \( l = L/2 \) for \( \Omega = \Phi \) (\( l \geq 3 \) is an odd integer in this case). Let \( B \) be a rooted generalized map with \( 4n/L \) flags \((n/L \) edges) of the type that corresponds to \( \Omega(r) \) according to Lemma 4.3.

**Corollary.** There is a unique rooted planar map with \( n \) edges which is the \( \Omega(r) \)-lifting of \( B \).
5. General reductive enumeration formulae

5.1. According to Proposition 3.4 if $\mathcal{M}(n)$ is a set of planar maps satisfying the closure condition then

$$M^{(L)}(n) = \sum_{\Omega \in K} M^{(\Omega, L)}(n), \quad n \geq 2, \quad L \geq 2,$$

where $M^{(\Omega, L)}(n)$ is the number of $L$-rooted maps in $\mathcal{M}(n)$ for which the corresponding $L$-automorphism $L^\circ$ is of type $\Omega$ (see Section 2.7).

5.2. Let $\mathcal{M}_{\Omega, r}(t)$ be the set of $t$-edged generalized maps that are the quotient maps of the maps in $\mathcal{M}(n)$ with respect to the $\Omega$-automorphisms of order $L$. Here $t = n/L$ is an integer or half-integer number. $L = 2$ in the cases of $O$ and $X$, $L = l$ in the cases of $I(l)$ and $\Theta(l)$, $L = 2l$ in the case of $\Phi(l)$. The forms of these generalized maps are described in Lemma 4.3.

According to Lemma 4.4 and Corollary 4.5,

$$M^{(\Omega, L)}(n) = M^{(\Omega, L)}_{t/2}(n/L), \quad L \mid 2n, \quad \Omega \in K,$

where $M^{(\Omega, L)}_{t/2}(n)$ is the number of rooted maps in $\mathcal{M}_{\Omega, r}(t)$.

5.3. The main result of this paper is expressed as follows.

**Theorem.** The number of non-isomorphic planar maps in $\mathcal{M}(n)$ is determined by the formula

$$M(n) = \frac{1}{4n} \left[ M'(n) + M'_{1,2}(n/2) + \sum_{\ell \mid n} \varphi(l) M'_{1,1}(n/l) + M_{O}(n/2) + M'_{X}(n/2) \right]$$

$$+ 2M_{\Phi, 4}(n/4) + \sum_{\ell \mid n, \ell \geq 6 \text{ even}} \varphi(l) M'_{\Phi, 1}(n/l) + \sum_{\ell \mid n, \ell \text{ odd}, \ell \geq 3} \varphi(l) M'_{\Phi, 2}(n/2l) \right].$$

Moreover, here

$$M'_{1,2}(n/2) = \begin{cases} M'_{1,2,1}(n/2), & n \text{ odd}, \\ M'_{1,2,0}(n/2) + M'_{1,2,2}(n/2), & n \text{ even}, \end{cases}$$

$M'_{x}(n/2) = \emptyset$ if $n$ is odd and

$$M_{\Phi, 4}(n/4) = \begin{cases} 0, & n \text{ odd}, \\ M'_{\Phi, 4,2}(n/4), & n \equiv 2 \pmod{4}, \\ M'_{\Phi, 4,0}, & 4 \mid n, \end{cases}$$

where the third index denotes the number of singular axial vertices.

Proof follows directly from Theorem 2.7 and formulas 5.1 and 5.2. It should also be noted that in all cases $\varphi(L) = \varphi(l)$. 
Thus, if $n > 1$ is odd, then only $M', M_1$ (including $M'_{1,2,1}$), $M_0$ and $M_\Phi$ contribute; if $4|n$, then $M', M_1$ (including $M'_{1,2,2}$), $M_\Phi$ (including $M'_{\Phi,4,0}$) and $M_\Phi$ (unless $n = 2^k$) contribute while if $n = 2 (\mod 4)$, then $M', M_1$ (including $M'_{1,2,2}$), $M_0, M_\chi, M_\phi$ (including $M'_{\Phi,4,2}$) and $M_\Phi$ contribute. In the simplest case when $n = p \geq 3$ is a prime number, we have

$$M(p) = \frac{1}{4p} \left[ M'(p) + M'_{1,2,1}(p/2) + (p - 1)M_1(p/2) + M_0(p/2) + M_{\Phi,2p}(1/2) \right].$$

Moreover, for ordinary maps (that is, for $\mathcal{M}(p) \subseteq \mathcal{U}(p)$, see Section 6) $M_1(p) \leq 2$, $M_{\Phi,2p}(\frac{3}{2}) \leq 2$.

This general theorem reduces the enumeration of non-isomorphic planar maps to that of planar, projective and circular rooted maps of several kinds. Irrespective of $l$ we obtain the following six classes of rooted quotient maps to count:

- $\mathcal{M}$ — original planar;
- $\mathcal{M}_t$ — planar punctured (and subclasses with 0, 1 or 2 singular axial vertices);
- $\mathcal{M}_x$ — projective;
- $\mathcal{M}_o$ — projective punctured (and subclasses with the normal or singular axial vertex);
- $\mathcal{M}_o$ — circular;
- $\mathcal{M}_\Phi$ — circular punctured.

In general, these classes, different topologically, may also differ considerably from each other by their internal properties, even $\mathcal{M}$ from $\mathcal{M}_t, \mathcal{M}_x$ from $\mathcal{M}_o$ and $\mathcal{M}_o$ from $\mathcal{M}_\Phi$. Moreover, for a natural class $\mathcal{M}$, the classes of quotient maps may turn out rather artificial and cumbersome. The original map structure is destroyed most of all by $O$ and $\Phi$.

In general, we must also take into account properties of the axial and equatorial cells.

5.4. When enumerating up to sense-preserving symmetries, we obtain similarly from Theorem 2.8 the following.

Theorem.

$$M^+(n) = \frac{1}{2n} \left[ M'(n) + M'_{1,2,1}(n/2) + \sum_{l \geq 3, n/l} \varphi(l) M_1(n/l) \right].$$

In a slightly different form (taking into account kinds and properties of the axial cells) this was given earlier in [10, 11].

5.5. The same considerations hold for maps on the plane, i.e., planar maps with a distinguished external ('infinite') face. The external face remains always fixed, thus,
the rotation axes and equators pass through it. Therefore, automorphisms of types $X$, $\Phi$ and $\Theta$ are impossible (the symmetries belong now to the dihedral group, cf. [2]). Let $\mathfrak{M}(n)$ denote a class of maps on the plane. Then $\mathfrak{M}_I(n)$ is the class of $I$-quotient maps, which are punctured maps on the plane, that is, maps with a selected second end of the rotation axis. $\mathfrak{M}_O(n)$ is the class of $O$-quotient maps which are, clearly, circular maps with a selected equatorial quasi-edge (it corresponds to the external face of the plane). In these designations (together with a standard abuse of notations) we obtain immediately the following.

**Theorem.**

$$ M(n) = \frac{1}{4n} \left[ M'(n) + M'_{1,2}(n/2) + \sum_{l \geq 3, l \mid n} \varphi(l) M_{1,1}(n/l) + \mathcal{M}_O(n/2) \right]. $$

It is worth noting that if class $\mathfrak{M}$ corresponds to $\mathfrak{M}$, then, in general, $M'(n) \neq M'(n)$ because of various ways of selecting the external face. But $M'(n)$ is usually expressed easily through $M'(n)$.

5.6. Finally, if maps on the plane are considered up to sense-preserving homeomorphisms ($M^+(n)$), then we need only to replace 4n with 2n in Theorem 5.5 and to drop the last summand:

**Theorem.**

$$ M^+(n) = \frac{1}{2n} \left[ M'(n) + M'_{1,2}(n/2) + \sum_{l \geq 3, l \mid n} \varphi(l) M_{1,1}(n/l) \right]. $$

6. **Further reduction for the class of all maps**

In the case of all planar maps without restrictions (step 7 of the program proposed in [12]) it is possible to transform the result to a more convenient form (with less unknown summands). Of course, some of these actions may be applied similarly for other subclasses of maps.

6.1. Let $\mathbb{U}(n)$ stand for the set of all (1-connected) $n$-edged planar maps, $n \geq 1$. Then according to Theorem 5.3 the number $A(n)$ of non-isomorphic maps in $\mathbb{U}(n)$ is expressed by the following formula (for $n = 1$ it is verified directly):

$$ A(n) = \frac{1}{4n} \left[ A'(n) + A'_{1,2}(n/2) + \sum_{l \geq 3, l \mid n} \varphi(l) A'_{1,1}(n/l) + A'_O(n/2) + A'_X(n/2) \\
+ 2A'_{O,4}(n/4) + \sum_{l \geq 6 \text{ even, } l \mid n} \varphi(l) A'_{O,1}(n/l) + \sum_{l \geq 3 \text{ odd, } l \mid n} \varphi(l) A'_{O,2}(n/2l) \right]. $$
6.2. There is a famous formula for \( A'(n) \) due to Tutte [17] (cf. [8, Section 2.9]):
\[
A'(n) = \frac{2(2n)!3^n}{n!(n+2)!}, \quad n \geq 0.
\]

6.3. According to the Euler formula, a planar map with \( t \) edges has \( t + 2 \) other cells. \( I \)-quotient maps of arbitrary maps are obviously arbitrary punctured planar maps, so one needs only to select two axial cells. Therefore
\[
\begin{align*}
A'_1(n/l) &= \frac{1}{2}((n/l) + 2)((n/l) + 1)A'_1(n/l), \quad l \geq 3, \quad l|n, \\
A'_1,2,0(n/2) &= \frac{1}{2}((n/2) + 2)((n/2) + 1)A'_1(n/2), \quad 2|n.
\end{align*}
\]

6.4. As in Theorem 5.3,
\[
A'_1,2(n/2) = \begin{cases} 
A'_1,2,1(n/2), & n \text{ odd}, \\
A'_1,2,0(n/2) + A'_1,2,3(n/2), & n \text{ even}.
\end{cases}
\]

As was shown in [11] (where we counted a singular edge as one edge rather than as a half-edge), for quotient maps with one or two singular axial vertices,
\[
\begin{align*}
A'_1,2,1(t) &= t(2t + 3)A'(t - \frac{1}{2}), \quad t > 0 \text{ is half-integer}, \\
A'_1,2,2(t) &= t(2t - 1)A'(t - 1), \quad t > 0 \text{ is integer}.
\end{align*}
\]
Thus, we have for \( n \geq 1 \)
\[
A'_1,2(n/2) = \begin{cases} 
\frac{1}{4}n(n + 3)A'((n - 1)/2), & n \text{ odd}, \\
\frac{1}{8}(n + 2)(n + 4)A'(n/2) + \frac{1}{2}n(n - 1)A'((n - 2)/2), & n \text{ even}.
\end{cases}
\]

6.5. Let \( P'(t) \) denote the number of all rooted projective maps with \( t \) edges. Such a map has \( t + 1 \) other cells. Clearly, for arbitrary planar maps, \( X \)-quotient maps are arbitrary projective maps and \( \Theta \)-quotient maps are arbitrary punctured projective maps. Thus,
\[
\begin{align*}
A'_X(n/2) &= P'(n/2), \quad n \text{ even}, \\
A'_{\Theta,4}(n/l) &= ((n/l) + 1)P'(n/l), \quad l \geq 6 \text{ even, } l|n, \\
A'_{\Theta,4,0}(n/4) &= A'_{\Theta,4,3}(n/4) = \frac{1}{4}(n + 4)P'(n/4), \quad 4|n.
\end{align*}
\]

6.6. For the case of the singular axial vertex, by repeating arguments from [11] we obtain
\[
A'_{\Theta,4}(t) = A'_{\Theta,4,2}(t) = 2tP'(t - \frac{1}{2}), \quad t \text{ half-integer}.
\]
Indeed, let \( B \) be a punctured projective map with \( t, \ t > 1, \) edges one of which is singular (incident to the singular pendant axial vertex). Then \( t \) is half-integer. Let \( r \) denote the root of \( B. \) Remove the axial vertex and the singular edge. We obtain an
ordinary projective map with \( m = t - \frac{1}{2} \) edges. It has the same root unless \( r \) is a flag of the removed edge. In the latter case, we adopt \( \psi(r) \) as the new root. This is always possible because the root end is not pendant and thus \( \psi(r) \) does not belong to the same singular edge. On the contrary, to any rooted projective map with \( m \) edges one can attach a new pendant singular edge in \( 2m \) ways inserting it to one of \( 4m/2 \) corners. \( r \) is preserved as the root in all cases except for the corner \((r, \psi(r))\) when two flags may be taken as the root: \( r \) or the flag neighbouring to \( r \) in the inserted edge. This generates \( 2m + 1 = 2t \) punctured rooted maps with the singular axial vertex. Therefore,

\[
A'_{\Theta,4}(n/4) = \frac{1}{2} nP'((n - 2)/4), \quad n \geq 6, \quad n \equiv 2 \pmod{4}.
\]

6.7. The above reduction for \( X \) and \( \Theta \) is quite effective in the sense that for \( P'(n) \) a rather simple expression in generating functions and a recursive formula have recently been obtained in [1]. Namely,

\[
p(x) = (2R + 1 - \sqrt{3R(R + 2)})/6x,
\]

where \( p(x) = \sum_{n=0}^{\infty} P'(n)x^n \), \( R = x/1 - 12x \), and

\[
(n + 1)n(n - 1)P'(n) - 4n(n - 1)(8n - 13)P'(n - 1) + 144(n - 1)(n - 2)(2n - 5)P'(n - 2) + 216(2n - 5)(4n - 9)P'(n - 3) - 1728(2n - 5)(2n - 7)(n - 3)P'(n - 4) = 0, \quad n \geq 4.
\]

6.8. Clearly \( A'_\Theta(n/2) \) is the number of all possible rooted circular maps without restrictions. We will now derive a further reduction for it to a more familiar problem by generalizing a well-known notion of the map quadrangulation (cf., for instance, [17, 11]).

Let \( B = B(D; V_n \cup V_s, E_n \cup E_b \cup E_s \cup E_q, F \cup F_b) \) be a circular map where \( D \) stands for the closed disc, \( V_n \) and \( V_s \) stand for the sets of normal (i.e., non-singular) and singular (boundary) vertices, respectively, \( E_n, E_b, E_s \) and \( E_q \) stand for the sets of normal (internal), boundary, singular (internal) edges and quasi-edges, respectively, \( F \) and \( F_b \) stand for the sets of internal and boundary faces of \( B \), respectively. According to property \( C2 \), there is a bijection between \( F_b \) and \( E_q \).

We distinguish a point inside any internal face (from \( F \)) and any quasi-edge (from \( E_q \)). These points are called **normal facial vertices** and form the set \( W_n \). Now we draw a line that connects an original vertex \( v \) (from \( V_n \cup V_s \)) with a facial vertex whenever the corresponding face (including its quasi-edge, if any) is incident to \( v \). The line lies inside this internal or boundary face. Moreover, the line that connects an end of a quasi-edge with the added point on it forms a part of the quasi-edge. Two vertices may be connected by several lines that form a complete set of pairwise **non-homotopic** paths on the face. Finally, we require that these lines do not intersect each other (inside).
Similarly, we distinguish a point inside any boundary edge (from Eq). It partitions this edge to two lines that connect it with both ends of the edge. These points are called singular facial vertices and form the set $W_s$.

All the above-mentioned lines form the set $E_\square$ of added edges. Now we erase the original edges and thus obtain the required circular map $B_\square = B_\square (D; V_n \cup V_s \cup W_n \cup W_s, E_\square, F_\square)$ called the quadrangulation of $B$. An example is shown in Fig. 1.

**Lemma.** (i) The set $F_\square$ of faces of $B_\square$ corresponds bijectively to the set $E_n \cup E_b \cup E_s$ of edges in such a way that a normal or singular edge of $B$ lies inside the corresponding face of $F_\square$ while a boundary edge lies on its boundary.

(ii) Singular vertices $V_s \cup W_s$ are of valency 2 and are pairwise non-adjacent in $B_\square$.

(iii) Each face of $B_\square$ is of valency 4, i.e., has 4 sides.

(iv) The boundary of $D$ is covered by edges and vertices of $B_\square$.

A similar assertion is well known for planar maps and, thus, these properties need to be verified in addition, only with respect to boundary elements. This is done directly by construction. The 2-valence of vertices in $V_s$ follows from property $C2$.

**Remark.** In general, $B_\square$ is a 1-connected map, i.e., a face of it may contain coincident or even pendant vertices (isthmuses) on its boundary.

6.9. Conversely, let $B_\square$ be a quadrangular dissection of the disc, just described, with more than 1 cell (cf. [2] where only 2-connected quadrangular dissections were considered). It is a planar map with all faces, including the external face, of even valencies. Therefore (see [18]), it is bipartite. So we select one part and declare it

---

Fig. 1. Dotted lines represent quasi-edges. Vertices: round — original, square — facial, hollow — singular.
as the set of the original vertices; then the other part consists of the facial vertices. Now map $B$ is uniquely restored from $B^\Box$. Namely,

1. in each face $f$ of $B^\Box$ we draw an edge $e$ of $B$ that connects both original vertices $v_1$ and $v_2$ on the 4-boundary of $f$ (if they coincide then a loop, non-shrinkable in $f$, arises);

2. such an edge $e$ goes along the boundary of the disc iff $f$ is incident to a singular facial vertex (in this case $v_1$ and $v_2$ lie on the boundary);

3. moreover, two original boundary vertices are connected with a boundary quasi-edge iff they both are adjacent to a boundary normal facial vertex.

To obtain $B$ it remains to erase all the facial vertices and edges of $B^\Box$.

The resulting construction is indeed a circular map that satisfies all the conditions $C1$–$C5$.

In order to finish constructing the required correspondence, two more questions should be settled: the overall number of edges and a choice of the root.

In the case of planar maps there is a $1 : 2$ correspondence between edges of $B$ and $B^\Box$. To ensure this in our case we adopt the following rules of weighting the edges:

- any internal edge of $B^\Box$ contributes 1;
- any boundary edge of $B^\Box$ with no singular ends contributes $\frac{1}{2}$; and
- any (boundary) edge with a singular end contributes 0.

Accordingly, they contribute $4t, 2\beta$ and $\alpha$ flags. Then, by definition, $B$ has $t = \beta + \beta/2 + \alpha/2$ edges ($t \geqslant 2$) and $4\beta + 2\beta + 2\alpha$ flags. On the other hand, $B^\Box$ has the following parameters:

- $\beta$ faces that correspond to normal edges of $B$ and each of which contributes 8 flags (along the sides and directed inside the face);
- $\sigma$ faces that are incident to singular original vertices and correspond to singular edges of $B$; such a face contributes 4 flags (again from the two non-boundary sides).

Thus, we have here $8\beta + 4\sigma$ flags. Q.E.D.

As to roots, we adopt the rule that the root-flag of $B^\Box$ lies on the 4-gon which contains the root-edge $r$ of $B$, is directed to the same vertex (and, thus, necessarily original!) and $r$ and inside this face. There are two such flags; we choose one to which $r$ is directed (when both maps are drawn simultaneously; this rule selects always an edge of $B^\Box$ possessing flags). And this rule is invertible.

Thus, the correspondence between these rooted maps proves to be bijective. Now we summarize the result.

Let $Q'(n)$ denote the number of all (1-connected) rooted generalized quadrangular dissections of the disc in which:

- some boundary mutually non-adjacent vertices of valency 2 are marked as singular;
- $n$ is equal to the number of internal edges plus half of the number of boundary edges with no singular ends ($n$ is always an integer).
**Lemma.** $A'_0(n/2) = Q'(n)$.

The quadrangulations with $n = 2$ are drawn in Fig. 2.

6.10. In the case of $\Phi^{(b)}$ we obtain the set of all punctured circular maps. And they are reduced similarly to punctured quadrangulations where one internal vertex (original or facial) is marked as axial. Let $Q'_1(n)$ denote the number of such rooted punctured quadrangulations with $n$ weighted edges (as defined above). We obtain likewise

**Lemma.** $A'_{p,2}(n/2l) = Q'_1(n/l), \quad l \geq 3$ odd, $l|n$.

This finishes our reduction for $A(n)$.

6.11. Initial numerical values are given in Table 1. The values of $Q'(n)$ and $Q'_1(n)$ for $n \leq 4$ were found by the exhaustive search; the other values of $Q'(n)$ (in parentheses) were obtained reversely from the known values of $A(n)$ given in [19]. Values for $n = 0$ are conventional.

Note also that for a prime odd $n = p$, each unlabelled generalized quadrangular dissection has $p$, $2p$ or $4p$ rootings unless it is the $2p$-wheel (with $p$ singular vertices) which has only $2$ rootings. Thus,

$$Q'(p) \equiv 2 \pmod{p}, \quad Q'_1(p) \equiv 2 \pmod{p}.$$
6.12. In essence, our quadrangular dissections differ from usual 1-connected quadrangular dissections of the disc only by the availability of singular vertices. But we do not know a way to get rid of them. Moreover, their set is not empty if \( n \) is odd as will now be shown.

Let \( B^{\出入境} \) be a generalized quadrangular dissection of the disc with \( n \) weighted edges (as defined). Let \( m \) be the overall number of boundary vertices (sides), \( s \) be the number of singular vertices (\( s \leq m/2 \)), \( i \) be the number of internal vertices, \( f \) be the number of faces (not including the external face) and \( e \) be the overall number of edges of all the types. \( B^{\出入境} \) contains \( 2s \) singular boundary edges and \( m - 2s \) non-singular boundary edges. Then by definition of the parameter \( n \) we have

\[
\begin{align*}
\text{by definition of the parameter } n \text{ we have } e &= n + (m - 2s)/2 + 2s = n + s + m/2; \text{ moreover, } 2|m. \text{ Now by the Euler formula, } \\
&= e + 1 - (m + i) = n + s + m/2 + 1 - m - i = n + s - m/2 - i + 1. \\
\end{align*}
\]

Counting sides of all the faces we obtain the equality \( m + 4f = 2e \) whence by the previous formulae,

\[
n + s = m + 2i - 2.
\]

In particular, \( n + s \) is even, i.e., \( s \equiv n \pmod{2} \).

Accordingly, for counting \( Q'(n) \) and \( Q'_1(n) \) it seems useful to refine them with parameters \( m = 2k \) and \( s \). Introducing the corresponding numbers \( Q'(n, 2k, s) \) of quadrangular dissections we have (recall that \( e = n + k + s \) and \( i = (n + s)/2 - k + 1 \))

\[
\begin{align*}
Q'(n) &= \sum_{k=1}^{n+1} \sum_{s \geq 0, 2 | (n + s)} Q'(n, 2k, s), \quad Q'_1(n) = \sum_{k=1}^{n} \sum_{s \geq 0, 2 | (n + s)} (\frac{1}{2}(n + s) - k + 1)Q'(n, 2k, s).
\end{align*}
\]

It is evident that \( Q'(n, 2n, n) = 2. \) Moreover, \( f = (n + s)/2. \)

Remark. At present I know only one rather evident though cumbersome way to count \( Q' \) and \( Q'_1 \) in principle. Namely, in addition, one can take into account compositions of \( b = m - s \) ordinary boundary vertices into \( s \) parts (the case of \( s = 0 \) is treated separately) according as they lie on the circle and follow clockwise, starting from a (rooted) vertex. Deletion of a certain singular vertex reduces such a quadrangular map to a similar map or decomposes it into two maps with \( s - 1 \) singular vertices.
This gives a recurrent formula. Furthermore, quadruangulations with no singular vertices can be counted by reducing them to non-separable ones (cf. [8, Section 2.9]). But details go beyond the scope of the present work.

References