On the Use of Degree Theory for Nonlinear Multiparameter Eigenvalue Problems*

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It is shown how existence questions for general multiparameter eigenvalue problems can be treated quite simply using degree theory. The equations to be solved are \( W_n(\lambda)x_n = 0 \neq x_n, \ n = 1, 2, ..., k \), where \( \lambda \in \mathbb{R}^k \) and each \( W_n(\lambda) \) is a self-adjoint linear operator on a Hilbert space \( H_n \). The \( W_n \), which may be unbounded, depend continuously on \( \lambda \) in a suitable sense. A coercivity condition for large \( \| \lambda \| \) is used, and is shown to be equivalent, in the "linear" case, to a standard determinantal definiteness condition.

1. INTRODUCTION

We propose solving a type of eigenvalue problem of the form

\[
W_n(\lambda)u_n = 0, \quad \|u_n\| = 1, \ n = 1, 2, ..., k \tag{1.1}
\]

where, for each \( n \), \( W_n(\lambda) \) is a self-adjoint linear operator on a Hilbert space \( H_n \). Throughout, the symbols \( \lambda \) and \( u \) will denote elements of \( \mathbb{R}^k \) and \( \bigoplus_{n=1}^k H_n \) respectively, where \( u = (u_1, ..., u_k) \) satisfies \( \|u_n\| = 1 \). If such a pair satisfies (1.1) then \( \lambda \) and \( u \) will be called an eigenvalue and eigenvector.

The "linear" case, where

\[
W_n(\lambda) = T_n + \sum_{m=1}^k V_{nm}\lambda_m \tag{1.2}
\]

the \( T_n \) and \( V_{nm} \) being self-adjoint, has been investigated by many authors. In particular, if the \( T_n \) have compact resolvent and the \( V_{nm} \) are bounded, then most of the known applications to differential and difference equations (see e.g., [1]–[3]) are included. Other "singular" cases can be treated by extra limiting devices [8].

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Various methods have been used for the existence of solutions to (1.1) in the linear case (1.2). Even in finite dimensions (see e.g., [2]–[4], [12], [15]) the arguments are in some ways more complicated than those here. Further, the passage from finite to infinite dimensions, as carried out by Pell [15], Faierman [13] and Browne [7], is rather technical. Browne [10] and Sleeman [17] have tackled the problem directly via the spectral theory of several commuting operators induced by (1.2) on $\bigotimes_{n=1}^{k} H_n$. Although such methods are conceptually based on Cramer's rule for eliminating the $\lambda_n$, a good deal of background is needed. Binding and Browne [5] have used a variational approach, based on eliminating the $\lambda_n$ one at a time. Although simpler than previous methods, it still involves a preliminary transformation and ancillary continuous dependence arguments.

The approach here has the advantages of generality, simplicity and little background—we use only elementary properties of the degree (see, e.g., [14, Chap. 6], [16; Chap. 3]). We shall establish existence theorems for (1.1) under a variety of hypotheses. We also give a nonlinear generalization of a result of Atkinson ([3; Chap. 9]) permitting transformations of (1.2) to forms where the $V_{nm}$ have prescribed definiteness properties. It should, of course, be borne in mind that we leave aside several aspects of multiparameter theory, such as completeness, orthogonality and expansion relations for the eigenvectors. It would be of interest to extend such linear theory to our more general hypotheses, by working in $\bigotimes_{n=1}^{k} H_n$ instead of $\bigoplus_{n=1}^{k} H_n$.

Nonlinear problems have been investigated by Browne [9] and recently in several papers by Browne and Sleeman (e.g. [11]). In fact [11] also uses degree theory, but all these works are based on bifurcation theory and are quite different from the treatment here. In particular they assume existence of solutions for the linear case (1.2), nonlinear results being deduced by perturbation techniques.

2. Preliminaries

In this section we give the basic assumptions and some elementary consequences. Elements $\lambda \in \mathbb{R}^k$ and $x = (x_1, \ldots, x_k) \in \bigoplus_{n=1}^{k} H_n$ have norms $(\sum_{n=1}^{k} \lambda_n x_n)^{1/2}$ and $\sum_{n=1}^{k} ||x_n||$ respectively. $E$ will denote the "eigenspace" $\mathbb{R}^k \oplus \bigoplus_{n=1}^{k} H_n$, with $||(\lambda, x)|| = ||\lambda|| + ||x||$. We consider (1.1) as a system of equations on $E$, where each $W_n(\lambda)$ is a (perhaps unbounded) self-adjoint linear operator on a Hilbert space $H_n$.

Since quadratic forms will be used for much of the analysis, we introduce a general notation convention for them now. Let

$$w_n(\lambda, u) = (u_n, W_n(\lambda)u_n), \quad n = 1, 2, \ldots, k$$

(2.1)

where defined and let $w(\lambda, u) = (w_1(\lambda, u), \ldots, w_k(\lambda, u))$. This practice of using
bold face for vectors and vector valued functions in $\mathbb{R}^k$ will be repeated. In order to use a variational form of (1.1) we introduce the following

**Assumption I.**

$$\rho_n^0(\lambda) = \inf\{w_n(\lambda, u) : u_n \in D(W_n(\lambda))\}$$  \hspace{1cm} (2.2)

is attained as a finite minimum, for each $n$ and $\lambda$.

**Lemma 1.** Assuming I, if $\rho^0(\lambda) = 0$ is soluble for some $\lambda$, then (1.1) is soluble.

This is a standard result. Indeed, if $w_n(\lambda, u^0) = 0$ then $w_n(\lambda, u) \geq 0$ shows that $w_n(\lambda, u^s)$ has a minimum at $s = 0$, where

$$u_n^s = u_n^0 + sW_n(\lambda)u_n^0.$$

It readily follows that $W_n(\lambda)u^0 = 0$.

For the purposes of applications it is convenient to place most of the assumptions on differences between the $W_n(\lambda)$. We single out a particular $\lambda^* \in \mathbb{R}^k$ and write

$$T_n = W_n(\lambda^*), \quad W_n(\lambda) = T_n + V_n(\lambda)$$ \hspace{1cm} (2.3)

with evident similarity to (1.2). Corresponding to (2.1) we also introduce $v(\lambda, u)$ for the $V_n(\lambda)$ and $t(u)$ for the $T_n$.

**Assumption II.** For each $\lambda$, $v(\lambda, u)$ is continuous in $u$.

Equivalently, each $V_n(\lambda)$ is a bounded linear operator.

**Assumption III.** $v(\lambda, u)$ is continuous in $\lambda$, uniformly in $u$.

II and III are together equivalent to the $V_n(\lambda)$ being continuous in $\lambda$, in the (uniform) norm topology. Also, II and III are equivalent to $v$ being continuous on the eigenspace $E$, but the separation of these assumptions will prove useful in later sections.

**Lemma 2.** Assuming I and III, $\rho^0(\lambda)$ is continuous in $\lambda$.

Again this is a standard result (even without attainment in (2.2)). Indeed

$$\| \rho^0(\lambda) - \rho^0(\mu) \| \leq \sup_n \| v(\lambda, u) - v(\mu, u) \|.$$ \hspace{1cm} (2.4)

**Assumption IV.** $\| v(\lambda, u) \| \to \infty$ as $\| \lambda \| \to \infty$, uniformly in $u$. 
Let $B(r) = \{ \lambda \in \mathbb{R}^k : \| \lambda \| < r \}$. It follows from IV that, for some $r_0$,

$$\| \lambda \| = r > r_0 \Rightarrow \nu(\lambda, u) \neq 0.$$

Thus the degree $\text{deg}(\nu(\lambda, u), B(r), 0)$ is defined and invariant for large $r$, where $\nu(\lambda, u): \lambda \rightarrow \nu(\lambda, u)$.

**Assumption V.** For some $u^*$, $\text{deg}(\nu(\lambda, u), B(r), 0) \neq 0$ for large $r$.

### 3. The Continuous Case

We now have enough assumptions for the existence theorem. First we introduce the homotopies to be used. Let

$$\nu_n = \inf \{ \nu_n(\lambda, u) : \| u_n - u_n^* \| \leq 2\alpha, 0 \leq \alpha \leq 1 \},$$

$$\nu_n = \inf \{ (\alpha - 1) \nu_n(u) + \nu_n(\lambda, u) : u_n \in D(T_n) \}, 1 \leq \alpha \leq 2. \quad (3.2)$$

**Theorem 1.** Assuming I-V, (1.1) has a solution for which $\nu^0(\lambda) = 0$.

**Proof.** From $I'$ we know that $\text{deg}(\nu, B(r), 0) \neq 0$ for large $r$. We shall show that $\nu_n(\lambda)$ is jointly continuous in $\alpha$ and $\lambda$ and does not vanish for large $\| \lambda \|$. It will then follow that $\text{deg}(\nu^2, B(r), 0) \neq 0$, so Lemma 1 will complete the proof.

We first consider $0 \leq \alpha \leq 1$. Continuity of $\nu_n(\lambda)$ in $\alpha$ follows from II, and continuity in $\lambda$, uniformly in $\alpha$, comes from (2.4). Also, $\nu_n(\lambda) = 0$ for $\| \lambda \| > r_0$ violates IV.

It remains to consider $1 \leq \alpha \leq 2$. Continuity of $\nu$ in $\lambda$, uniformly in $\alpha$, follows as before. We shall now prove continuity in $\alpha$. Fix $n$ and $\lambda$ and note that $\nu_n(\lambda)$ increases with $\alpha$.

First let $\alpha = 1$. Fix $\epsilon > 0$ and consider

$$S = \{ u : \nu_n(\lambda, u) < \nu_n^1(\lambda) + \epsilon \}$$

Obviously $S$ is an open set by virtue of II. Now $T_n$ is self-adjoint so $t_n$ is densely defined and we can find $u_\infty \in S$ with $t_n(u_\infty)$ finite. Then for $1 \leq \beta \leq 1 + \epsilon t_n(u_\infty)$,

$$\nu_n^\beta(\lambda) \leq \nu_n^1(\lambda) + 2\epsilon$$

so $\nu_n^\alpha$ is continuous at $\alpha = 1$.

Second let $\alpha \geq 1$, fix $\epsilon > 0$ and choose $u_\alpha$ so that

$$(\alpha - 1)t_n(u_\alpha) + \nu_n(\lambda, u_\alpha) \leq \nu_n^\alpha(\lambda) + \epsilon.$$
Thus for $\beta > \alpha$, \[ 0 \leq \iota_n^\beta(\lambda) - \iota_n^\alpha(\lambda) \leq (\beta - \alpha)t_n(u_\lambda) + \varepsilon. \]

Since $t_n(u_\lambda)$ is finite, continuity on the right of $\alpha$ is clear. Further \[ (\alpha - 1)t_n(u_\lambda) \leq \iota_n^\beta(\lambda) + \varepsilon - \| V_n(\lambda) \| = 2\gamma, \]
say. Thus if we fix $\beta$ and take $2\gamma > 1 + \beta$, then \[ 0 \leq \iota_n^\beta(\lambda) - \iota_n^\alpha(\lambda) \leq \varepsilon + (\beta - \alpha)\gamma/(\beta - 1), \]
which yields continuity on the left of $\beta$.

Now using IV, suppose $r_1$ is chosen so large that \[ \| \lambda \| > r_1 : \max_n \| v_n(\lambda, u) \| > \| \rho(\lambda^*) \| \]
— see (2.2), (2.3). Observe that $t_n(u) \geq -\| \rho(u) \| \geq -\| \rho(\lambda^*) \|$ for each $n$. Hence \[ \max_n |(\alpha - 1) t_n(u) + v_n(\lambda, u)| > 0 \]
for each $u$ and each $\lambda$ satisfying $\| \lambda \| > r_1$. Therefore $\iota^\gamma(\lambda)$ cannot vanish for large $\lambda$. Q.E.D.

Note that IV prevents the number of solutions from changing at the boundary of $B(r)$ during the homotopy. Bifurcation inside $B(r)$ is quite possible, however, so we make no assertion about the number of solutions.

Cases do arise satisfying I yet with partly continuous spectra—cf. [18; Chap. 1]. Nevertheless, many applications satisfy the following condition.

**Assumption I'.** Each $T_n$ has compact resolvent and is bounded below, i.e., \[ t_n(u) \geq \gamma \]
for some $\gamma \in \mathbb{R}$, and for each $n$ and $u$.

While this assumption explicitly depends on $\lambda^*$ via the $T_n$, it turns out that any $\lambda^*$ will do, as the following shows.

**Lemma 3.** [5; Lemma 1 and Corollary]. I' and II imply that each $W_n(\lambda)$ has compact resolvent and is bounded below.

As a consequence, \[ \rho_n'(\lambda) = \max \{ \min \{ w_n(\lambda, u) : u_n \in \mathcal{D}(W_n(\lambda)), (u_n, y_1) = 0 \} : y_1 \in H_n, 1 \leq I \leq J \} \] (3.3)
exists finite, and indeed is the jth eigenvalue of \( W_n(\lambda) \), counted from \( j = 0 \) according to multiplicity—see [5], [18; Chap. 3]. Consequently, we can improve the existence theorem as follows.

**Corollary 1.** Assuming I' and II-V, for each multiindex \( j \in \mathbb{R}^k \) where each \( j_n \) is a nonnegative integer, there exists a solution \( \lambda^1, u^1 \) to (1.1), satisfying

\[
\rho_n^*(\lambda^1) = 0 \quad n = 1, 2, \ldots, k. \tag{3.4}
\]

The proof is a straightforward extension of that of Theorem 1, replacing infima over \( u_n \) by infima subject to \( (u_n, y_l) = 0, 1 \leq l \leq j_n \), and using (3.3). Again there could be several solutions for each \( j \). On the other hand, (1.1) forces each \( W_n(\lambda) \) to have a zero eigenvalue, so all solutions of (1.1) are characterized by Corollary 1.

Binding and Browne [5; Theorem 2] have also used the equations (2.4) to solve (1.1) in the linear case (1.2), but the proof is based on a specialised elimination process which cannot be applied here.

### 4. The Continuously Differentiable Case

We now strengthen III to

**Assumption III'.** \( v(\lambda, u) \) has a derivative \( v'(\lambda, u) \) with respect to \( \lambda \), and \( v' \) is continuous on \( E \).

This will be the case, for example, if each \( V_n \) is continuously differentiable (in the uniform norm topology). We shall write

\[
\Delta(\lambda, u) = \det v'(\lambda, u), \quad N(\lambda, u) = ||[v'(\lambda, u)]^{-1}||
\]

where the uniform norm is used and \( N \) is infinite if \( v' \) is singular.

Assuming III' we can compute the degree for \( V \) via the integral formula

\[
\deg(v(\lambda, u^*), B(r), 0) = \int_{B(r)} h(\lambda(\lambda, u^*)) \Delta(\lambda, u^*) d\lambda \tag{4.1}
\]

where \( h \) is any continuous nonnegative valued function with compact support on \( B(\theta) \) for small enough \( \theta \), and \( \int_{B(\theta)} h = 1 \) [14; Chap. 6], [16; Chap. 3]. One rather brutal way of ensuring \( V' \) is to make the following

**Assumption V'.** \( N(\lambda, u^*) < \infty \) for all \( \lambda \).

Equivalently, \( \Delta(\lambda, u^*) \neq 0 \) and by continuity in \( \lambda \), \( \Delta(\lambda, u^*) \) has fixed sign.
Thus $V' \Rightarrow V$ trivially from (4.1). Actually IV and $V'$ together imply that $v(, u^*)$ is a homeomorphism of $\mathbb{R}^k$ [14; Theorem 5.3.8]. In this case, then,

$$\text{deg}(v(, u^*), B(r), 0) = \pm 1$$

and $v(\lambda, u^*) = 0$ has exactly one solution. (1.1) may of course still have several solutions—see the remark after Theorem 1.

It turns out that if we take $V'$ a step further, then IV is included as well. More precisely, we introduce

**Assumption IV'.** $N(\lambda, u) \leq g(\|\lambda\|)$ for all $\lambda$ and $u$, where $g: \mathbb{R} \to \mathbb{R}$ is nondecreasing and $\int^\infty dr/g(r)$ diverges.

It is also possible to write IV' in terms of $A(\lambda, u)$, but $\|v'(\lambda, u)\|$ is then involved as well (cf. Lemma 4 below). We shall pursue this for the linear case in the next section.

**Corollary 2.** Assuming I, II, III' and IV', (1.1) has a solution with $p^0(\lambda) = 0$.

**Proof.** Evidently IV' $\Rightarrow$ $V'$ which, as we have seen, implies V. Thus it is enough to demonstrate that IV is satisfied in order to apply Theorem 1.

If $g$ is a (positive) constant function, then [14; Theorem 5.3.10] applies to show that $v(, u)$ is a homeomorphism of $\mathbb{R}^k$ for each $u$. It is a simple extension to show that [14; Theorem 5.3.10] still holds for $g$ as assumed. The crux [14; p. 138] is to show that if

$$q(s) = sc + (1 - s)b, \quad 0 \leq s \leq 1$$

$$q(s) = v(p(s), u), \quad 0 \leq s < a \quad (4.2)$$

then $p(a)$ exists finite—this is Rheinboldt's "continuation property." From the chain rule,

$$p'(s) = [v'(p(s), u)]^{-1} q'(s),$$

so

$$d\|p(s)\|/ds \leq \|p'(s)\| \leq g(\|p(s)\|) \|c - b\|.\]$$

Writing $G(R) = \int_0^R dr/g(r)$, we therefore have

$$G(\|p(s)\|) \leq \|c - b\|, \quad (4.3)$$

whenever $0 \leq s < \alpha$. Thus IV' shows that $\|p(s)\|$ is bounded, and if $s, \uparrow a$ as $j \uparrow \infty$ then

$$\|p(s_j) - p(s_i)\| \leq \int_{s_i}^{s_j} g(\|p(s)\|) ds \|c - b\|,$$
cf. [14; p. 138, equation (2)]. It follows that $p(s_s)$ is Cauchy, so $p(s)$ does exist finite, and $v(s, u)$ is a homeomorphism of $\mathbb{R}^k$.

We complete the proof by using (4.3), which now holds for $0 \leq s \leq 1$. Let $p(0) = 0$ and $p(1) = \lambda$. From (4.3) we have

$$G(||\lambda||) \leq ||v(\lambda, u) - v(0, u)||$$

so

$$||v(\lambda, u)|| \geq G(||\lambda||) - \sup_u ||v(0, u)||$$

and IV is indeed satisfied. Q.E.D.

We might point out that however smooth $v(\lambda, u)$ is in $\lambda$, the functions $v^\alpha$ used for the proof of Theorem 1 are not smooth for $\alpha > 0$ unless the range of $v(\lambda, )$ is a point. This is equivalent to each $V_n(\lambda)$ being a multiple of the identity. Thus in general the degree integral cannot be applied directly, say, to $\rho^0 = v^2$. The same comment holds, of course, for the homeomorphism criteria [14; Theorems 5.3.8, 5.3.10] used above. Note that the homotopy (3.1) essentially "expands" the range of $v(\lambda, )$ outwards from the point $v(\lambda, u*)$.

5. The Linear Case

We shall employ our final assumption for this section.

**Assumption III'**. For each $u$, $v(\lambda, u)$ is linear in $\lambda$.

It follows that

$$v(\lambda, u) = V(u)\lambda \quad (5.1)$$

where $V(u)$ is a $k \times k$ matrix with $(n, m)$th entry $v_n(e_m, u)$, $e_1, \ldots, e_k$ being the coordinate vectors in $\mathbb{R}^k$. Writing $V_{nm} = V_n(e_m)$, we are evidently in the setting of (1.2).

We shall need the following lemma, where we write $||M||_2$ for the spectral norm of a $k \times k$ matrix $M$. Thus $||M||_2$ is the maximum eigenvalue of $M^TM$.

**Lemma 4.** Let $\kappa = k - 1$ and $\delta = \det M > 0$. Then

$$\delta = \min\{||M\lambda||: ||\lambda|| = 1\} = ||M^{-1}||^{-1} \quad \text{if} \quad \kappa = 0,$$

$$\delta||M||_2^\kappa \leq ||M^{-1}||^{-1} \leq (\delta||M||_2)\frac{1}{\kappa} \quad \text{if} \quad \kappa > 0.$$

**Proof.** Let the eigenvalues of $M^TM$ be

$$\mu_1 \leq \mu_2 \leq \cdots \leq \mu_k.$$
The case $\kappa = 0$ is trivial. When $\kappa > 0$,

$$\delta^2 = \prod \mu_j \leq \mu_1 \mu_1^\kappa$$

so

$$\delta^2 \mu_1^{-\kappa} \leq \mu_1 \leq \| M \lambda \|^2.$$ 

This gives the first inequality. The second is similar—for a related result see [6; Lemma 1]. Q.E.D.

We now consider the consequences of III" for our previous assumptions. A simple calculation shows that $v'(\lambda, u) = V(u)$ so $\Delta(\lambda, u)$ and $N(\lambda, u)$ are constant in $\lambda$. Thus IV' reduces to

$$N(\lambda, u) \leq \nu$$

for some $\nu > 0$, for all $u$.

It follows from Lemma 4 with $M = V(u)$ and $\| M^{-1} \| = \nu$ that IV and IV' are each equivalent, assuming III", to the following condition which is standard in the literature.

**Assumption IV".** $\Delta(\lambda, u) \geq \delta$ for some $\delta > 0$, for all $u$.

Thus from Corollary 2 we may conclude the following result for the linear case.

**Corollary 3.** Assuming I, II, III" and IV", (1.1) has a solution satisfying $\rho(\lambda) = 0$.

We point out that corresponding results involving (3.4) hold for Corollaries 2 and 3 under assumption I', In particular. Corollary 3 then leads to an abstract version of Klein's oscillation theorem. Alternative proofs may be found in [5], [10] and [17]—see section 1.

### 6. On Prescribing Definiteness

This final section treats a somewhat different problem in multiparameter theory, but the methods used are close to those of section 3. In the linear case (1.2) there is a useful result which allows the $V_{nm}$ to be taken definite (with prescribed signs) after a suitable linear transformation of the $\lambda$ space. This result is proved in [3; Theorem 9.4.1] for finite dimensions, and in [6; Theorem 1] for infinite dimensions, by means of Borsuk's theorem.

The purpose of this section is to prove a direct nonlinear generalisation of the cited result via degree theory. It should be pointed out that the application of Borsuk's theorem requires $v(-\lambda, u) = -v(\lambda, u)$ for each $\lambda$ and $u$. There is
also a second method of proof in the linear case \([3; \text{Theorem 9.8.1}]\) which depends on the properties of the cones in \(\mathbb{R}^k\) formed by the rows of the \(V'(u)\) as \(u\) varies. In the nonlinear case these cones cannot be constructed so Atkinson's second method does not apply.

We now present the nonlinear result.

**Theorem 2.** Let \(\epsilon_n = \pm 1, \ n = 1, 2, \ldots, k\). Assuming \(II-1'\), there exists \(\lambda \in \mathbb{R}^k\) so that

\[
\epsilon_n V_n(\lambda, u) \geq 1
\]

for each \(n\) and \(u\). In particular, \(\epsilon_n V_n(\lambda)\) is positive definite for each \(n\).

**Proof.** The proof is similar to that of Theorem 1. We define, for \(0 \leq \alpha \leq 1\),

\[
\sigma_n^0(\lambda) = \begin{cases} 
\sup \{v_n(\lambda, u) : \|u_n - u_n^*\| \leq 2\alpha\} & \text{if negative} \\
\inf \{v_n(\lambda, u) : \|u_n - u_n^*\| \leq 2\alpha\} & \text{if positive} \\
0 & \text{otherwise.}
\end{cases}
\]

Then \(\sigma^0(\lambda) = v(\lambda, u^*)\), so \(\deg(\sigma^0, B(r), 0) \neq 0\) for large \(r\) by \(I'\).

Now continuity of \(\sigma^0(\lambda)\) in \(\alpha\) and \(\lambda\) follows as for Theorem 1, while \(\sigma^0(\lambda) = 0\) for large \(\|\lambda\|\) contradicts \(IV\). We therefore obtain \(\deg(\sigma^1, B(r), 0) \neq 0\). Using \(IV\) again, we in fact have \(\sigma^1(\lambda) \neq \alpha e\) for \(0 \leq \alpha \leq 1\) and large \(\|\lambda\|\). Hence

\[
\deg(\sigma^1, B(r), e) \neq 0
\]

for large \(r\). Thus indeed \(\sigma^1(\lambda) = e\) for some \(\lambda \in B(r)\). Q.E.D.

**References**


