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Closed categories, star-autonomy, and monoidal comonads

Craig Pastro ^{*,1}, Ross Street ¹*Department of Mathematics, Macquarie University, New South Wales 2109, Australia*

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Abstract

This paper determines what structure is needed for internal homs in a monoidal category \mathcal{C} to be liftable to the category \mathcal{C}^G of Eilenberg–Moore coalgebras for a monoidal comonad G on \mathcal{C} . We apply this to lift $*$ -autonomy with the view to recasting the definition of quantum groupoid.

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Introduction

It was recognized by Szlachányi [Sz03] that Takeuchi's \times_R -bialgebras (bialgebroids) could be described as opmonoidal monads. Brzeziński and Militaru [BM02] developed this further and dualized the notion. The dual concept was called quantum category in [DS04] and was expressed in terms of a monoidal comonad; however the main point of the paper was to obtain a definition of quantum groupoid which involved $*$ -autonomy in the sense of Barr [B95]. This $*$ -autonomy amounts to an antipode in the case of a bialgebra (which is a “one object” quantum category). The paper [DS04] expressed the generalized antipode as a structure on a generating

* Corresponding author.

E-mail addresses: craig@ics.mq.edu.au (C. Pastro), street@ics.mq.edu.au (R. Street).

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monoidal adjunction (“basic data”) for the comonad, rather than giving this antipode in terms of the monoidal comonad itself. Motivation for the present paper was to clarify the latter possibility.

The problem leads to one that can be stated for monads T on ordinary monoidal categories \mathcal{A} . It was pointed out in [M02] that the category \mathcal{A}^T of Eilenberg–Moore algebras for an opmonoidal monad T becomes monoidal in such a way that the underlying functor $U : \mathcal{A}^T \rightarrow \mathcal{A}$ becomes strict monoidal. We ask when internal homs in \mathcal{A} can be lifted to \mathcal{A}^T . More specifically, we ask under what extra structure on T does the Eilenberg–Moore category \mathcal{A}^T become $*$ -autonomous if \mathcal{A} is.

In the meantime, the paper [BV06] came to our notice, solving the autonomous case. An autonomous category in the sense of [JS91] (also, well before that, called “compact” and “rigid” in the symmetric case) admits a left and right dual for each object. A common generalization of antipode for a bialgebra and autonomy for a monoidal category was obtained in [DMS03] and called “dualization.” The concept of antipode ν for an opmonoidal monad T on an autonomous monoidal category \mathcal{A} is defined in [BV06] and the pair (T, ν) is there called a “Hopf monad.” Autonomy is a special case of $*$ -autonomy so [BV06] answered our questions in an important special case.

Our present paper answers the question of lifting $*$ -autonomy. Our motivation from quantum groupoids causes us to write in terms of a monoidal comonad G on a monoidal category \mathcal{C} rather than an opmonoidal monad T . Since we are interested in abstracting our work to monoidal comonads in monoidal bicategories, this duality is not a serious point of difference. There are some new subtleties required in the non-autonomous case arising from the lack of unit and counit morphisms involved with duals in \mathcal{C} ; we must be content with the coevaluation and evaluation morphisms associated with the weaker duality of $*$ -autonomy.

To support our claim (following [DS04]) that quantum groupoids lead to $*$ -autonomous monoidal comonads, we should explain how an ordinary groupoid \mathbf{A} with finite homsets $\mathbf{A}(a, b)$ leads to such a comonad on some non-braided $*$ -autonomous monoidal category. Let \mathcal{V} be a complete and cocomplete $*$ -autonomous monoidal category; we have in mind the monoidal category obtained from the category of (not necessarily finite dimensional) vector spaces by applying the Chu construction [B95]. Let A denote the set of morphisms and let C denote the set of objects of the groupoid \mathbf{A} . Let \mathcal{A} be the monoidal category of A -indexed families of objects of \mathcal{V} with tensor product defined by

$$(V \otimes W)_\gamma = \sum_{\beta \circ \alpha = \gamma \in \mathbf{A}} V_\alpha \otimes W_\beta.$$

It follows from unpublished work of Brian Day (on $*$ -autonomous promonoidal categories) that \mathcal{A} is $*$ -autonomous. A special case of this construction is when \mathbf{A} is chaotic: that is, where each homset is a singleton and so $A = C \times C$. So we also have the $*$ -autonomous monoidal category \mathcal{C} of $C \times C$ -indexed families of objects of \mathcal{V} . There is a $*$ -autonomous monoidal comonad G on \mathcal{C} which is defined on objects by $G(V)_{(a,b)} = \mathbf{A}(a, b) \cdot V_{(a,b)}$, where $S \cdot X$ (for a set S) denotes the coproduct of S copies of X . Furthermore, \mathcal{A} is the category of Eilenberg–Moore algebras for the comonad G .

In Section 1 we review closed and $*$ -autonomous categories from the point of view of what we are calling “raisers.” Section 2 reviews monoidal comonads and describes what is required to lift a raiser from a category \mathcal{C} to the category \mathcal{C}^G of Eilenberg–Moore G -coalgebras for a monoidal comonad G . In Section 3 we define what it means for a monoidal comonad G to be (left) $*$ -autonomous and prove the main result of our paper, viz., that \mathcal{C}^G is (left) $*$ -autonomous

if G is. Section 4 starts from a monoidal adjunction and investigates what is required on the adjunction to reproduce the results of Section 3 for the induced comonad. In Section 5 we show that a Hopf algebra in a braided $*$ -autonomous category gives an example of a $*$ -autonomous comonad.

1. Internal homs and raisers

Let D be an object of a monoidal category \mathcal{C} . A *left internal hom* for objects B and D is a representing object ${}^B D$ (or $[B, D]_l$) for the functor

$$\mathcal{C}(- \otimes B, D) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}.$$

This means that the object ${}^B D$ comes equipped with an isomorphism

$$\varpi_{A,B} : \mathcal{C}(A, {}^B D) \xrightarrow{\cong} \mathcal{C}(A \otimes B, D)$$

which is natural in $A \in \mathcal{C}$. By taking $A = {}^B D$ and evaluating at the identity, we obtain an *evaluation morphism*

$$e_B : {}^B D \otimes B \rightarrow D.$$

The Yoneda lemma states that natural transformations out of a representable functor are in bijection with their values at the identity morphism of the representing object. This means that $\varpi_{A,B}$ is recaptured as the composite

$$\mathcal{C}(A, {}^B D) \xrightarrow{- \otimes B} \mathcal{C}(A \otimes B, {}^B D \otimes B) \xrightarrow{\mathcal{C}(1, e_B)} \mathcal{C}(A \otimes B, D).$$

For $B = I$, the unit for \otimes on \mathcal{C} , we always have the choice ${}^I D = D$ with $e_I : D \otimes I \rightarrow D$ equal to the right unit isomorphism.

Our object D is called a *left raiser* when there is a choice of ${}^B D$ for all $B \in \mathcal{C}$. Again by Yoneda, we obtain a unique functor

$$S = {}^{-}D : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

defined on objects by $S B = {}^B D$ and such that $\varpi_{A,B}$ becomes natural in $B \in \mathcal{C}$. This last is equivalent to saying that e_B is natural in B in the sense of Eilenberg and Kelly [EK66]. We can easily modify the tensor product to make the unit I strict, so we can ensure that

$$S I = D, \quad \varpi_{A,I} = 1_{\mathcal{C}(A,D)}, \quad \text{and} \quad e_I = 1_D.$$

The composite natural isomorphism

$$\begin{aligned} \mathcal{C}(A, S(B \otimes C)) &\xrightarrow{\varpi_{A,B \otimes C}} \mathcal{C}(A \otimes (B \otimes C), D) \\ &\xrightarrow{\cong} \mathcal{C}((A \otimes B) \otimes C, D) \xrightarrow{\varpi_{A \otimes B, C}^{-1}} \mathcal{C}(A \otimes B, S C) \end{aligned}$$

will be denoted by

$$\omega_{A,B,C} : \mathcal{C}(A, S(B \otimes C)) \xrightarrow{\cong} \mathcal{C}(A \otimes B, SC).$$

It follows that, if D is a left raiser, so too is $SC = {}^C D$ with ${}^B SC = S(B \otimes C)$. Note also that $\omega_{A,I,C} = 1_{\mathcal{C}(A,SC)}$ and $\omega_{A,B,I} = \varpi_{A,B}$.

Assume that D is a left raiser for the remainder of this section.

By taking $A = S(B \otimes C)$ and evaluating at the identity, the isomorphism $\omega_{A,B,C}$ defines a morphism

$$e_{B,C} : S(B \otimes C) \otimes B \rightarrow SC.$$

By Yoneda, $\omega_{A,B,C}$ is recovered as the composite

$$\mathcal{C}(A, S(B \otimes C)) \xrightarrow{-\otimes B} \mathcal{C}(A \otimes B, S(B \otimes C) \otimes B) \xrightarrow{\mathcal{C}(1, e_{B,C})} \mathcal{C}(A \otimes B, SC).$$

In particular, $e_{I,C} = 1_{SC}$. From the definition of ω in terms of ϖ and ϖ^{-1} , we obtain the commutativity of the triangle²

$$\begin{array}{ccc} S(B \otimes C) \otimes B \otimes C & \xrightarrow{e_{B,C} \otimes 1} & SC \otimes C \\ & \searrow e_{B \otimes C} & \swarrow e_C \\ & & D. \end{array}$$

In particular, $e_{B,I} = e_B : SB \otimes B \rightarrow D$.

We define a natural isomorphism

$$\rho_{A,B} = \omega_{I,A,B}^{-1} : \mathcal{C}(A, SB) \xrightarrow{\cong} \mathcal{C}(I, S(A \otimes B)).$$

Taking $A = SB$ and evaluating at the identity, we obtain a morphism

$$n_B : I \rightarrow S(SB \otimes B)$$

natural in B . By Yoneda, $\rho_{A,B}$ is the composite

$$\mathcal{C}(A, SB) \xrightarrow{S(-\otimes B)} \mathcal{C}(S(SB \otimes B), S(A \otimes B)) \xrightarrow{\mathcal{C}(n_B, 1)} \mathcal{C}(I, S(A \otimes B)).$$

Using the formula for $\omega_{I,A,B}$ in terms of $e_{A,B}$, we obtain the commutativity of the triangle

$$\begin{array}{ccc} SB & \xrightarrow{n_B \otimes 1} & S(SB \otimes B) \otimes SB \\ & \searrow 1_{SB} & \swarrow e_{SB,B} \\ & & SB. \end{array}$$

² We are now writing as if \mathcal{C} were strict monoidal, however this is not necessary.

Proposition 1.1. *The following triangle commutes.*

$$\begin{array}{ccc}
 \mathcal{C}(I, S(A \otimes B \otimes C)) & \xrightarrow{\omega_{I,A,B \otimes C}} & \mathcal{C}(A, S(B \otimes C)) \\
 \searrow \omega_{I,A \otimes B,C} & & \swarrow \omega_{A,B,C} \\
 & \mathcal{C}(A \otimes B, SC) &
 \end{array}$$

Proof. This is verified by the following calculation.

$$\begin{aligned}
 \omega_{A,B,C} \omega_{I,A,B \otimes C} &= \varpi_{A \otimes B,C}^{-1} \varpi_{A,B \otimes C} \varpi_{A,B \otimes C}^{-1} \varpi_{I,A \otimes B \otimes C} \\
 &= \varpi_{A \otimes B,C}^{-1} \varpi_{I,A \otimes B \otimes C} \\
 &= \omega_{I,A \otimes B,C} \quad \square
 \end{aligned}$$

Corollary 1.2. *The following triangles commute.*

(i)

$$\begin{array}{ccc}
 \mathcal{C}(A, S(B \otimes C)) & \xrightarrow{\omega_{A,B,C}} & \mathcal{C}(A \otimes B, SC) \\
 \searrow \rho_{A,B \otimes C} & & \swarrow \rho_{A \otimes B,C} \\
 & \mathcal{C}(I, S(A \otimes B \otimes C)) &
 \end{array}$$

(ii)

$$\begin{array}{ccc}
 I & \xrightarrow{n_B} & S(SB \otimes B) \\
 \searrow n_{A \otimes B} & & \swarrow S(e_{A,B}) \\
 & S(S(A \otimes B) \otimes A \otimes B) &
 \end{array}$$

(iii)

$$\begin{array}{ccc}
 I & \xrightarrow{n_I} & SSI \\
 \searrow n_A & & \swarrow S(e_A) \\
 & S(SA \otimes A) &
 \end{array}$$

Proposition 1.3. *The inverse of $\varpi_{I,B}$ is the composite*

$$\mathcal{C}(B, SI) \xrightarrow{S} \mathcal{C}(SSI, SB) \xrightarrow{\mathcal{C}(n_I, 1)} \mathcal{C}(I, SB).$$

Proof. $\varpi_{I,B} = \omega_{I,B,I}$ has inverse $\rho_{B,I}$ and this composite is the formula for $\rho_{B,I}$ in terms of n_I . \square

Corollary 1.4.

(i) *The inverse of $\varpi_{A,B}$ is the composite*

$$\begin{aligned} \mathcal{C}(A \otimes B, D) &\xrightarrow{S} \mathcal{C}(SD, S(A \otimes B)) \xrightarrow{\mathcal{C}(n_I, 1)} \mathcal{C}(I, S(A \otimes B)) \\ &\xrightarrow{-\otimes A} \mathcal{C}(A, S(A \otimes B) \otimes A) \xrightarrow{\mathcal{C}(1, e_{A,B})} \mathcal{C}(A, SB). \end{aligned}$$

(ii) A left inverse for $S : \mathcal{C}(B, SI) \rightarrow \mathcal{C}(SSI, SB)$ is the composite

$$\mathcal{C}(SSI, SB) \xrightarrow{\mathcal{C}(n_I, 1)} \mathcal{C}(I, SB) \xrightarrow{\varpi_{I,B}} \mathcal{C}(B, SI).$$

Proof. (i) From Proposition 1.1, we have

$$\begin{aligned} \varpi_{I, A \otimes B} &= \omega_{I, A \otimes B, I} \\ &= \omega_{A, B, I} \circ \omega_{I, A, B} \\ &= \varpi_{A, B} \circ \omega_{I, A, B}. \end{aligned}$$

So the result follows from Proposition 1.3 and the formula for $\omega_{I, A, B}$ in terms of $e_{A, B}$.

(ii) This is a reinterpretation of Proposition 1.3. \square

We also introduce the natural family of morphisms

$$\pi_{A, B, C} = (\mathcal{C}(A \otimes B, C) \xrightarrow{S} \mathcal{C}(SC, S(A \otimes B)) \xrightarrow{\omega_{SC, A, B}} \mathcal{C}(SC \otimes A, SB)).$$

Taking $C = A \otimes B$ and evaluating at the identity, we obtain the natural transformation

$$e_{A, B} : S(A \otimes B) \otimes A \rightarrow SB.$$

By Yoneda, it follows that $\pi_{A, B, C}$ is the composite

$$\mathcal{C}(A \otimes B, C) \xrightarrow{S(-) \otimes A} \mathcal{C}(SC \otimes A, S(A \otimes B) \otimes A) \xrightarrow{\mathcal{C}(1, e_{A, B})} \mathcal{C}(SC \otimes A, SB).$$

Proposition 1.5. *The natural transformation π is invertible if and only if S is fully faithful.*

Proof. If S is fully faithful then each $\pi_{A, B, C}$ is invertible (from the definition, using invertibility of $\omega_{SC, A, B}$). Conversely, if π is invertible, we may take $A = I$ in the definition of $\pi_{A, B, C}$ to obtain $S : \mathcal{C}(B, C) \rightarrow \mathcal{C}(SC, SB)$ which is consequently invertible. So S is fully faithful. \square

A right internal hom for objects A and E of \mathcal{C} is a representing object E^A (or $[A, E]_r$) for the functor $\mathcal{C}(A \otimes -, E) : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.

Corollary 1.6. *If S is fully faithful then $(SB)^{SC} \cong {}^B C$.*

Proof. The representability of $\mathcal{C}(SC \otimes -, SB)$ by ${}^B C$ is guaranteed by the invertibility of π . \square

An object E is called a right raiser when there exists a choice of E^A for all $A \in \mathcal{C}$.

Proposition 1.7. *A left raiser D is a right raiser if and only if the functor*

$$S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$$

has a left adjoint $S' : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$.

Proof. To say S has a left adjoint means that, for each object A , there is an object $S'A$ and an isomorphism $\mathcal{C}(A, SB) \cong \mathcal{C}(B, S'A)$, natural in B . However, we have the natural isomorphism $\mathcal{C}(A, SB) \cong \mathcal{C}(A \otimes B, D)$, and therefore $S'A \cong D^A$. \square

Notice that the existence of a family of “commutativity” isomorphisms $c_{A,B} : A \otimes B \cong B \otimes A$ in \mathcal{C} , which only need to be natural in one of the indices A or B , implies that every left raiser D is automatically also a right raiser; moreover, $D^A \cong {}^A D$. This is the case when \mathcal{C} is braided, or, *a fortiori*, symmetric.

We call D a *raiser* when it is both a left and right raiser. In this case, the unit and counit for the adjunction $S' \dashv S$ are natural families of morphisms $\alpha_A : A \rightarrow SS'A$ and $\beta_B : B \rightarrow S'SB$ in \mathcal{C} . (The apparent wrong direction of the counit β is explained by the contravariantness of S ; Peter Freyd has called this situation “a contravariant adjunction on the right.”)

A monoidal category is *left closed* when every object is a left raiser. It is *closed* when every object is a raiser.

Following Chapter 12 of [S07] we call the object D *left dualizing* when it is a raiser and each $\alpha_A : A \rightarrow SS'A$ is invertible. By Proposition 1.7, this is equivalent to requiring D to be a left raiser for which S has a fully faithful left adjoint. We call D *dualizing* when it is a left raiser and S is an equivalence. Since an equivalence has a fully faithful left adjoint, it follows that D is also a right raiser.

A monoidal category is *left *-autonomous* when it is equipped with a left dualizing object. It is **-autonomous* [B95] when it is equipped with a dualizing object.

Each left *-autonomous category is left closed since

$${}^B A \cong {}^B(SS'A) \cong S(B \otimes S'A).$$

Each *-autonomous category is closed since it is left *-autonomous and so left closed, and (by looking at \mathcal{C} with the reversed tensor product) has right internal hom defined by

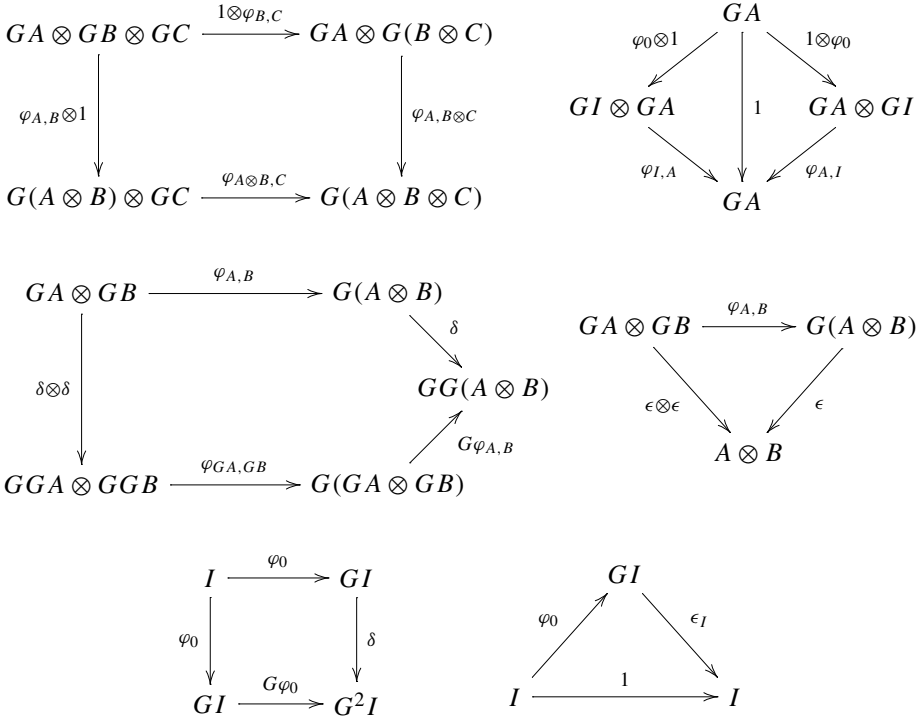
$$B^C \cong S'(SB \otimes C).$$

2. Monoidal comonads

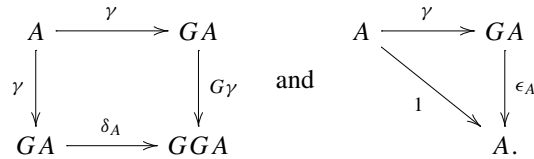
A monoidal comonad on a monoidal category \mathcal{C} consists of a comonad $G = (G, \delta, \epsilon)$ on \mathcal{C} such that $G : \mathcal{C} \rightarrow \mathcal{C}$ is a monoidal functor and $\delta : G \rightarrow GG$ and $\epsilon : G \rightarrow 1_{\mathcal{C}}$ are monoidal natural transformations. So, apart from the comonad axioms, we have a natural transformation

$$\varphi_{A,B} : GA \otimes GB \rightarrow G(A \otimes B)$$

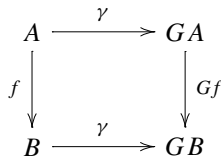
and a morphism $\varphi_0 : I \rightarrow GI$ satisfying the following conditions (where we continue to write as if \mathcal{C} were strict monoidal).



Let \mathcal{C}^G denote the category of Eilenberg–Moore coalgebras for the comonad G . Objects are pairs $(A, \gamma : A \rightarrow GA)$, called G -coalgebras, satisfying



Morphisms $f : (A, \gamma) \rightarrow (B, \gamma)$ in \mathcal{C}^G are morphisms $f : A \rightarrow B$ in \mathcal{C} such that the square



commutes.

We make a note of the following fact:

Proposition 2.1. *If $G : \mathcal{C} \rightarrow \mathcal{C}$ is a comonad with a left adjoint $T : \mathcal{C} \rightarrow \mathcal{C}$, then T becomes a monad and $\mathcal{C}^G \cong \mathcal{C}^T$. Furthermore, if G is a monoidal comonad then T is an opmonoidal monad.*

It is well known [M02] that, if G is a monoidal comonad, then \mathcal{C}^G becomes monoidal in such a way that the underlying functor $U : \mathcal{C}^G \rightarrow \mathcal{C}$ becomes strict monoidal. The tensor product for \mathcal{C}^G is defined by

$$(A, \gamma) \otimes (B, \gamma) = (A \otimes B, A \otimes B \xrightarrow{\gamma \otimes \gamma} GA \otimes GB \xrightarrow{\varphi_{A,B}} G(A \otimes B))$$

and the unit object is (I, φ_0) .

In the dual setting of opmonoidal monads, the paper of A. Bruguières and A. Virelizier [BV06] provides the structure on the monad in order for left (or right) autonomy of \mathcal{C} to lift to the category of Eilenberg–Moore algebras. Here we are interested in lifting $*$ -autonomy from \mathcal{C} to \mathcal{C}^G . We begin with structure weaker than $*$ -autonomy.

Assume we merely have a functor $S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ that we would like to lift to \mathcal{C}^G .

$$\begin{array}{ccc} (\mathcal{C}^G)^{\text{op}} & \xrightarrow{\quad S \quad} & \mathcal{C}^G \\ U^{\text{op}} \downarrow & & \downarrow U \\ \mathcal{C}^{\text{op}} & \xrightarrow{\quad S \quad} & \mathcal{C} \end{array}$$

By [S72], we require a G -coaction on SU^{op} ; that is, a natural transformation

$$\hat{\nu} : SU^{\text{op}} \rightarrow GSU^{\text{op}}$$

satisfying two conditions. Since U has a right adjoint R defined by $RA = (GA, \delta_A)$, such $\hat{\nu}$ are in bijection with natural transformations

$$\nu : S \rightarrow GSG^{\text{op}}$$

(where we use the fact that $G = UR$) satisfying:

$$\begin{array}{ccc} & GSG & \\ \nu \nearrow & & \searrow \epsilon_{SG} \\ S & \xrightarrow{S\epsilon} & SG, \end{array} \tag{Axiom 1}$$

$$\begin{array}{ccc} S & \xrightarrow{\nu} & GSG \\ \downarrow \nu & & \searrow \delta_{SG} \\ GSG & \xrightarrow{G\nu_G} & GGSGG. \\ & & \nearrow GG\delta \end{array} \tag{Axiom 2}$$

Proposition 2.2. *If $S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ is a functor and $v : S \rightarrow GSG$ is a natural transformation satisfying Axioms 1 and 2 then a functor $\bar{S} : (\mathcal{C}^G)^{\text{op}} \rightarrow \mathcal{C}^G$ is defined by*

$$\bar{S}(A, \gamma) = (SA, GS\gamma \circ v_A), \quad \bar{S}f = Sf.$$

Now suppose we also have a natural transformation

$$\omega_{A,B,C} : \mathcal{C}(A, S(B \otimes C)) \rightarrow \mathcal{C}(A \otimes B, SC).$$

By Yoneda, such natural transformations are in bijection with natural transformations

$$e_{B,C} : S(B \otimes C) \otimes B \rightarrow SC.$$

The bijection is determined by $e_{B,C} = \omega_{S(B \otimes C), B, C}(1_{S(B \otimes C)})$ and $\omega_{A,B,C}$ is the composite

$$\mathcal{C}(A, S(B \otimes C)) \xrightarrow{- \otimes B} \mathcal{C}(A \otimes B, S(B \otimes C) \otimes B) \xrightarrow{\mathcal{C}(1, e_{B,C})} \mathcal{C}(A \otimes B, SC).$$

As we shall see, the condition that e is a G -coalgebra morphism is encapsulated in the following axiom.

$$\begin{array}{ccc}
 S(A \otimes B) \otimes GA & \xrightarrow{1 \otimes \epsilon_A} & S(A \otimes B) \otimes A & \xrightarrow{e_{A,B}} & SB \\
 \downarrow v_{A \otimes B} \otimes \delta_A & & & & \downarrow v_B \\
 GSG(A \otimes B) \otimes GGA & & & & GSGB \quad \text{(Axiom 3)} \\
 \downarrow \varphi_{SG(A \otimes B), GA} & & & & \uparrow Ge_{GA, GB} \\
 G(SG(A \otimes B) \otimes GA) & \xrightarrow{G(S\varphi_{A,B} \otimes 1)} & G(S(GA \otimes GB) \otimes GA) & &
 \end{array}$$

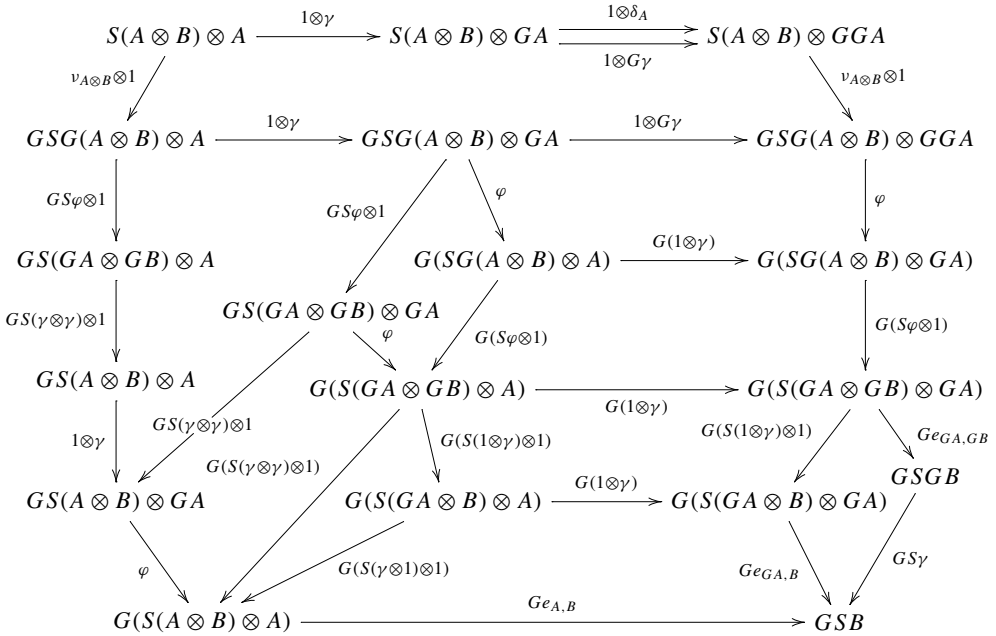
Proposition 2.3. *Assuming Axioms 1, 2, and 3, the morphism $e_{A,B}$ becomes a G -coalgebra morphism*

$$e_{A,B} : \bar{S}((A, \gamma) \otimes (B, \gamma)) \otimes (A, \gamma) \rightarrow \bar{S}(B, \gamma)$$

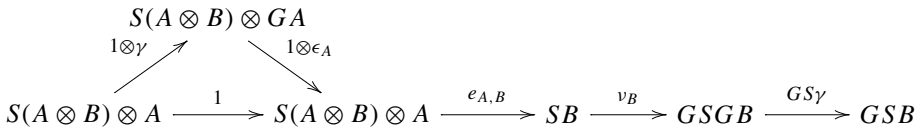
for G -coalgebras (A, γ) and (B, γ) .

Conversely, if $e_{X,Y}$ is a G -coalgebra morphism when $X = (GA, \delta_A)$ and $Y = (GB, \delta_B)$ are cofree G -coalgebras, then Axiom 3 holds.

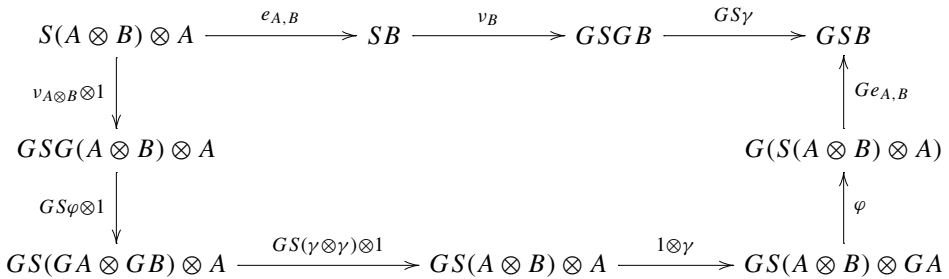
Proof. The following diagram commutes.



By Axiom 3, the top route around this diagram is

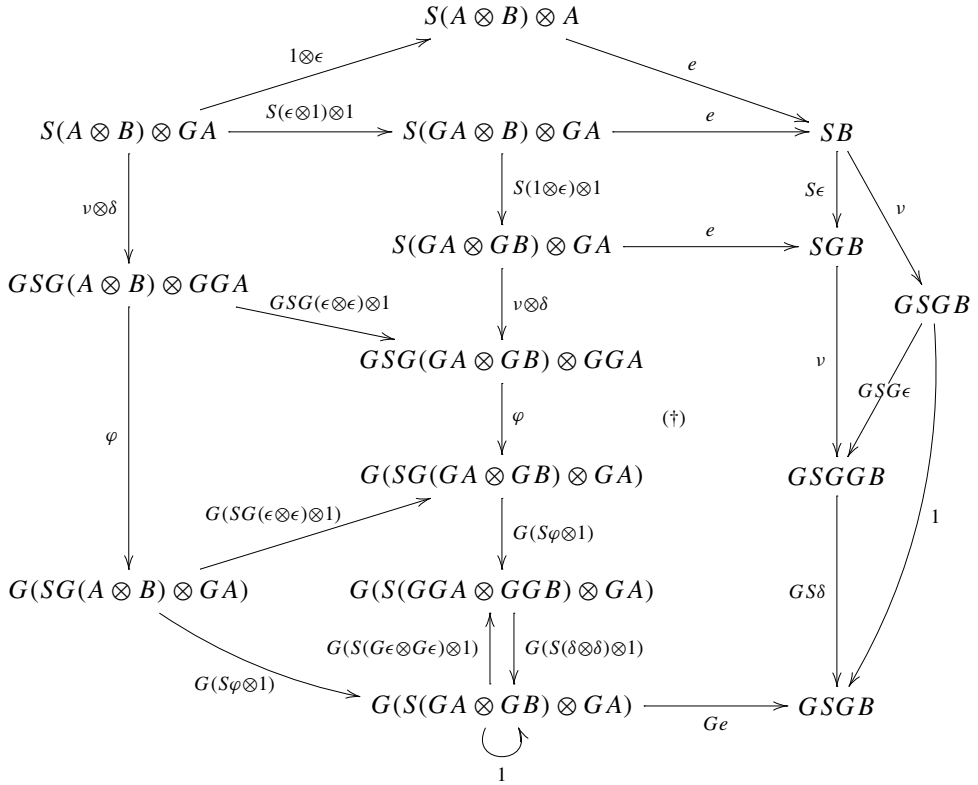


and, therefore, the following diagram commutes



which is precisely the condition for $e_{A,B}$ to be a G -coalgebra morphism.

To prove the converse statement we observe that the diagram



commutes. The outside of the diagram is Axiom 3 and the region labeled by (†) exactly expresses that $e_{GA,GB}$ is a G -coalgebra morphism. \square

Corollary 2.4. *If A , B , and C are G -coalgebras then there is a natural transformation $\omega_{A,B,C}$ such that the following square commutes.*

$$\begin{array}{ccc}
 \mathcal{C}^G(A, \bar{S}(B \otimes C)) & \xrightarrow{\omega_{A,B,C}} & \mathcal{C}^G(A \otimes B, \bar{S}C) \\
 \downarrow & & \downarrow \\
 \mathcal{C}(UA, S(UB \otimes UC)) & \xrightarrow{\omega_{UA,UB,UC}} & \mathcal{C}(UA \otimes UB, SUC)
 \end{array}$$

The condition that a morphism

$$n_I : I \rightarrow SSI$$

should be a G -coalgebra morphism is given by

$$\begin{array}{ccccc}
 I & \xrightarrow{\varphi_0} & GI & \xrightarrow{Gn_I} & GSSI \\
 n_I \downarrow & & & & \uparrow GSv_I \\
 SSI & \xrightarrow{v_{SI}} & GSGSI & \xrightarrow{GSGS\varphi_0} & GSGSGI.
 \end{array} \tag{Axiom 4}$$

Theorem 2.5. Suppose D is a left raiser in the monoidal category \mathcal{C} and define S , e , and n as in Section 1. Suppose (G, δ, ϵ) is a monoidal comonad on \mathcal{C} . If $v : S \rightarrow GSG$ is a natural transformation satisfying Axioms 1, 2, 3, and 4 then $(D, GS\varphi_0 \circ v_I)$ is a left raiser in \mathcal{C}^G .

Proof. By Proposition 2.3 and the fact that $e_B = e_{B,I}$, we have that $e_B : \bar{S}(B, \gamma) \otimes (B, \gamma) \rightarrow \bar{S}(I, \varphi_0)$ is a G -coalgebra morphism. By Axiom 4 and Corollary 1.2(iii), we have that $n_A : (I, \varphi_0) \rightarrow \bar{S}(\bar{S}(A, \gamma) \otimes (A, \gamma))$ is a G -coalgebra morphism. From the formula for $\rho_{A,B}$ in terms of n , we see that $\rho_{A,B}$ restricts as follows:

$$\begin{array}{ccc}
 \mathcal{C}^G(A, \bar{S}B) & \xrightarrow{\rho_{A,B}} & \mathcal{C}^G(I, \bar{S}(A \otimes B)) \\
 \downarrow & & \downarrow \\
 \mathcal{C}(UA, SUB) & \xrightarrow{\rho_{UA,UB}} & \mathcal{C}(I, S(UA \otimes UB)).
 \end{array}$$

Since $\rho_{A,B} = \omega_{I,A,B}^{-1}$, it follows from Corollary 1.2(i) that the component $\omega_{I,A,B} : \mathcal{C}^G(I, \bar{S}(A \otimes B)) \rightarrow \mathcal{C}^G(A, \bar{S}B)$ is invertible. By Proposition 1.1, it follows that

$$\omega_{A,B,C} : \mathcal{C}^G(A, \bar{S}(B \otimes C)) \rightarrow \mathcal{C}^G(A \otimes B, \bar{S}C)$$

is invertible. Taking $C = (I, \varphi_0)$, we have that $\bar{S}(I, \varphi_0) = (D, GS\varphi_0 \circ v_I)$ is a left raiser in \mathcal{C}^G , as required. \square

In other words, we have

$$\begin{array}{ccc}
 \mathcal{C}^G(A \otimes B, C) & \xrightarrow{\pi_{A,B,C}} & \mathcal{C}^G(\bar{S}C \otimes A, \bar{S}B) \\
 \downarrow & & \downarrow \\
 \mathcal{C}(UA \otimes UB, UC) & \xrightarrow{\pi_{UA,UB,UC}} & \mathcal{C}(SUC \otimes UA, SUB)
 \end{array}$$

for $A, B, C \in \mathcal{C}^G$. It follows that if the π for \mathcal{C} is injective, then so is the π for \mathcal{C}^G .

3. Star-autonomous monoidal comonads

Suppose G is a monoidal comonad on the monoidal category \mathcal{C} . Suppose $S : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ has a left adjoint $S' : \mathcal{C} \rightarrow \mathcal{C}^{\text{op}}$ with unit $\alpha : 1 \rightarrow SS'$ and counit $\beta : 1 \rightarrow S'S$. Suppose we have $\nu : S \rightarrow GSG$ and $\nu' : S' \rightarrow GS'G$ each satisfying Axioms 1 and 2 so that we obtain liftings

$$\bar{S} : (\mathcal{C}^G)^{\text{op}} \rightarrow \mathcal{C}^G \quad \text{and} \quad \bar{S}' : \mathcal{C}^G \rightarrow (\mathcal{C}^G)^{\text{op}}.$$

Consider the following conditions:

$$\begin{array}{ccc}
 G & \xrightarrow{G\alpha} & GSS' \\
 \alpha_G \downarrow & & \uparrow GS\nu' \\
 SS'G & \xrightarrow{\nu_{S'G}} & GSGS'G,
 \end{array}
 \tag{Axiom 5}$$

$$\begin{array}{ccc}
 G & \xrightarrow{G\beta} & GS'S \\
 \beta_G \downarrow & & \uparrow GS'\nu \\
 S'SG & \xrightarrow{\nu'_{S'G}} & GS'GSG.
 \end{array}
 \tag{Axiom 6}$$

Proposition 3.1. *The unit $\alpha_A : (A, \gamma) \rightarrow \bar{S}\bar{S}'(A, \gamma)$ is a G -coalgebra morphism for each G -coalgebra $\gamma : A \rightarrow GA$ if and only if Axiom 5 holds. The counit $\beta_A : (A, \gamma) \rightarrow \bar{S}'\bar{S}(A, \gamma)$ is a G -coalgebra morphism for each G -coalgebra $\gamma : A \rightarrow GA$ if and only if Axiom 6 holds. Consequently, if Axioms 5 and 6 hold, then \bar{S}' is a left adjoint for \bar{S} .*

Proof. The outside of the following diagram expresses that $\alpha_A : (A, \gamma) \rightarrow \bar{S}\bar{S}'(A, \gamma)$ is a G -coalgebra morphism.

$$\begin{array}{ccccc}
 A & \xrightarrow{\gamma} & GA & & \\
 \alpha_A \downarrow & & \alpha_{GA} \swarrow & & G\alpha_A \searrow \\
 SS'A & \xrightarrow{SS'\gamma} & SS'GA & (\dagger) & GSS'A \\
 \nu_{S'A} \downarrow & & \nu_{S'GA} \swarrow & & GS\nu'_A \searrow \\
 GSGS'A & \xrightarrow{GSGS'\gamma} & GSGS'GA & &
 \end{array}$$

The two unlabeled regions commute by naturality and the region labeled by (\dagger) is simply Axiom 5. Conversely, for all objects B of \mathcal{C} , taking $A = GB$ and $\gamma = \delta_B$ in the G -coalgebra condition, we obtain the commutativity of the region labeled by (\ddagger) in the following diagram.

$$\begin{array}{ccccc}
 GB & \xrightarrow{\delta_B} & GGB & \xrightarrow{G\epsilon_B} & GB \\
 \alpha_{GB} \downarrow & & \downarrow G\alpha_{GB} & & \downarrow G\alpha_B \\
 SS'GB & (\ddagger) & GSS'GB & \xrightarrow{GSS'\epsilon_B} & GSS'B \\
 \downarrow \nu_{S'GB} & & \uparrow GS\nu'_{GB} & & \uparrow GS\nu'_B \\
 GSGS'GB & \xrightarrow{GSGS'\delta_B} & GSGS'GGB & \xrightarrow{GSGS'G\epsilon_B} & GSGS'GB
 \end{array}$$

The unlabeled squares in the diagram commute by functoriality and naturality, so the outside of the square commutes and is seen to be Axiom 5.

The second sentence is dealt with symmetrically. \square

Definition 3.2. A monoidal comonad G on a left $*$ -autonomous monoidal category \mathcal{C} is *left $*$ -autonomous* when it is equipped with natural transformations $\nu : S \rightarrow GSG$ and $\nu' : S' \rightarrow GS'G$, each satisfying Axioms 1 and 2, with ν satisfying Axioms 3 and 4, and together satisfying Axioms 5 and 6. If \mathcal{C} is $*$ -autonomous, we also say G is $*$ -autonomous when it is left $*$ -autonomous.

Although we obtain the following corollary as an immediate consequence of our results, it is the desired object of this paper.

Corollary 3.3. *If \mathcal{C} is a left $*$ -autonomous monoidal category and G is a left $*$ -autonomous monoidal comonad on \mathcal{C} then the monoidal category \mathcal{C}^G is left $*$ -autonomous and the strict monoidal underlying functor $U : \mathcal{C}^G \rightarrow \mathcal{C}$ preserves left internal homs. If \mathcal{C} is $*$ -autonomous then so is \mathcal{C}^G and U preserves left and right internal homs.*

4. Monoidal adjunctions and monoidal comonads

Now we step back a bit and work in the reverse direction. Suppose $U \dashv R : \mathcal{C} \rightarrow \mathcal{A}$ with unit $\eta : 1 \rightarrow RU$ and counit $\epsilon : UR \rightarrow 1$, such that U is strong monoidal and the square

$$\begin{array}{ccc}
 \mathcal{A}^{\text{op}} & \xrightarrow{U} & \mathcal{C}^{\text{op}} \\
 S \downarrow & & \downarrow S \\
 \mathcal{A} & \xrightarrow{U} & \mathcal{C}
 \end{array}$$

commutes. Then we obtain a monoidal comonad $G = UR$ on \mathcal{C} as

$$\begin{aligned}
 \delta &= U\eta_R : UR \rightarrow URUR, \\
 \epsilon &= \epsilon : UR \rightarrow 1,
 \end{aligned}$$

$$\begin{aligned} \varphi &= (URA \otimes URB \xrightarrow{\varphi} U(RA \otimes RB) \xrightarrow{U\eta_{RA \otimes RB}} URU(RA \otimes RB) \\ &\xrightarrow{UR\varphi^{-1}} UR(URA \otimes URB) \xrightarrow{UR(\epsilon_A \otimes \epsilon_B)} UR(A \otimes B)), \\ \varphi_0 &= (I \xrightarrow{\varphi_0} UI \xrightarrow{U\eta_I} URUI \xrightarrow{UR\varphi_0^{-1}} URI). \end{aligned}$$

There is also a candidate for $\nu : S \rightarrow GSG$, viz.,

$$S \xrightarrow{S\epsilon} SUR = USR \xrightarrow{U\eta_{SR}} URUSR = URSUR. \tag{★}$$

Proposition 4.1. *Axioms 1 and 2 hold for the data above.*

Proof. The following diagram shows that Axiom 1 is satisfied:

$$\begin{array}{ccccccc} S & \xrightarrow{S\epsilon} & SUR & \xrightarrow{=} & USR & \xrightarrow{U\eta_{SR}} & URUSR & \xrightarrow{=} & URSUR \\ & & & & & \searrow 1 & \downarrow \epsilon_{USR} & & \downarrow \epsilon_{SUR} \\ & & & & & & USR & \xrightarrow{=} & SUR, \end{array}$$

and from the diagram

$$\begin{array}{ccccc} URSUR & \xrightarrow{U\eta_{RSUR}} & URURSUR & & \\ \downarrow URS\epsilon_{UR} & \searrow 1 & \downarrow U\eta_{RSUR} & & \uparrow U\eta_{RSUR} \\ URSURUR & \xrightarrow{URSU\eta_R} & URSUR & & \\ \downarrow = & & \downarrow = & \searrow 1 & \\ URUSRUR & & URUSR & \xrightarrow{=} & URSUR \\ \downarrow URU\eta_{SRUR} & & \downarrow URU\eta_{SR} & & \uparrow UR\epsilon_{SUR} \\ URURUSRUR & \xrightarrow{URURUS\eta_R} & URURUSR & \xrightarrow{=} & URURSUR \end{array}$$

we may conclude that $\delta_{SG} = GGS\delta \circ G\nu_G$ so that Axiom 2 is satisfied. \square

Now suppose \mathcal{A} and \mathcal{C} are equipped with natural transformations $e_{A,B} : S(A \otimes B) \otimes A \rightarrow SB$ and that U preserves these; that is, the following diagram commutes.

$$\begin{array}{ccc}
 U(S(A \otimes B) \otimes A) & \xrightarrow{Ue_{A,B}} & USB \\
 \nearrow \varphi \cong & & \searrow = \\
 US(A \otimes B) \otimes UA & & SUB \\
 \searrow = & & \nearrow e_{UA,UB} \\
 SU(A \otimes B) \otimes UA & \xrightarrow{S\varphi \otimes 1} & S(UA \otimes UB) \otimes UA
 \end{array}$$

Proposition 4.2. *In this case, Axiom 3 holds.*

Proof. This leads us to examine the diagram in Fig. 1. It can be seen that the unlabeled areas of the diagram commute. The area labeled by (C) commutes by the above assumption and the area labeled by (A) is seen to commute by examining the following diagram.

$$\begin{array}{ccccc}
 & & URUSR(A \otimes B) \otimes URURA & & \\
 & & \nearrow U\eta \otimes U\eta & & \\
 USR(A \otimes B) \otimes URA & & \downarrow \varphi & & URSUR(A \otimes B) \otimes URURA \\
 & & U(RUSR(A \otimes B) \otimes RURA) & & \downarrow \varphi \\
 \downarrow \varphi & \nearrow U(\eta \otimes \eta) & \downarrow U\eta & \searrow = & U(RSUR(A \otimes B) \otimes RURA) \\
 U(SR(A \otimes B) \otimes RA) & & URU(RUSR(A \otimes B) \otimes RURA) & & \downarrow U\eta \\
 \downarrow U\eta & \nearrow URU(\eta \otimes \eta) & \downarrow UR\varphi^{-1} & \searrow = & URU(RSUR(A \otimes B) \otimes RURA) \\
 URU(SR(A \otimes B) \otimes RA) & & UR(URUSR(A \otimes B) \otimes URURA) & & \downarrow UR\varphi^{-1} \\
 \downarrow UR\varphi^{-1} & \nearrow UR(U\eta \otimes U\eta) & \downarrow UR(\epsilon \otimes \epsilon) & \searrow = & UR(URSUR(A \otimes B) \otimes URURA) \\
 UR(USR(A \otimes B) \otimes URA) & & \downarrow 1 & \searrow = & UR(\epsilon \otimes \epsilon) \\
 & & UR(USR(A \otimes B) \otimes URA) & \xrightarrow{=} & UR(SUR(A \otimes B) \otimes URA)
 \end{array}$$

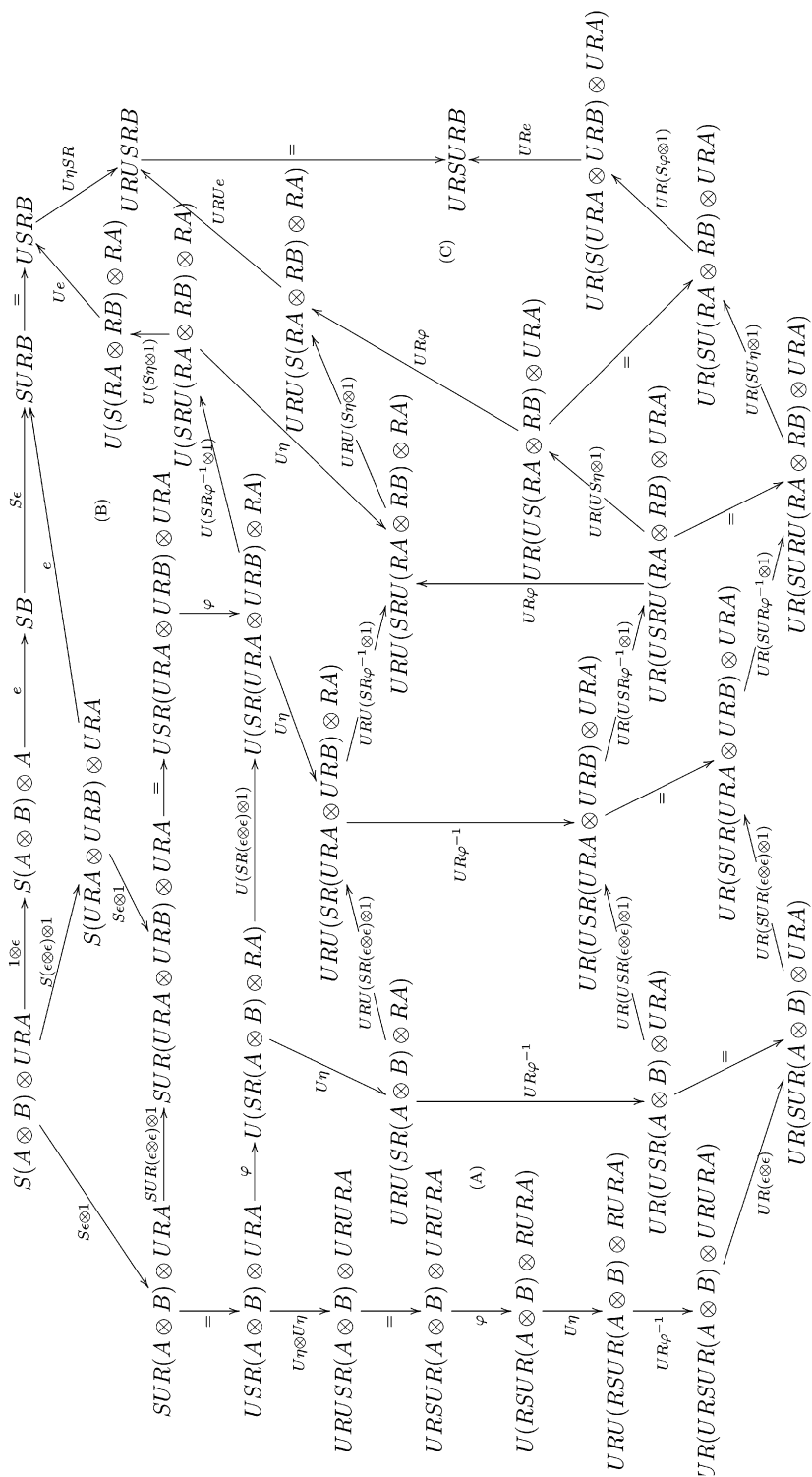
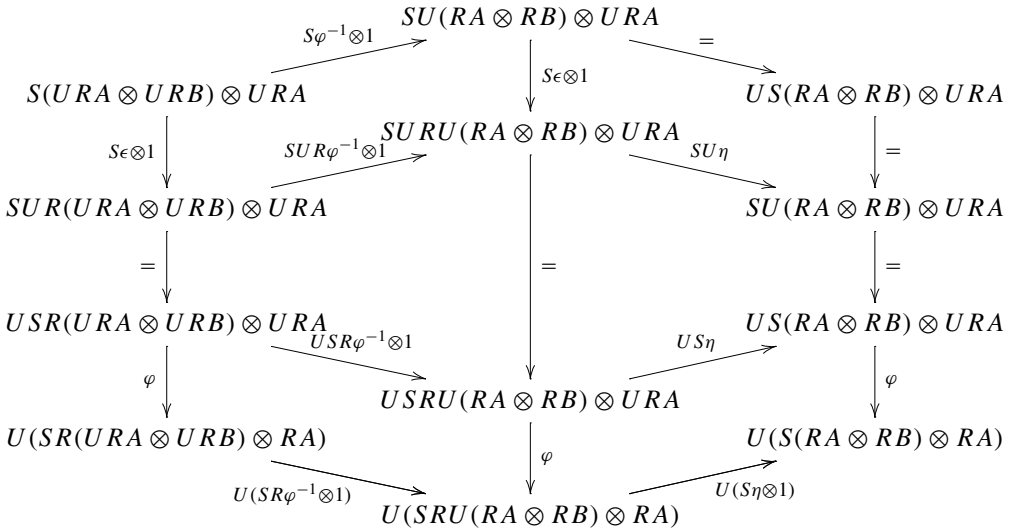


Fig. 1. Proof of Proposition 4.2.

To see that the region labeled by (B) commutes observe that the following diagram commutes.



By our assumption that U preserves e the upper route of the above diagram is

$$S(URA \otimes URB) \otimes URA \xrightarrow{e} SURB \xrightarrow{1} USRB \xleftarrow{Ue} U(S(RA \otimes RB) \otimes RA)$$

which then shows the commutativity of the region labeled by (B). \square

Remark 4.3. By Yoneda, Axiom 3 is equivalent to the commutativity of



Proposition 4.4. The formula for ν given in (\star) recovers the original ν when applied to the adjunction $U \dashv R : \mathcal{C} \rightarrow \mathcal{C}^G$ in the setting of Proposition 4.1.

Proof. We have that $U(A, \gamma) = A$, $RX = (GX, \delta_X)$, $\epsilon : UR \rightarrow 1$ is just the counit ϵ of the comonad, and $\eta : 1 \rightarrow RU$ has components $\gamma : (A, \gamma) \rightarrow (GA, \delta_A)$. So the ν given in (\star) becomes

$$\begin{array}{ccccc}
 SX & \xrightarrow{\nu_X} & GSGX & & \\
 \downarrow S\epsilon_X & & \downarrow GSG\epsilon_X & \searrow 1 & \\
 SGX & \xrightarrow{\nu_{GX}} & GSGGX & \xrightarrow{GS\delta_X} & GSGX. \quad \square
 \end{array}$$

Now suppose \mathcal{A} and \mathcal{C} are equipped with morphisms $n_I : I \rightarrow SSI$ and that U preserves this; that is, the following diagram commutes.

$$\begin{array}{ccccc}
 I & \xrightarrow{n_I} & SSI & \xrightarrow{SS\varphi} & SSUI \\
 \downarrow \varphi_0 & & & & \uparrow = \\
 UI & \xrightarrow{Un_I} & USSI & &
 \end{array}$$

Proposition 4.5. *In this case, Axiom 4 holds.*

Proof. In the diagram in Fig. 2 the unlabeled regions are easily seen to commute, the two regions labeled by (A) commute by our assumption that U preserves n_I , and the region labeled by (B) is seen to commute from the following diagram.

$$\begin{array}{ccccc}
 US^2I & \xrightarrow{=} & S^2UI & \xrightarrow{S^2U\eta} & S^2URUI \\
 \downarrow U\eta & \searrow US^2\eta & & \searrow = & \downarrow S^2UR\varphi_0^{-1} \\
 & & US^2RUI & & S^2URI \\
 & & \downarrow U\eta & \searrow US^2R\varphi_0^{-1} & \downarrow = \\
 URUS^2I & \xrightarrow{URUS^2\eta} & URUS^2RUI & & US^2RI \\
 \downarrow = & & \downarrow = & \searrow URUS^2R\varphi_0^{-1} & \downarrow U\eta \\
 URS^2UI & \xrightarrow{URS^2U\eta} & URS^2URUI & & URUS^2RI \\
 & \searrow 1 & \downarrow URS^2\epsilon & & \downarrow = \\
 & & URS^2UI & & URS^2URI \\
 & & \downarrow URS^2\varphi_0^{-1} & \swarrow URS^2\epsilon & \\
 & & URS^2I & &
 \end{array}$$

□

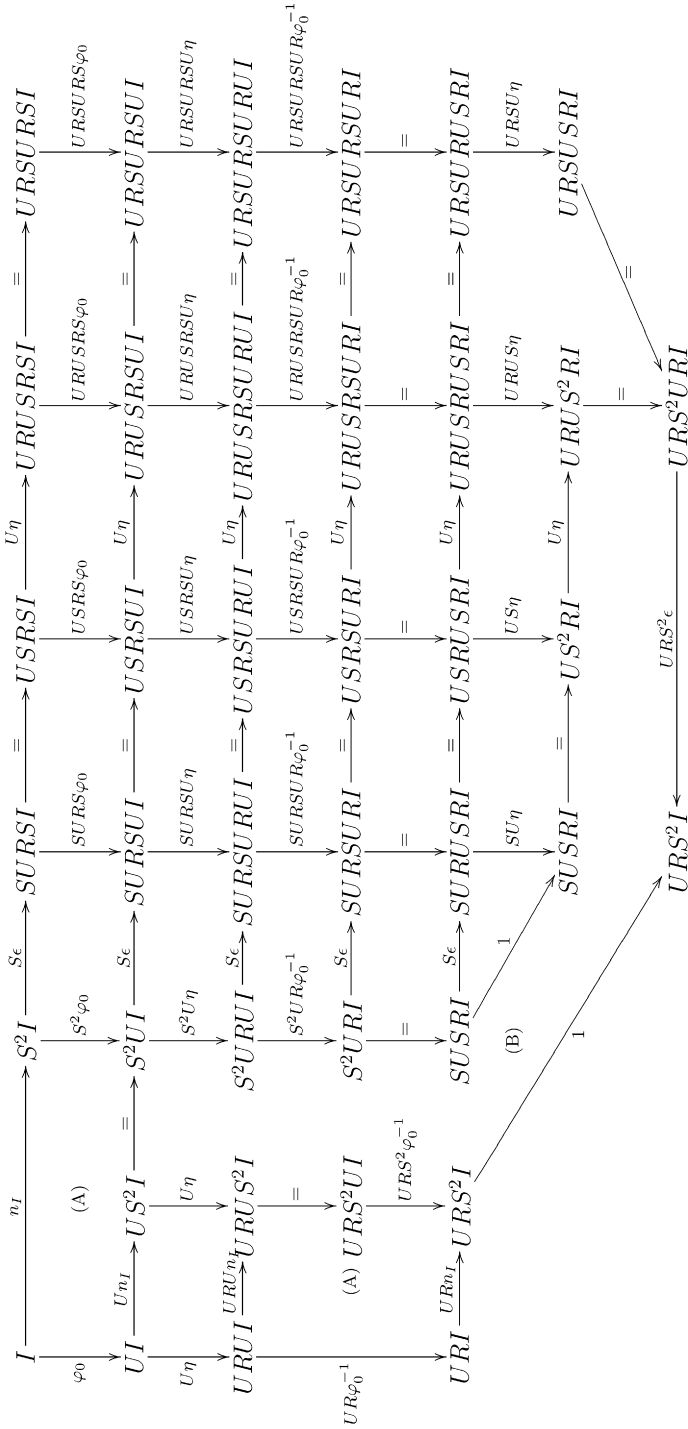


Fig. 2. Proof of Proposition 4.5.

Finally, suppose that in \mathcal{A} and \mathcal{C} we have $S' \dashv S$ with unit $\alpha : 1 \rightarrow SS'$ and counit $\beta : 1 \rightarrow S'S$ and that U preserves these, meaning both

$$\begin{array}{ccc}
 U & \xrightarrow{U\alpha} & USS' \\
 & \searrow \alpha_U & \downarrow = \\
 & & SS'U
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 U & \xrightarrow{U\beta} & US'S \\
 & \searrow \beta_U & \downarrow = \\
 & & S'SU
 \end{array}$$

commute.

Proposition 4.6. *In this case, both Axioms 5 and 6 hold.*

Proof. The commutativity of the following diagram proves that Axiom 5 holds and Axiom 6 is proved with a similar diagram.

$$\begin{array}{ccccc}
 & & URUR & \xrightarrow{UR\epsilon} & UR \\
 & \nearrow U\eta_R & & \searrow UR\alpha_{UR} & \\
 G = UR & \xrightarrow{U\alpha_R} & USS'R & \xrightarrow{URU\alpha_R} & URSS'UR \\
 \alpha_{UR} \downarrow & & \downarrow = & \nearrow U\eta_{SS'R} & \downarrow UR\alpha \\
 SS'UR & \xrightarrow{=} & SUS'R & \xrightarrow{URU\alpha_R} & URSS'UR \\
 S\epsilon_{S'UR} \downarrow & & \downarrow S\epsilon_{US'R} & \nearrow 1 & \downarrow URSS'\epsilon \\
 SURS'UR & \xrightarrow{=} & SURUS'R & \xrightarrow{SU\eta_{S'R}} & SUS'R \\
 \downarrow = & & \downarrow = & \nearrow U\eta_{SS'R} & \downarrow = \\
 USRS'UR & \xrightarrow{=} & USRUS'R & \xrightarrow{US\eta_{S'R}} & USS'R \\
 U\eta_{SRS'UR} \downarrow & & \downarrow U\eta_{SRUS'R} & \nearrow U\eta_{SS'R} & \downarrow U\eta_{SS'R} \\
 URUSRS'UR & \xrightarrow{=} & URUSRUS'R & \xrightarrow{URUS\eta_{S'R}} & URUSS'R \\
 \downarrow = & & \downarrow = & \nearrow = & \downarrow = \\
 URSURS'UR & \xrightarrow{=} & URSURUS'R & \xrightarrow{URSU\eta_{S'R}} & URSUS'R
 \end{array}$$

5. The Hopf algebra example

In this section we will show that any Hopf algebra with bijective antipode $H = (H, \mu, \eta, \delta, \epsilon, \nu)$ in a braided $*$ -autonomous category \mathcal{C} gives rise to a $*$ -autonomous monoidal comonad $G : \mathcal{C} \rightarrow \mathcal{C}$ defined on objects by $GX = {}^H X$. A left adjoint for this G is given by $T = H \otimes -$ and so, by Proposition 2.1, there is a bijection between G -coalgebras $X \rightarrow X^H$ and T -algebras $H \otimes X \rightarrow X$. As tensors are easier to work with here than homs, we will take this latter view in the remainder of this section and show that T is a $*$ -autonomous opmonoidal monad.

For the functor $T = H \otimes - : \mathcal{C} \rightarrow \mathcal{C}$, the data of a $*$ -autonomous monad is given as follows:

$$\begin{aligned} \mu &= H \otimes H \otimes X \xrightarrow{\mu \otimes 1} H \otimes X, \\ \eta &= X \xrightarrow{\eta \otimes 1} H \otimes X, \\ \psi &= (H \otimes X \otimes Y \xrightarrow{\delta \otimes 1 \otimes 1} H \otimes H \otimes X \otimes Y \xrightarrow{1 \otimes c \otimes 1} H \otimes X \otimes H \otimes Y), \\ \psi_0 &= H \xrightarrow{c} I, \\ \nu &= (H \otimes S(H \otimes X) \xrightarrow{c} S(H \otimes X) \otimes H \xrightarrow{1 \otimes \nu} S(H \otimes X) \otimes H \xrightarrow{e} SX), \\ \nu' &= (H \otimes S'(H \otimes X) \xrightarrow{\nu^{-1} \otimes S'c^{-1}} H \otimes S'(X \otimes H) \xrightarrow{e'} S'X). \end{aligned}$$

That T forms an opmonoidal monad is standard and so in the remainder of this section we concentrate on showing that this data satisfies the axioms of a $*$ -autonomous monad.

5.1. Lifting the internal hom

We begin with Axiom 3. The remainder of the axioms are straightforward and will be proved in Section 5.2. Axiom 3 is meant to express that $e_{A,B} : S(A \otimes B) \otimes A \rightarrow SB$ is a morphism of T -algebras, i.e., a morphism of left H -modules. We will prove this statement directly, instead of proving Axiom 3. It follows from the more general statement in Proposition 5.1 below.

In this section we need only assume that \mathcal{C} is a braided closed category. Suppose H is a Hopf algebra in \mathcal{C} and that $M, N \in \mathcal{C}$ are left H -modules. Denote by $e_{M,N} : {}^M N \otimes M \rightarrow N$ the morphism obtained by evaluating the isomorphism $\mathcal{C}(L, {}^M N) \cong \mathcal{C}(L \otimes M, N)$ at the identity.

Note that we have the composites

$$\begin{aligned} H \otimes {}^M N \otimes M &\xrightarrow{1 \otimes e} H \otimes N \xrightarrow{\mu} N, \\ {}^M N \otimes H \otimes M &\xrightarrow{1 \otimes \mu} {}^M N \otimes M \xrightarrow{e} N \end{aligned}$$

which, under the isomorphism $\mathcal{C}(L \otimes M, N) \cong \mathcal{C}(L, {}^M N)$, become respectively left and right actions of H on ${}^M N$:

$$\begin{aligned} \mu_l &: H \otimes {}^M N \rightarrow {}^M N, \\ \mu_r &: {}^M N \otimes H \rightarrow {}^M N. \end{aligned}$$

It is not too difficult to see that these actions make ${}^M N$ into an H -bimodule or a left $H \otimes H^{op}$ -module. Restriction of scalars along the algebra morphism

$$H \xrightarrow{\delta} H \otimes H \xrightarrow{1 \otimes \nu} H \otimes H^{op}$$

makes ${}^M N$ into a left H -module.

Proposition 5.1. *The evaluation morphism $e_{M,N} : {}^M N \otimes M \rightarrow N$ is a morphism of H -modules.*

Proof. By the definition of μ_l and μ_r the following two diagrams commute.

$$\begin{array}{ccc}
 H \otimes M_N \otimes M & \xrightarrow{1 \otimes e} & H \otimes N \\
 \mu_l \otimes 1 \downarrow & & \downarrow 1 \otimes e \\
 M_N \otimes M & \xrightarrow{e} & N
 \end{array}
 \qquad
 \begin{array}{ccc}
 M_N \otimes H \otimes M & \xrightarrow{1 \otimes \mu} & M_N \otimes M \\
 \mu_r \otimes 1 \downarrow & & \downarrow e \\
 M_N \otimes M & \xrightarrow{e} & N
 \end{array}$$

Using these facts it is possible to see that the diagram (where we have dropped the “ \otimes ”)

$$\begin{array}{ccccc}
 H^M N M & \xrightarrow{1} & H^M N M & \xrightarrow{1 e} & H N & \xrightarrow{\mu} & N \\
 \delta^3 1 1 \downarrow & \searrow \delta 1 1 & & \searrow \mu_l 1 & & & \uparrow e \\
 H H H^M N M & & H H^M N H M & \xrightarrow{\mu_l \mu} & M N M & & M N M \\
 1 c_{HH, M_N} 1 \downarrow & \searrow 1 1 \epsilon 1 & & \searrow 1 1 \eta 1 & & & \uparrow \mu_r 1 \\
 H^M N H H M & & H^M N H H M & \xrightarrow{\mu_l 1 \mu} & M N H M & & M N H M \\
 & \swarrow 1 1 \delta 1 & & \swarrow 1 1 \mu 1 & & & \uparrow 1 \mu \\
 & & H^M N H H M & \xrightarrow{1 1 \nu 1} & H^M N H H M & \xrightarrow{\mu_l 1 \mu} & M N H M
 \end{array}$$

commutes and expresses that $e_{M,N}$ is a morphism of H -modules. \square

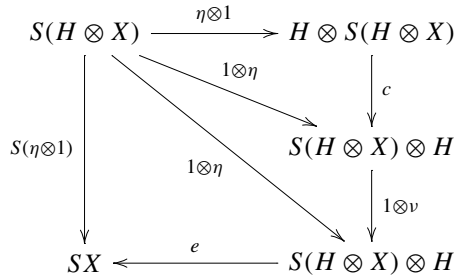
That $e_{A,B} : S(A \otimes B) \otimes A \rightarrow SB$ becomes a morphism of H -modules follows by choosing $M = A$ and $N = {}^B D$.

5.2. The remainder of the axioms

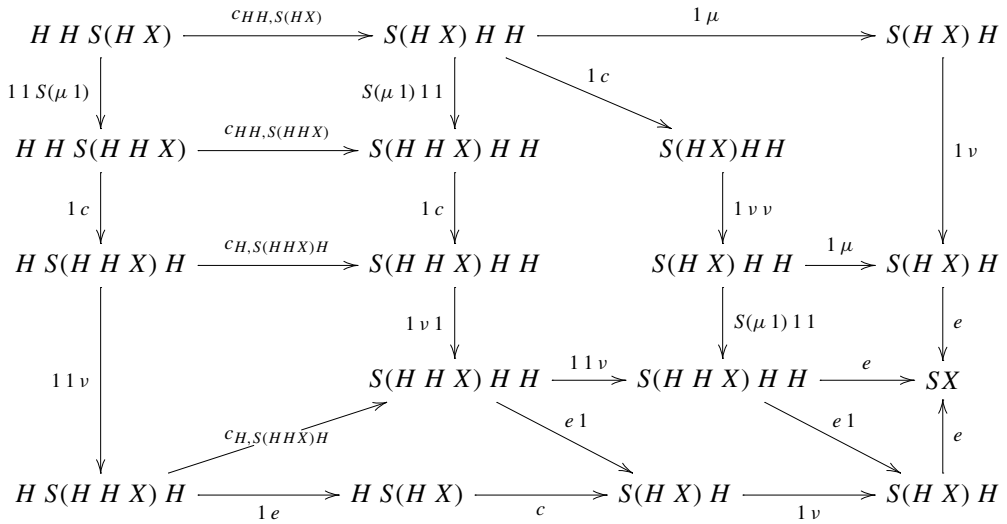
It is left to show that the remainder of the axioms hold. Axioms 1 and 2 may respectively be expressed in terms of a monad as follows:

$$\begin{array}{ccc}
 ST & \xrightarrow{S\eta} & S \\
 \eta_{ST} \searrow & & \nearrow v \\
 & TST, &
 \end{array}
 \qquad
 \begin{array}{ccccc}
 TTST & \xrightarrow{\mu_{ST}} & TST & \xrightarrow{v} & S \\
 TTS\mu \downarrow & & & & \uparrow v \\
 TTSTT & \xrightarrow{T\nu_T} & TST & &
 \end{array}$$

The diagram

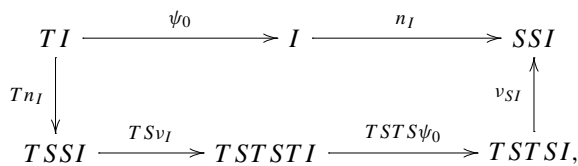


shows that Axiom 1 holds and Axiom 2 is seen to be satisfied since the following diagram commutes (we have dropped the “ \otimes ” symbol).

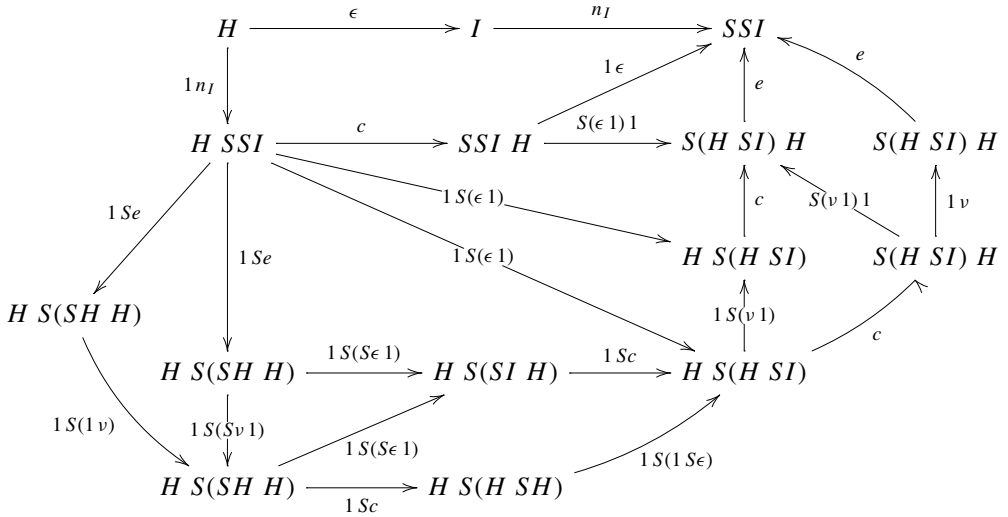


Similar diagrams show that S' and v' also satisfy Axioms 1 and 2.

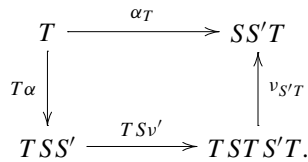
Axiom 4 expressed in terms of a monad is



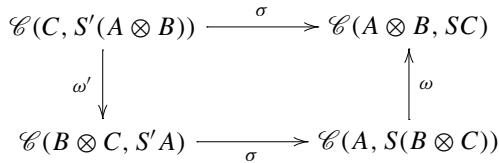
and that this holds may be seen from the following diagram (where we have again dropped the “ \otimes ”).



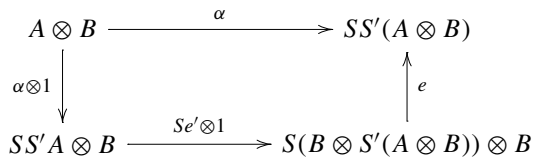
Axiom 5 may be expressed in terms of a monad as



Note that the commutativity of the diagram



implies that



commutes, and therefore, that the following diagram expressing Axiom 5 commutes.

$$\begin{array}{ccccc}
 H \otimes X & \xrightarrow{c} & X \otimes H & \xrightarrow{\alpha} & SS'(X \otimes H) & \xrightarrow{SS'c^{-1}} & SS'(H \otimes X) \\
 \downarrow 1 \otimes \alpha & & \downarrow \alpha \otimes 1 & & \uparrow e & & \uparrow e \\
 H \otimes SS'X & \xrightarrow{c} & SS'X \otimes H & & S(H \otimes S'(H \otimes X)) \otimes H & & \\
 \downarrow 1 \otimes Se' & & \downarrow Se' \otimes 1 & & \uparrow S(1 \otimes S'c^{-1}) \otimes 1 & & \uparrow S(v \otimes 1) \otimes 1 \\
 H \otimes S(H \otimes S'(X \otimes H)) & \xrightarrow{c} & S(H \otimes S'(X \otimes H)) \otimes H & & S(H \otimes S'(H \otimes X)) \otimes H & & \\
 & & \uparrow e & & \uparrow 1 \otimes v & & \\
 & & & & & & \uparrow c \\
 H \otimes S(H \otimes S'(X \otimes H)) & \xrightarrow{1 \otimes S(v^{-1} \otimes S'c^{-1})} & H \otimes S(H \otimes S'(H \otimes X)) & & & &
 \end{array}$$

Axiom 6 may be handled similarly. Thus $T = H \otimes -$ is a $*$ -autonomous monad.

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