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On Modules with Cyclic Vertices in the Auslander–Reiten Quiver

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In studying indecomposable modules of a finite-dimensional K-algebra where K is a field, the Auslander–Reiten quiver Γ has proved to be a powerful tool.

If the algebra is a group algebra KG (or a block B), then properties of the group are related to the graph structure of the Auslander–Reiten quiver. For example, the graph $\Gamma(KG)$ (or $\Gamma(B)$) is finite if and only if a Sylow-p-subgroup of G (or a defect group of B) is cyclic, where p is the characteristic of K.

It is now well known that an indecomposable KG-module has a vertex, which is a minimal subgroup Q of G such that M is Q-projective; and Q is unique up to G-conjugation [7]. We ask whether there are constraints on the position of modules with a given vertex in the Auslander–Reiten quiver. Here we are concerned with this problem for modules with cyclic vertices which are not projective.

There is not much to say in the case when the defect group of the block is cyclic; in this case, all its modules have cyclic vertices. We assume therefore that the block containing these modules has a non-cyclic defect group.

Assume that M is indecomposable with a cyclic vertex, and that M is not projective. It is well known that M is Ω-periodic, hence τ-periodic. (For group algebras, the Auslander–Reiten translate τ is the same as Ω^2, where Ω is the usual Heller operator.) By a more general theorem [2, p. 163], the component—Θ say—of the stable Auslander–Reiten quiver containing M, is a "tube" [15]. Given such a tube Θ, then the vertices in each row form a single τ-orbit. The "end" of the tube is the unique row where each vertex has only one predecessor. That is, if M is a module corresponding to a vertex $[M]$ in Θ and

$$0 \to \tau M \to E \to M \to 0 \qquad (*)$$

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is the Auslander–Reiten sequence of M, then $[M]$ lies at the end if and only if E has a unique indecomposable non-projective summand.

Our main result is now:

THEOREM. *Assume that G is a finite group, K a field of characteristic p where $p>2$. Let M be an indecomposable non-projective KG-module with a cyclic vertex which lies in a block B whose defect group is not cyclic. Then M lies at the end of a tube.*

COROLLARY. *Assume that a p-block B contains modules M_1, M_2 with cyclic vertices such that there is an irreducible map $\varphi: M_1 \to M_2$. If $p>2$ then the defect group of B must be cyclic.*

We should like to add two remarks. First, if the vertex of M is normal in G then the theorem holds also in the case $p=2$ (2.2). Second, if $(*)$ is the Auslander–Reiten sequence of the module M in the theorem, then E must be indecomposable. If E had a projective summand then by a general theorem [13, p. 111], $\Omega(M)$ would have to be simple, and then [4] implies that the defect group of B would be cyclic. In Section 1, we study Auslander–Reiten sequences; the results hold for arbitrary finite-dimensional algebras. The second section contains the proof of the theorem in the case where the vertex of M is normal, for an arbitrary prime p; and in the third Section we prove the theorem in the general case, when $p>2$.

For any module M, the largest semisimple submodule of M is denoted by $\operatorname{soc}(M)$; and $\operatorname{soc}_k(M)$ is defined by the property $\operatorname{soc}_k(M)/\operatorname{soc}_{k-1}(M) = \operatorname{soc}(M/\operatorname{soc}_{k-1}(M))$. Also, $M/\operatorname{rad} M$, or "top(M)" is the largest semisimple factor module.

We denote the composition length of M by $|M|$ and the vertex of M by $\operatorname{vx}(M)$. We write "$M \mid N$" if M is isomorphic to a direct summand of N. Any other notation should be standard; for modular representations we refer to [5, 8, 11]; and relevant material on Auslander–Reiten theory may be found in [2, 6, 11, 14].

1. On Auslander–Reiten Sequences

In this chapter, we study the situation where a τ-orbit consists entirely of modules with simple socles and tops. Here, Λ is any finite-dimensional K-algebra.

(1.1) LEMMA. *Let $0 \to \tau X \to^{\beta} E \to^{\alpha} X \to 0$ be an AR-sequence of Λ-modules. Assume that X and τX have simple socles and tops, and that E is decomposable. Then*

(a) *$E = L \oplus S$ where L and S have simple socles and tops*

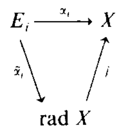
(b) $|X| = |\tau X| = |S| + 1 = |L| - 1$. *(In particular, X is not simple.)*

Proof. (a) Since $\mathrm{soc}(E)$ is isomorphic to a submodule of $\mathrm{soc}(X) \oplus \mathrm{soc}(\tau X)$, it has length at most 2; therefore E can only have two indecomposable summand, E_1 and E_2, say, and these must have simple socles. Similarly, $E/\mathrm{rad}\, E$ has length $\leqslant 2$, which forces $E_i/\mathrm{rad}\, E_i$ to be simple.

(b) Denote by $\alpha_i = \alpha|_{E_i}$, and $\beta_i = \pi_i \circ \beta$, where π_i are the canonical projections corresponding to $E = E_1 \oplus E_2$. Then the maps α_i, β_i are irreducible and therefore either one-to-one or onto (not both).

(b_1) *If α_i is one-to-one then $E_i \cong \mathrm{rad}\, X$; and if α_i is onto then $X \cong E_i/\mathrm{soc}(E_i)$:*

Assume that α_i is one-to-one. Then α_i is not onto, and since E_i has a unique maximal submodule, there is a factorization



where j is the inclusion map and $\tilde{\alpha}_i = \alpha_i$. Since α_i is irreducible and j does not split, it follows that $E_i \cong \mathrm{rad}\, X$, as both modules are indecomposable.

The other part is proved similarly; in the same way one sees that

(b_2) *If β_i is one-to-one then $\tau X \cong \mathrm{rad}\, E_i$, and if β_i is onto then $E_i \cong \tau X/\mathrm{soc}(\tau X)$.*

Since X has a unique maximal submodule, at least one of α_1, α_2 must be onto. Similarly, at least one of β_1, β_2 must be one-to-one. Assume that β_1 is one-to-one; then $X = \mathrm{rad}\, E_1$. Hence $|E_2| = |X| + |\tau X| - |E_1| = |X| - 1 < |X|$. Thus α_2 is one-to-one, and consequently α_1 must be onto. Now (b) follows immediately if we set $L = E_1$ and $S = E_2$.

(1.2) PROPOSITION. *Assume that Λ is a finite-dimensional K-algebra. Let M be an indecomposable Λ-module such that all modules $\tau^k M$ ($k \in \mathbb{Z}$) are defined and have simple socles and tops. Let Θ be the component of M. Then one of the following holds*:

(a) *There is some $k \in \mathbb{Z}$ such the middle term E in the AR-sequence $0 \to \tau^{k+1} M \to E \to \tau^k M \to 0$ is indecomposable.*

(b) *Θ contains a τ-orbit consisting of simple modules. Moreover, Θ is finite.*

That is, either M lies "at the end" of its component, or else the "block" of Λ containing M is of finite representation type, by Auslander's theorem [1, 14].

Proof. Assume that (a) does not hold. Then (1.1) applies with $X=\tau^k M$, for all $k\in\mathbb{Z}$. Let $m=|M|$. Take any module Y in Θ whose distance in Θ to $\{\tau^k M/k\in\mathbb{Z}\}$ is d. We shall first prove that

(1) If Y is not projective then the AR-sequence of Y is of the form $0\to\tau Y\to L\oplus S\to Y\to 0$ (possibly $S=0$), where

(i) $|Y|=|\tau Y|=m\pm d\equiv|L|-1=|S|+1$

(ii) the modules Y, τY, L, S (if nonzero) have simple socles and tops

(1*) If Y is not injective then the AR-sequence ending in Y is of the form $0\to Y\to L\oplus S\to\tau^{-1}Y\to 0$ (possibly $S=0$) where

(i) $|Y|=|\tau^{-1}Y|=m\pm d=|L|-1=|S|+1$

(ii) the modules Y, $\tau^{-1}Y$, L, S (if nonzero) have simple socles and tops.

(2) If Y is projective or injective then $|Y|=m+d$, and any predecessor (or successor) of Y whose distance to $\{\tau^k M|k\in\mathbb{Z}\}$ is $\gneqq d$ is projective or injective.

We prove (1), (1*), and (2) by induction on d. If $d=0$ then $Y=\tau^k M$ for some k; thus (1), (1*) follow from (1.1), and (2) holds vacuously. Note also that $|\tau^k M|=|M|$ for all $k\in\mathbb{Z}$.

Assume now that $d=1$. Then there are irreducible maps $\alpha\colon Y\to\tau^k M$ and $\beta\colon\tau^{k+1}M\to Y$ for some $k\in\mathbb{Z}$. The socle and top of Y are simple, by (1) for "$d=0$."

(2) Assume that Y is projective. Then β must be a monomorphism, hence $|Y|=|\tau^{k+1}M|+1=m+1$. The predecessors of Y are the summands of rad Y. This module must be indecomposable since soc Y is simple; hence there are no predecessors whose distance to $\{\tau^k M|k\in\mathbb{Z}\}$ is >1. Assume that Y has a successor, Z say, with $Z\not\cong\tau^k M$. Then Z must be projective, since otherwise τZ would be defined, and Y would have a predecessor $\not\cong\tau^{k+1}M$. If Y is injective then (2) is proved by similar arguments.

(1) Now assume that Y is not projective. Let $0\to\tau Y\to E\to Y\to 0$ be the AR-sequence of Y. Then the socle and top of τY are also simple, by (1) for "$d=0$," since there are irreducible maps $\tau Y\to\tau^{k+1}M$ and $\tau^{k+2}M\to\tau Y$. Also $\tau^{k+1}M|E$. Either E is decomposable, then the rest of (1) follows from (1.1). Otherwise, $|\tau^{k+1}M|=m=|Y|+|\tau Y|=(m\pm 1)+(m\pm 1)$ which implies that $m=2$ and $|Y|=|\tau Y|=1$. Hence we obtain (1) with $S=0$.

(1*) is proved similarly.

Assume that $d \geqslant 2$. Then there is an irreducible map $\alpha: Y \to W$ or an irreducible map $\beta: W' \to Y$ where W, W' have distance $d-1$ from $\{\tau^k M / k \in \mathbb{Z}\}$. In fact, both must hold: Say α exists, then by (2) of the induction hypothesis, W is not projective. Therefore τW is defined, and there is an irreducible map $\beta: W' \to Y$ where $W' = \tau W$. Now (2) and (1), (1*) follow as in the case $d = 1$.

Proof of Proposition (1.2)(b). By (1)(i), all modules in a τ-orbit $\subseteq \Theta$ have the same length. Moreover, whenever $|Y| > 1$ there is a module of length $|Y| - 1$ in Θ. Thus Θ contains modules of lengths $m-1, m-2, \ldots, 1$.
The τ-orbit consisting of simple modules must be finite. From the description of the AR-sequences in (1), (1*) it follows that then all τ-orbits in Θ are finite. (To see this, one could also apply Riedtmann's structure theorem [13]). Moreover, the lengths of Λ-modules with simple tops are bounded and therefore also the distance in Θ to the τ-orbit $\{\tau^k M / k \in \mathbb{Z}\}$ by (1)(i). Also, there are only finitely many injective or projective modules, hence Θ must be finite.

2. Modules with Cyclic Normal Vertices

Assume that K is a field of characteristic p and that G is a finite group. We recall the following facts on the sructure of KG-modules with cyclic normal vertices.

(2.1) Lemma. *Assume that Q is a normal p-subgroup of G such that $Q = (x)$. Let M be an indecomposable KG-module. Then*

(a) $\mathrm{vx}(M) = Q$ *if and only if* $M \cong eKG/e(1-x)^s KG$ *where e is a primitive idempotent and $1 \leqslant s < |Q|$ such that $p \nmid s$ and $e(1-x)^s KG \neq \{0\}$.*

Assume that $\mathrm{vx}(M) = Q$. *Then*

(b) $\mathrm{soc}(M)$ *and* $\mathrm{top}(M)$ *are simple.*

(c) *Let* $M_r = \mathrm{soc}_r(M_Q)$ $(r = 1, 2, \ldots, s)$. *Then M_r and M/M_r are indecomposable KG-modules whose vertex is $\leqslant Q$.*

(d) *M and τM have the same source.*

Proof. (a) [4, 12].

(b) By (a), M is a quotient of an indecomposable projective module, thus $M/\mathrm{rad}\, M$ is simple. Also, $\mathrm{soc}(M)$ is simple since $\mathrm{soc}(M) \cong (M^*/\mathrm{rad}\, M^*)^*$ and $\mathrm{vx}(M) = \mathrm{vx}(M^*)$.

(c) With the notation of (a), $M_r \cong fKG/f(1-x)^r KG$ for some primitive idempotent f of KG. Apply (a).

(In case $p \mid r$, let $r = up^a$ such $p \nmid u$. Then $(1-x)^r = (1-x^{p^a})^u$.) The statement on M/M_r follows similarly:

(d) This is well known; recall that $\tau M \cong \Omega^2(M)$.

(2.2) *Proof of the Theorem in the Case Where the Vertex of M is Normal in G*

Let $\mathfrak{A}: 0 \to \tau M \to F \to M \to 0$ be the AR-sequence of M. We will prove that F is indecomposable.

It is known that Ω preserves vertices [11], hence all modules in the τ-orbit of M have cyclic normal vertices, and by Lemma (2.1)(b), the hypothesis of Proposition (1.2) is satisfied. Since the defect group of the block B containing M is not cyclic, B is not of finite representation type [3], and therefore, by Auslander's theorem [1, 14], the AR-quiver of B does not have a finite component. Thus Proposition (1.2)(a) must hold.

By the remark in the introduction, there are no projective modules occuring in the AR-sequence of $\tau^k M$ (and of M). If E is as in Proposition (1.2)(a) then $\tau\ ^{k}E \cong F$, and F is indecomposable.

For the proof of the theorem in the general case, we shall need some information about the action of Q on F and also on the vertex of F. The following is presumably well known:

(2.3) Lemma. *Assume that G is an arbitrary finite group, and that $\gamma: R \to S$ is an irreducible map where R, S are indecomposable KG-modules. Then either* $\mathrm{vx}(R) \leqslant_G \mathrm{vx}(S)$, or $\mathrm{vx}(S) \leqslant_G \mathrm{vx}(R)$.

Proof. Let $V = \mathrm{vx}(R)$ and $Q = \mathrm{vx}(S)$. There is a factorization of the form

$$\begin{array}{ccc} R & \xrightarrow{\ \gamma\ } & S \\ & {\scriptstyle\rho}\searrow \quad \nearrow{\scriptstyle\mu} & \\ & (S_V)^G & \end{array}$$

Since γ is irreducible, it follows that either μ is a split epimorphism, or ρ is a split monomorphism. In the first case, S is V-projective and therefore $Q \leqslant_G V$. If $R \mid (S_V)^G$ the S_V must have a summand whose vertex is V. Assume that T is a source of S in Q. Considering the Mackey decomposition of $(T^G)_V$ gives that $V \leqslant Q^g \cap V$ for some $g \in G$, as required.

(2.4) Lemma. *Let M, F, α be as in* (2.2), *and let* $Q = \mathrm{vx}(M)$. *Assume that a source of M has dimension s. Then*

(a) *$F_X \cong (M \oplus \tau M)_X$ if and only if $Q \not\leqslant X$.*

(b) *F_Q is a sum of indecomposable modules of dimensions s, $s+1$, $s-1$.*

If $s>1$ then $s+1$ and $s-1$ both occur.

Proof. (a) More generally, by comparing dimensions of spaces of homomorphisms, one proves

(2.4.1) LEMMA. *Suppose that $0 \to A \to B \to C \to 0$ is a short exact sequence of modules over a finite dimensional algebra. If $B \cong A \oplus C$ then the sequence splits.*

(This can be traced back to M. Auslander.) Using this (a) is obtained from [2, p. 84].

(b) For $s=1$, (b) follows directly by restricting the AR-sequence of M. Assume that $s>1$.

We denote by V_d the d-dimensional indecomposable Q-module. Then M_Q is a direct sum of copies of V_s, $M_Q \cong aV_s$ (say), where $a \geqslant 1$. Also $\tau M_Q \cong bV_s$ for some $b \geqslant 1$. First we claim that

(I) *there is a s.e.s. of Q-modules*

$$0 \to bV_s \oplus aV_{s-1} \to F_Q \to aV_1 \to 0. \qquad (*)$$

The module M_{s-1} is a proper submodule of M. Hence the sequence $0 \to \tau M \to \alpha^{-1}(M_{s-1}) \to^{\alpha} M_{s-1} \to 0$ splits [6, Proposition 1.6], and there is a s.e.s. $0 \to \tau M \oplus M_{s-1} \to F \to M/M_{s-1} \to 0$. Restricting this to Q gives $(*)$ (see (2.1)).

(II) *F_Q has $a+b$ indecomposable summands.* Since $s>1$, we know that M is not isomorphic to a summand to K_Q^G. Therefore, by the definition of an AR-sequence,

$$0 \to \mathrm{Hom}_G(K_Q^G, \tau M) \to \mathrm{Hom}_G(K_Q^G, F) \to \mathrm{Hom}_G(K_Q^G, M) \to 0$$

is exact. Now, $\mathrm{Hom}_G(K_Q^G, X) \cong_K \mathrm{Hom}_Q(K, X)$, and the dimension of this space is the same as the number of indecomposable direct summands of X_Q when Q is cyclic.

Since F_Q and the kernel in $(*)$ have the same number of indecomposable summands, we must have that

$$F_Q \cong cV_s \oplus dV_{s-1} \oplus eV_{s+1}, \qquad \text{where} \quad c+d+e=a+b;$$

this proves the first part of Lemma (2.4)(b).

Also $\dim F_Q = s(a+b) = sc + (s-1)\,d + (s+1)\,e$, hence $d=e$. If d were zero, then it would follow that $F_Q \cong M_Q \oplus \tau M_Q$ and $\mathfrak{A}_Q$ would split, by Lemma (2.4.1), contrary to (a).

(2.5) LEMMA. *Assume that* F *is as in* (2.2) *and that* $p>2$. *Let* $V=\mathrm{vx}(F)$ *and* $Q=\mathrm{vx}(M)$. *Then* $Q \lneq V$, *and* V *is not cyclic.*

Proof. (i) If $Q \nleq V$ then by (2.3), V is normal and cyclic. Then (2.2) may be applied to F as well. Consequently, the middle of the AR-sequence of F is indecomposable and therefore isomorphic to τM. This is not possible, since the composition lengths do not add up.

(ii) Assume for contradiction that V is cyclic. Let T be a source of F in V. Then $F_Q \mid (T^G\mid_Q)$, and by the Mackey decomposition, $T^G\mid_Q \cong \sum \oplus T_Q$. Let dim $T=r+up^a$, where $p^a=|V:Q|$ and $0 \leqslant r < p^a$. Then the indecomposable summands of T_Q have dimensions u and $u+1$, that is, these differ by 1. However, by Lemma (2.4)(b), we know that F_Q has summands whose dimensions differ by 2; this is a contradiction. (We may assume that $s \neq 1$; otherwise we replace M by ΩM.)

3. THE ARBITRARY CASE

In this chapter, G is an arbitrary finite group. For the proof of the theorem, we need a result of "Clifford type."

(3.1) Assume that L is a normal subgroup of a group Y and that $L \subseteq H \subseteq Y$. Take an indecomposable H-module W, and let

$$W_L = \sum_{i \in I} \oplus n_i W_i$$

where each W_i is indecomposable, such that $W_i \not\cong W_j$ for $i \neq j$, and n_i is the multiplicity of W_i as a direct summand of W_L. Let I_0 be the set of all $i \in I$ such that $W_i \otimes t$ is isomorphic to a direct summand of W_L, for some $t \in Y \backslash H$. Then the result is

LEMMA. *Let* $W^Y = A \oplus C$ *such that* $W \mid (A_H)$. *Then* C_L *is isomorphic to a direct sum of* W_i*'s with* $i \in I_0$. *In particular, if* $I_0 = \varnothing$ *then* W^Y *must be indecomposable.*

The proof is a variation of [10, VII, Thereom 9.6].

(3.2) *Middle Terms in Induced AR-sequences*

Let $N=N_G(Q)$. Assume that U is a non-projective indecomposable N-module with vertex Q and that

$$\mathfrak{A}_0 : 0 \to \tau U \to D \to U \to 0 \text{ is the } AR\text{-sequence of } U.$$

If we induce this to G we obtain $(\mathfrak{A}_0)^G \cong \mathfrak{A} \oplus \mathfrak{C}$, where $\mathfrak{A}$ is the AR-

sequence of gU, and $\mathfrak{E}$ is a split sequence [2, p. 93]. We are interested in the middle term of $\mathfrak{A}$. Let F be an indecomposable summand of D whose vertex V satisfies $V \gneqq Q$. For a subgroup X of V containing Q, consider the following condition:

($\mathfrak{H}_X$) Assume $C(X)X = L \leqslant N^g$ for some $g \in G$; and $t \in N_G(V) \backslash N^g$. If Z is indecompsable with $Z | (F^g)_L$ and also $Z | (F^g)_L \otimes t$ then $vx(Z) \subsetneqq Q^g$.

PROPOSITION. *Assume that F is an indecompsable summand of D with vertex V such that $V \gneqq Q$:*

(a) *If F satisfies ($\mathfrak{H}_V$) then F^G has a unique summand whose vertex is V.*

(b) *If F satisfies ($\mathfrak{H}_X$) for $Q \leqslant_G X <_G V$ then F^G has no indecomposable summand whose vertex is X.*

Proof. We shall use the following basic fact: Given that $Q \lhd G$ and $V \leqslant G$. If U is Q-projective then U_V is $Q \cap V$-projective.

(a) Let $Y = N_G(V)$. By the Burry–Carlson theorem [2, p. 61] we know that F^G and F^G_Y have the same number of summands whose vertex is V. Now

$$F^G|_Y \cong \sum \oplus (F^g_{N^g \cap Y})^Y$$

where g runs through a system of representatives of the (N, Y)-double cosets in G. Take a fixed g, set $H = Y \cap N^g$, and $T = (F^g)^Y_H$; we study the module T.

Case 1. $g = 1$. Then we claim that

(1) *T has a unique summand whose vertex is V.* If $g = 1$ then $H = N_N(V)$, and by the Green correspondence, $F_H \cong fF \oplus \mathfrak{Y}(F)$ where fF is indecomposable with $\mathrm{vx}(fF) = V$, and $\mathfrak{Y}(F)$ is $\mathfrak{Y}$-projective where $\mathfrak{Y} = \{V^y \cap H \mid y \in N \backslash H\}$. Then $\mathfrak{Y}(F)^Y$ is still $\mathfrak{Y}$-projective, hence does not have a summand whose vertex is V (recall that V is normal in Y). To prove (1), it suffices to show that fF^Y is indecomposable. This will follow from (3.1), with $W = fF$ (and $L = VC_G(V)$) if we can prove

(1*) Let $t \in Y \backslash H$. Then W_L and $(W \otimes t)_L$ do not have a common summand.

Any indecomposable summand W_1 of W_L is a summand of fF_L, and therefore it satisfies $F | W_1^N$. If W_1 also occurred as a summand of $(W \otimes t)_L$ then by the condition ($\mathfrak{H}_V$), the vertex of W_1 would be $\subsetneqq Q$, hence $vx(F) \subsetneqq Q$, contradiction.

Case 2. Assume that $g \in NY$. Our aim is to prove that T does not have a summand whose vertex is V. We start with some simplifications concerning the groups involved:

(1) *We may assume that* $V \subseteq N^g$. If $N \nsubseteq N^g$ then $V \nsubseteq H^h \cap V$ for any $h \in Y$ since V is normal in Y. Therefore, no summand of T_V can have vertex V, by the Mackey decomposition. Consequently, T does not have a summand whose vertex is V. Therefore, let now $V \subseteq N^g$.

(2) *Without loss of generality* $Q^g \subseteq V$. Assume that $Q^g \nsubseteq V$. Then $(\mathfrak{A}_0)_V^g$ splits (by Lemm (2.4), note that $Q^g \lhd N^g$), and therefore $(F^g)_V$ is a summand of $(U \oplus \tau U)_V^g$, hence $(F^g)_V$ is $Q^g \cap V$-projective. Assume for contradiction that T has a summand whose vertex is V; then T_V has one, too. Now, $T_V \cong \sum \oplus (F^g)^y_{H^y \cap V}{}^V \cong \sum \oplus [(F^g)^y]_V$ since $y \in Y$ and $V \lhd Y$. It follows that $(F^g)_V$ has a summand with vertex V. This is not so; we have proved above that $(F^g)_V$ is $Q^g \cap V$-projective. Hence we assume now that $Q^g \leqslant V$, so that we have

(3) $Q^g \lneqq V \leqslant N^g$, *and* $L \subseteq H$ *where* $L = VC_G(V)$. Now we return to our problem. The module T is a direct sum of modules W^Y where W is an indecompsable summand of $(F^g)_H$. Since $\mathrm{vx}(F^g) = V^g \neq V$ and also $H = N_{N^g}(V)$, the Burry–Carlson theorem tells us that $\mathrm{vx}(W) \neq V$. Now, W^Y has an indecomposable summand, A, say, such that $W | (A\,|_H)$. Then $\mathrm{vx}(A) = \mathrm{vx}(W)$ [5, p. 113 (4.6)(ii)], hence $\mathrm{vx}(A) \neq V$. Let $W^Y = A \oplus C$. We are left to prove that

(4) *C does not have a summand whose vertex is V.* Consider C_L where $L = VC_G(V)$. Let Z be an indecompsable summand of C_L, then Z is H-conjugate to a summand of W_L, hence of $(F^g)_L$. By (3.1), there is some $t \in Y \backslash H$ such that $(Z \otimes t) | W_L$. Thus $Z | F^g_L$ and also $(F^g)_L \otimes t^{-1}$. By the hypothesis $(\mathfrak{H}_V)$, $\mathrm{vx}(Z) \subsetneqq_H Q^g$. That is, no summand of C_L has vertex V, and (4) follows.

Proof of (b). Let $Q \leqslant X \lneqq V$, and let $Y = N_G(X)$. By the Burry–Carlson theorem, F^G and $(F^G)_Y$ have the same number of indecompsable summands whose vertex is X. Define the module T as in the proof of (a). Then T does not have a summand whose vertex is X; this follows as in Case 2 of the proof of (a).

(3.3) *Proof of the Theorem.* Assume that $\mathfrak{A}: 0 \to \tau M \to E \to M \to 0$ is the AR-sequence of M. We have to prove that E is indecomposable. Let $N = N_G(Q)$, where $Q = \mathrm{vx}(M)$. We use the Green correspondence $f\colon G \to N$. Since $Q = vx(M)$ and since the defect group of the block containing fM is not cyclic [9], we obtain from (2.2) that the AR-sequence of fM is of the form $\mathfrak{A}_0: 0 \to \tau(fM) \to F \to fM \to 0$ where F is indecomposable. If we induce this to G we obtain $(\mathfrak{A}_0)^G \cong \mathfrak{A} \oplus \mathfrak{E}$ where $\mathfrak{E}$ is a split sequence [2, p. 93]. Let $V = \mathrm{vx}(F)$, then $Q \lneqq V$ by (2.5). We shall prove

(I) F satisfies $(\mathfrak{H}_X)$ for any $Q \leqslant_G X \lneqq V$,

(II) E does not have a summand whose vertex is $\lneqq_G Q$.

Then (3.2) and (II) imply that $E \cong gF$, by (2.3).

(3.3*) *Remark.* It follows then from (2.5) that $\mathrm{vx}(E)$ is not cyclic, and $Q \lneqq_G \mathrm{vx}(E)$:

(I) Let $L = XC_G(X)$, $Y = N_G(X)$, and $t \in Y \backslash N^g$. Assume that $Z \mid (F^g)_L$ and also $Z \otimes t \mid (F^g)_L$. Denote by $\tilde{Q} = tQ^g t^{-1}$, let $p^a = |Q^g : \tilde{Q} \cap Q^g|$, and let S be a source of fM in Q. Write dim $S = s = p^a u + r$ $(1 \leqslant r < p^a)$. Then

(1) *Every indecomposable summand of* $(Z)_{Q^g}$ *has dimension* $p^a u$ *or* $p^a(u+1)$. The action of Q^g on Z is the same as the action of $\tilde{Q}$ on $Z \otimes t$. By the hypothesis, $(Z \otimes t)|_{\tilde{Q}}$ is a summand of $(F^g)_{\tilde{Q}}$ which is isomorphic to $(fM \oplus \tau fM)^g_{\tilde{Q}}$ (by Lemma (2.4)). We know that S^g is a source of fM^g and of τfM^g. Hence any indecomposable summand of $(Z \otimes t)_{\tilde{Q}}$ is a summand of $[(S^g)_{Q^g \cap \tilde{Q}}]^{\tilde{Q}}$; this implies (1).

On the other hand, since $(Z|_{Q^g}) | (F^g)_{Q^g}$, we obtain from (2.4):

(2) *Every indecomposable summand of* Z_{Q^g} *has dimension* s *or* $s+1$ *or* $s-1$. Combining (1) and (2) and using the fact that $p > 2$ we must have that $s > 1$ and that either $s \equiv 1$ or $s \equiv (-1) \pmod{p^a}$. We may assume that $s \equiv 1 \pmod{p^a}$; otherwise we replace M by ΩM. Then

(3) $s = p^a u + 1$, and every summand of Z_{Q^g} has dimension $p^a u$.

Now let $R = \mathrm{soc}_{s-1}(fM^g_{Q^g})$. In the AR-sequence, let $\tilde{F}$ be the inverse image of R in F under α. Since $R \subsetneqq fM^g$, the AR-sequence splits by restriction on $\tilde{F}$ [6, Proposition 1.6], that is, $\tilde{F} \cong R \oplus \tau fM^g$. Now, Z is a summand of $(F^g)_L$ which is contained in $\tilde{F}$ (by (3)), hence $Z \mid \tilde{F}$, as an L-module. By (3) we must have $Z \mid R$, and $(\mathfrak{H}_X)$ is proved.

Proof of (II). The module E has no projective summand, as we remarked in the introduction. In particular, if $|Q| = p$ then (II) holds.

We continue by induction on a where $|Q| = p^a$. Note that whenever (II) is satisfied then the theorem follows in this case. Let $a > 1$, and assume for contradiction that R is an indecomposable summand of B such $\mathrm{vx}(R) \lneqq Q$. If $0 \to R \to T \to \tau^{-1} R \to 0$ is the AR-sequence then by the induction hypothesis, T is indecomposable and therefore $T \cong M$. However, by (3.3*), the vertex of T is not cyclic, which is a contradiction.

Proof of the Corollary. Assume for contradiction that the defect group of the block is not cyclic. Then the theorem applies with $M = M_2$, and it follows that the AR-sequence of M_2 is of the form

$$0 \to \tau M_2 \to M_1 \overset{\varphi}{\to} M_2 \to 0.$$

But then, by (3.3*), the vertex of M_1 is not cyclic, contrary to the hypothesis.

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