A Taylor series approach to the numerical analysis of the M/D/1/N queue

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Abstract

This paper presents a functional approximation of the M/D/1/N built on a Taylor series approximation. Numerical examples are carried out to illustrate the performance of our approach.

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1. Introduction

In this paper we consider the M/D/1/N queue with Poisson arrival process, deterministic service times, first-come first-served (FCFS) service discipline, and a capacity of in total N customers (the one in service included). Customers that do not find an empty place at the queue upon their arrival are lost.

Let $\pi^*$ denote the stationary distribution of the continuous-time queue-length process of the M/D/1/N queue. It is well known that $\pi^*$ can be expressed via the stationary distribution of the Markov chain embedded at departure points of customers, denoted by $\pi$. Specifically, let $\rho$ denote the traffic rate, then it holds that

$$
\pi^*(i) = \frac{\pi(i)}{\pi(0) + \rho}, \quad i = 0, \ldots, N - 1,
$$

and

$$
\pi^*(N) = \frac{1}{\rho} \left( \rho - 1 + \frac{\pi(0)}{\pi(0) + \rho} \right) = \frac{\pi(0) + \rho - 1}{\pi(0) + \rho},
$$

see [2] for details.

Unfortunately, the stationary distribution $\pi^*$ of the embedded jump chain can be obtained in a closed-form only for special cases. This has lead to a rich literature on approximations of the stationary distribution $\pi^*$. A very efficient
numerical approach is to approximate the M/D/1/N queue by the M/D/1/∞ queue, see [10, 9]. However, this approach is limited to the cases where the traffic load is less than one (so that the corresponding M/D/1/∞ model is stable). For a detailed overview on numerical approaches we refer to the excellent survey in [8].

Note that although obtaining a (simple) closed-form expression for \( \pi^* \) is a hard problem, using (1) and (2), \( \pi^* \) can be numerically evaluated via \( \pi \), where \( \pi \) can be computed as the solution of \( \pi P = \pi \) for \( P \) being the transition matrix of the embedded Markov chain.

In this paper, we choose a different point of view. We consider \( \pi^* \) as a mapping of some real-valued parameter \( \theta \), in notation \( \pi^*_\theta \). For example, \( \theta \) may denote the mean service time of the queue. We are interested in obtaining the functional dependence of \( \pi^*(\theta) \) on \( \theta \) in a simplified form. For our approach we will compute \( \pi^*_\theta \) for some parameter value \( \theta \) numerically. However, then we will approximate the function \( \pi^*(\theta + \Delta) \) on some \( \Delta \)-interval. More specifically, we will approximate \( \pi^*(\theta + \Delta) \) by a polynomial in \( \Delta \).

To achieve this we will use the Taylor series expansion approach established in [3]. More specifically, let \( \pi_\theta \) denote the stationary distribution of the queue-length process embedded at departure epochs in the M/D/1/N queue, where \( \theta \in \mathbb{R} \) denotes a control parameter. Under quite general conditions it holds that \( \pi_{\theta + \Delta} \) can be developed into the following Taylor series

\[
\pi_{\theta + \Delta} = \sum_{n=0}^{\infty} \frac{\Delta^n}{n!} \pi^{(n)}_\theta,
\]

where \( \pi^{(n)}_\theta \) denotes the \( n \)-th order derivative of \( \pi_\theta \) with respect to \( \theta \). We call

\[
H_\theta(k, \Delta) = \sum_{n=0}^{k} \frac{\Delta^n}{n!} \pi^{(n)}_\theta
\]

the \( k \)-th order Taylor approximation of \( \pi_{\theta + \Delta} \) at \( \theta \), and

\[
R_\theta(k, \Delta) = \pi_{\theta + \Delta} - H_\theta(k, \Delta)
\]

the \( k \)-th order remainder term at \( \theta \). Let \( \eta \) denote the half-width of the interval \( (\theta - \eta, \theta + \eta) \) on which the stationary distribution has to be approximated. The Taylor series approximation will only be of practical value if it converges fast, i.e., if for small \( k \) it holds that \( \sup_{\theta \in \mathbb{R}} |R_\theta(k, \Delta)| \) is sufficiently small.

The current paper will investigate the use of \( H_\theta(k, \Delta) \) for numerical purposes for the M/D/1/N queue. We will present the algorithm and provide a simplified representation of the derivatives. In particular, we will identify a recursive form of the derivatives which simplifies coding the algorithm. Numerical examples illustrate the performance of the new algorithm.

The paper is organized as follows. The embedded Markov chain of the M/D/1/N model is presented in Section 2. Our series expansions approach is detailed in Section 3. Numerical examples are provided in Section 4.

2. The M/D/1/N Model

Consider the M/D/1 queue with arrival rate \( \lambda \) and deterministic service time \( \theta \). Let \( X(t) \) denote the number of customers in the M/D/1/N queue at time \( t \), for \( t \geq 0 \). Note that the queue-length processes \( \{X(t) : t \geq 0\} \) of the M/D/1/N system fails to be a Markov process because the service time distribution does not have the memoryless property. Since we have assumed that customers that do not find an empty buffer place upon their arrival are lost, the stationary distribution of \( \{X(t) : t \geq 0\} \), denoted by \( \pi \), exists (independent of the traffic rate). Let \( \{X_n : n \in \mathbb{N}\} \) denote the queue–length process embedded right after the departure of the \( n \)-th customer, see [2]. Note that \( X_n \) has state-space \( \{0, \ldots, N-1\} \) as after the departure of a customer the system cannot be full. Then \( \{X_n : n \in \mathbb{N}\} \) is a Markov chain with
transition matrix

\[
P_\theta = \begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \cdots & a_{N-2} & 1 - \sum_{k=0}^{N-2} \frac{a_k}{k!}
\end{pmatrix},
\]

where

\[
a_k = \frac{(\lambda \theta)^k}{k!} e^{-\lambda \theta}, 0 \leq k \leq N - 2.
\]

In words, \(a_k\) is the probability of \(k\) Poisson arrivals during a service.

3. The Taylor Series Expansion

For the M/D/1/N queue, \(P_\theta\) is infinitely often differentiable with respect to \(\theta\). Hence, it holds that the first \(n\) derivatives of \(P_\theta\) exists. Let \(P^{(k)}_\theta\) denote the \(k\)th order derivative of \(P_\theta\) with respect to \(\theta\), then it holds that

\[
P^{(k)}_\theta = \frac{d^k}{d\theta^k} P_\theta = \begin{pmatrix}
a_0(k) & a_1(k) & a_2(k) & a_3(k) & \cdots & a_{N-2}(k) & 1 - \sum_{j=0}^{N-2} \frac{a_j(k)}{j!}
\end{pmatrix},
\]

where

\[
a_j(k) = \frac{d^k}{d\theta^k} a_j, \quad 0 \leq j \leq N - 2.
\]

Let \(\pi_\theta\) denote the stationary distribution of the embedded chain, where \(\theta\) denotes the parameter of interest, and denote the deviation matrix by \(D_\theta\) defined by

\[
D_\theta = \sum_{n=0}^{\infty} (P^{(k)}_\theta - \Pi_\theta),
\]

where \(\Pi_\theta\) is a square matrix with rows equal to \(\pi_\theta^T\), with \(x^T\) denoting the transposed of vector \(x\). As shown in [5], for any finite-state aperiodic Markov chain the deviation matrix exists. However, the deviation matrix is only known in explicit form for the M/M/1 and the M/M/1/N queue and we will compute it for our purposes numerically by matrix inversion using the fact that \(D_\theta = (I - P_\theta + \Pi_\theta)^{-1} - \Pi_\theta\), where \(I\) denotes the identity matrix.

The following theorem is an adaptation of Theorem 4 in [3] to the special case of the M/D/1/N queue.

**Theorem 1.** Let \(\theta \in \Theta\) and let \(\Theta_0 \subset \Theta\) be a closed interval with \(\theta\) an interior point such that the queue is stable on \(\Theta_0\). Provided that the entries of \(P_\theta\) are \(n\)-times differentiable with respect to \(\theta\), it holds that

\[
\pi^{(n)}_{\theta} = \sum_{l_1, l_2, \ldots, l_n} \frac{n!}{l_1! \cdots l_n!} \pi_\theta \prod_{k=1}^{n} \left( P^{(l_k)}_\theta D_\theta \right).
\]
The derivatives in Theorem 1 enjoy a recursive structure in the sense that a \((k+1)\)-st order derivative is mainly constituted out of information already provided by the \(k\)-th order derivative. The following lemma provides the exact statement.

**Lemma 2.** Under the conditions put forward in Theorem 1 it holds for \(k < n\) that
\[
\pi^{(k+1)}_\theta = \sum_{m=0}^k \binom{k+1}{m} \pi^{(m)}_\theta \Delta^m P^{(k+1-m)} \mathbb{D}_\theta.
\]

For ease of reference we will provide in the following an explicit representation of the first derivatives of \(\pi_\theta\):
\[
\pi^{(1)}_\theta = \pi^{(1)}_\theta \mathbb{D}_\theta
\]
and
\[
\pi^{(2)}_\theta = \pi^{(2)}_\theta \mathbb{D}_\theta + 2\pi^{(1)}_\theta (\mathbb{D}_\theta)^2.
\]

Elaborating on the recursive formula for higher order derivatives in Lemma 2, the second order derivative can be written as
\[
\pi^{(3)}_\theta = \pi^{(3)}_\theta \mathbb{D}_\theta + 3\pi^{(2)}_\theta (\mathbb{D}_\theta)^2  + 3\pi^{(1)}_\theta (\mathbb{D}_\theta)^3.
\]

In the same vein, we obtain for the third order derivative
\[
\pi^{(4)}_\theta = \pi^{(4)}_\theta \mathbb{D}_\theta + 4\pi^{(3)}_\theta (\mathbb{D}_\theta)^2  + 6\pi^{(2)}_\theta (\mathbb{D}_\theta)^3  + 4\pi^{(1)}_\theta (\mathbb{D}_\theta)^4.
\]

In the following, denote by
\[
\|x\| = \max_i |x_i|, \quad x \in \mathbb{R}^n,
\]
the sup norm on \(\mathbb{R}^n\).

**Theorem 3.** Let \(\theta \in \Theta\) be an interior point of \(\Theta\). Provided that the entries of \(P_\theta\) are \((k+1)\)-times continuously differentiable with respect to \(\theta\) on \(\Theta\), it holds for \(\Delta > 0\) such that \(\theta + \Delta \in \Theta\):
\[
\|R_\theta(k, \Delta)\| \leq \int_{\theta}^{\theta + \Delta} \frac{\Delta^{k+1}}{(k+1)!} \left\|\pi^{(k+1)}_{\theta + \Delta} \right\| dx.
\]

**Proof:** An exact expression for the remainder term is given by the so-called Lagrange remainder:
\[
R_\theta(k, \Delta) = \int_{\theta}^{\theta + \Delta} \frac{\Delta^{k+1}}{(k+1)!} \pi^{(k+1)}_{\theta + \Delta} d\delta,
\]
where the above equation has to understood element-wise. For \(\Delta > 0\), the integration part \(\Delta^{k+1}\) is larger than zero and it holds for any element \(i\) of the vector \(R_\theta(k, \Delta)\)
\[
[R_\theta(k, \Delta)](i) \leq \int_{\theta}^{\theta + \Delta} \frac{\Delta^{k+1}}{(k+1)!} \left\|\pi^{(k+1)}_{\theta + \Delta}(i) \right\| d\delta.
\]
where \([\cdot](i)\) denotes the \(i\)-th element of vector \(x\). Taking norms yields the desired result. \(\square\)

The Taylor series approximation developed above applies to differentiable Markov kernels. This extends the case of linear \(\theta\) dependence that has been studied in the literature so far; see, for example, [1, 6, 7, 4, 5, 3]. More specifically, for linear perturbations the standard assumption is that \(P_\theta\), the Markov kernel of the embedded jump chain, is of the form
\[
P_\theta = \theta P + (1-\theta)\hat{P}, \quad \theta \in [0, 1],
\]
where \(P, \hat{P}\) with \(\hat{P} \neq P\) are two Markov kernels.

In contrast to the model in (3), the Taylor series expansions established in this section applies to non-linear perturbations. The case of a non-linear perturbation hadn’t been studied in the literature so far. This case is significantly more difficult than the linear case as in this case all higher-order derivatives of \(P_\theta\) may be different from zero. Moreover, the elements of the Taylor series in the linear case are of rather simple form, which stems from the fact that in the linear case all but the first derivative of \(P\) with respect to \(\theta\) are zero.
4. Numerical Examples

In this section we present numerical examples. We consider as main performance measure the blocking probability $\pi^*_\theta(N)$, which is, due to the fact that customers arrive according to a Poisson arrival stream, equal to the probability that an arriving customer is lost due to no available free waiting space. The traffic rate is given by $\rho(\theta) = \lambda \theta$.

Recall, that by (2), it holds that

$$
\pi^*_\theta(N) = \frac{\pi_{\theta+\Delta}(0) + \rho(\theta + \Delta) - 1}{\pi_{\theta+\Delta}(0) + \rho(\theta + \Delta)}.
$$

Inserting our Taylor series expansion for $\pi_{\theta+\Delta}(0)$ into the above expression yields a functional representation of $\pi^*_\theta(N)$ as a function in $\Delta$. Elaborating on (2), a similar procedure leads to a functional representation of the mean queue length and via Little’s law to one of the stationary waiting time.

The following numerical examples illustrate our approach. In all numerical examples we have set $\lambda = 1$ and $N = 5$. We first illustrate the numerical behavior of $\pi^*_\theta(n)$ for $0 \leq n \leq N - 1$ and $n = 1, 2, 3$, for the stationary distribution of the embedded jump chain of the M/D/1/N queue.

As a first example we apply a Taylor series of order 2, i.e.,

$$
H_{\theta}(2, \Delta) = \pi_0 + \Delta \pi^{(1)}_0 + \frac{\Delta^2}{2} \pi^{(2)}_0
$$

$$
= \pi_0 + \Delta \pi^{(1)}_0 \phi_0 + \frac{\Delta^2}{2} \pi^{(2)}_0 \phi_0 + \Delta^2 \pi_0 (\phi^{(1)}_0 \phi_0)^2.
$$

We let $\theta = 1$ and vary $\Delta$ by at most 20 percent of $\theta$, i.e., $|\Delta| \leq 0.2$. Figure 1 plots the relative error given by

$$
\frac{H_{\theta}(2, \Delta)(i) - \pi_{\theta+\Delta}(i)}{\pi_{\theta+\Delta}(i)}
$$
for each element \(i = 0, \ldots, 4\). Note that in this situation the traffic load of the system is one but, due to the fact that we consider a loss system, stability is still guaranteed.

Next, we plot the relative error of the Taylor series expansion for order 3 in Figure 2, where

\[
H_\theta(3, \Delta) = \pi_0 + \Delta \pi_\theta^{(1)} + \frac{\Delta^2}{2} \pi_\theta^{(2)}
\]

\[
= \pi_0 + \Delta \pi_\theta^{(1)} + \Delta^2 \frac{\pi_\theta^{(2)}}{2} + \Delta^3 \pi_\theta^{(3)}
\]

\[
+ \frac{\Delta^4}{6} \left( \pi_\theta^{(4)} + 3 \pi_\theta^{(2)} \pi_\theta^{(2)} + 3 \pi_\theta^{(1)} \pi_\theta^{(3)} \right)
\]

As can be seen from the figures, a third order Taylor series yields a remarkably good approximation of the stationary probabilities through the range \(\theta \pm 0.2 \times \theta\).

We now turn to the blocking probability. Starting point is the expression for the loss probability in (4). The traffic rate is given by \(\rho(\theta + \Delta) = \lambda(\theta + \Delta)\) and \(\pi_{\theta+\Delta}(0)\) is approximated via a Taylor series polynomial of degree 3, i.e., we replace \(\pi_{\theta+\Delta}(0)\) by \(H_{\theta}(3, \Delta)(0)\).

The approximation in case of saturation, i.e., \(\rho(\theta) = 1\), is illustrated in Figure 3. Figure 4 shows the behavior of our approximation in the case of over-saturation, i.e., \(\rho = 1.2\). Note that in both figures we only plot the values for \(|\Delta| \leq 0.1\) for better readability. As can be seen from the figures, the approximation yields a satisfying precision in predicting the loss probability \(\pi_{\theta+\Delta}(N)\). It is worth noting that the quality of the approximation increases with the traffic load. As Figure 4 suggests, for \(\rho > 1.2\) already a Taylor series of order 2 seems to yield a sufficient approximation.

We conclude the discussion of the \(M/D/1/N\) queue by providing a bound on the error of the Taylor series approximation for \(\pi_{\theta+\Delta}(N)\). Suppose that for given order \(k\) of the Taylor approximation, the error is bounded by \(R\) for any \(\Delta\) such that \(|\Delta| \leq \delta\), i.e., assume that

\[
|\pi_{\theta+\Delta}(0) - H_{\theta}(k, \Delta)| \leq R,
\]
Figure 3: The relative absolute error in predicting the loss probability in saturated traffic ($\rho = 1$).

see Theorem 3. Replacing $\pi_{\theta+\Delta}(0)$ in (2) or (4) by $H_\theta(k, \Delta)$ and noting that $\rho(\theta + \Delta) = \lambda \theta + \Delta \lambda$, implies thus that the true value for $\pi_{\theta+\Delta}(N)$ is bounded by

$$H_\theta(k, \Delta) - R \leq \pi_{\theta+\Delta}(N) \leq H_\theta(k, \Delta) + R.$$ 

After some calculations, one arrives at that the following lemma.

**Lemma 4.** Consider the M/D/1/N queue with arrival rate $\lambda$ and deterministic service time $\theta$. Suppose that for $k$ it holds for $|\Delta| \leq \delta$ that

$$|\pi_{\theta+\Delta}(0) - H_\theta(k, \Delta)| \leq R_\theta(k, \delta),$$

then

$$\sup_{|\Delta| \leq \delta} \left| \pi_{\theta+\Delta}(N) - \frac{H_\theta(k, \Delta)}{H_\theta(k, \Delta) + \lambda(\theta + \Delta)} \right| \leq \frac{R_\theta(k, \delta)\lambda(\theta + \delta)}{\lambda^2(\theta - \delta)^2 - (R_\theta(k, \delta))^2}.$$ 

We conclude this section with a discussion of the numerical bound on the error provided in Lemma 4. For $\theta = 1$, the remainder bound is given by

$$R_1(3, \delta) = 4.161 \times 10^{-3},$$

resulting in the error bound for $\pi_{\theta+\Delta}(N)$

$$1.314 \times 10^{-2},$$

with $1.280 \times 10^{-3}$ as true error.

For $\theta = 1.2$ the remainder bound is given by

$$R_{1.2}(3, \delta) = 1.215 \times 10^{-3},$$

resulting in the error bound for $\pi_{\theta+\Delta}(N)$

$$6.083 \times 10^{-3},$$
and $1.132 \times 10^{-3}$ is the true error.

As the numerical values indicate, a Taylor series of degree $k$ yields an approximation of the blocking probability on the interval $\Delta \in [-0.2, 0.2]$ with an error of no more than $10^{-4}$.

5. Conclusion

We have presented a new approach to the functional approximation of the $M/D/1/N$ queue. As illustrated by the numerical examples, the convergence rate of the Taylor series is such that already a Taylor polynomial of degree 2 or 3 yields good numerical results. This implies that the proposed Taylor series approximation can be of practical value. Future research will be on investigating the behavior of the series expansion for multi-server queues. Also a simple and efficiently computable bound on the remainder term of the Taylor series will be a topic of further research.

References