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# Existence of weak solutions for a quasilinear equation in $\mathbb{R}^N$

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ABSTRACT

This paper studies the $p$-Laplacian equation

$$-\Delta_p u + \lambda V_\lambda(x)\,|u|^{p-2}\,u = f(x,u) \text{ in } \mathbb{R}^N,$$

where $1 < p < N$, $\lambda \geq 1$ and $V_\lambda(x)$ is a nonnegative continuous function. Under some conditions on $f(x,u)$ and $V_\lambda(x)$, we prove the existence of nontrivial solutions for $\lambda$ sufficiently large.

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## 1. Introduction and main results

In this paper, we consider the following $p$-Laplacian equation

$$-\Delta_p u + \lambda V_\lambda(x)|u|^{p-2}u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $1 < p < N,\ \lambda \geq 1,\ V_\lambda \in C(\mathbb{R}^N, \mathbb{R})$ and $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$.

We assume that the potential $V_\lambda(x)$ and $f(x,u)$ satisfy the following conditions.

$(\mathcal{V}_1)$ $0 \leq V_\lambda(x)$ for all $x \in \mathbb{R}^N$ and $\lambda \geq 1$.

$(\mathcal{V}_2)$ There exists $M > 0$ such that for all $\lambda \geq 1,\ |\Omega_{M,\lambda}| < \infty$, where

$$\Omega_{M,\lambda} = \left\{x \in \mathbb{R}^N / V_\lambda(x) \leq M\right\}.$$

$(\mathcal{V}_3)$ $\lim_{\lambda\to\infty} V_\lambda(0) = 0$.

There exist a positive function $m(x) \in L^\infty_{loc}(\mathbb{R}^N)$ and constants $C_0,\ R_0 > 0,\ \alpha > 1$ such that

$(\mathcal{V}_4)$ $m(x) \leq C_0\left(1 + (V_\lambda(x))^{1/\alpha}\right)$ for all $|x| \geq R_0$ and $\lambda \geq 1$.

(f$_1$) There exists $q \in (p, p^\sharp)$, with $p^\sharp := p^\star - \frac{p^2}{\alpha(N-p)}$ and $p^\star := \frac{Np}{N-p}$, such that

$$|f(x,t)| \leq C_0 m(x)(1+|t|^{q-1}) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R}.$$

(f$_2$) $\frac{f(x,t)}{m(x)} = o(|t|^{p-1})$ as $t \to 0$ uniformly in $x$.

(f$_3$) There exist $\mu_0, \mu > p$ and a positive continuous function $\gamma_0(x)$ such that

$$F(x,t) \geq \gamma_0(x)|t|^{\mu_0} \quad \text{and} \quad \mu F(x,t) \leq tf(x,t) \quad \text{for all } x \in \mathbb{R}^N \text{ and } t \in \mathbb{R},$$

where $F(x,t) = \int_0^t f(x,s)ds$.

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An example of functions satisfying the assumptions $(f_1)$–$(f_3)$ is given by

$$f(x, u) = m_0(x)|u|^{s-2}u,$$

where $m_0(x)$ is a positive continuous function and $p < s < p^\sharp$.

Set

$$\mathcal{F} = \left\{\delta > p/\text{there exists a positive continuous function } \gamma(x) \text{ such that } F(x,t) \geq \gamma(x)|t|^\delta \text{ for } x \in \mathbb{R}^N,\ t \in \mathbb{R}\right\}.$$

By $(f_1)$ and $(f_3)$, we see that $\mu_0 \in \mathcal{F}$ and $\delta \leq q$ for all $\delta \in \mathcal{F}$.

The investigation of equations of the form (1.1) has been motivated by searching wave solutions for the nonlinear Schrödinger equations; see [1–3]. Many works have been devoted to the case $p = 2$; see [4–9]. The quasilinear case $p \in (1, N)$ appears in a variety of applications, such as non-Newtonian fluids, image processing, nonlinear elasticity and reaction–diffusion; see [10] for more details. In the paper [11], Liu consider the $p$-Laplacian equation $(1 < p < N)$

$$-\Delta_p u + V(x)|u|^{p-2}u = f(x, u), \quad x \in \mathbb{R}^N, \tag{1.2}$$

with a potential which is periodic or has a bounded potential well. Without assuming the Ambrosetti–Rabinowitz type condition and the monotonicity of the function $t \to \frac{f(x,t)}{|t|^{p-1}}$, the author proved the existence of ground states of (1.2). Another $p$-Laplacian equation with potential was considered by Wu and Yang [12]

$$-\Delta_p u + \lambda V(x)|u|^{p-2}u = |u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $2 \leq p < q < p^\star$ and the potential $V(x)$ is bounded. Using a concentration-compactness principle from critical point theory, they proved existence, multiplicity and concentration of solutions of (1.3). For more results we refer the reader to [13,14,12,15] and references therein. In the present paper, we are going to study the existence of nontrivial solutions of (1.1). The results of this paper may be considered as generalization of the results obtained by Sirakov [8]. Here we consider the situation when the potential is sufficiently large at infinity. Our method is mainly based on variational arguments.

The main results of this paper are the following theorems.

**Theorem 1.1.** *Assume that* $(f_1)$–$(f_3)$, $(\mathcal{V}_1)$–$(\mathcal{V}_4)$ *hold, and* $q \in \mathcal{F}$. *Then there exists* $\lambda_0 \geq 1$, *depending only on the various constants involved in the assumptions, such that* (1.1) *has a nontrivial solution, for any* $\lambda \geq \lambda_0$.

In the next theorem we will remove the hypothesis $q \in \mathcal{F}$ and strengthen $(\mathcal{V}_3)$ by replacing it with a more precise condition about the behaviour of $V_\lambda(x)$ near the origin, for $\lambda$ sufficiently large.

**Theorem 1.2.** *Assume that* $(f_1)$–$(f_3)$, $(\mathcal{V}_1)$, $(\mathcal{V}_2)$, $(\mathcal{V}_4)$ *hold, and* $(\mathcal{V}_5)$ *there exist constants* $C_1, \eta_0, \kappa > 0$ *such that*

$$V_\lambda(x) \leq C_1\left(|x|^\kappa + \lambda^{-\frac{\kappa}{\kappa+p}}\right) \quad \text{for all } |x| \leq \eta_0 \lambda^{-\frac{1}{\kappa+p}},$$

*and*

$$\frac{p}{\kappa+p}\left(\frac{\delta_0}{\delta_0 - p} - \frac{N}{p}\right) < \frac{q}{q-p} - \frac{N}{p},$$

*for some* $\delta_0 \in \mathcal{F}$.

*Then there exists* $\lambda_0 \geq 1$, *depending only on the various constants involved in the assumptions, such that* (1.1) *has a nontrivial solution, for any* $\lambda \geq \lambda_0$.

## 2. Preliminary results

We look for solutions of (1.1) in the following subspace

$$X_\lambda = \left\{u \in W^{1,p}(\mathbb{R}^N) \Big/ \int_{\mathbb{R}^N} V_\lambda(x)|u|^p dx < \infty\right\},$$

endowed with the norm

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx\right)^{1/p}.$$

**Remark 2.1.** It follows from $(\mathcal{V}_1)$, $(\mathcal{V}_2)$ and Poincaré's inequality for the set $\Omega_{M,\lambda}$ that there exists $C_\lambda > 0$ such that

$$\|u\|_{1,p} \leq C_\lambda \|u\|_\lambda \quad \text{for all } u \in X_\lambda,$$

where $\|.\|_{1,p}$ is the standard norm on $W^{1,p}(\mathbb{R}^N)$. Then the space $(X_\lambda, \|.\|_\lambda)$ is continuously embedded into $(W^{1,p}(\mathbb{R}^N), \|.\|_{1,p})$. Moreover, $(X_\lambda, \|.\|_\lambda)$ is a reflexive Banach space.

We consider the energy functional $J_\lambda : X_\lambda \mapsto \mathbb{R}$, given by

$$J_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx - \int_{\mathbb{R}^N} F(x,u)dx, \tag{2.1}$$

and the following weighted Lebesgue space

$$L^s_{m(x)}(\mathbb{R}^N) = \left\{ u : \text{ measurable function } \Big/ \int_{\mathbb{R}^N} m(x)|u|^s dx < \infty \right\},$$

with the norm

$$\|u\|_{L^s_{m(x)}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} m(x)|u|^s dx \right)^{1/s}.$$

**Lemma 2.1.** *Assume that* $(\mathcal{V}_1)$, $(\mathcal{V}_2)$ *and* $(\mathcal{V}_4)$ *hold. Then for every* $s \in [p, p^\sharp)$ *and* $\lambda \geq 1$ *there exists* $C_\lambda > 0$ *such that*

$$\|u\|^s_{L^s_{m(x)}(\mathbb{R}^N)} \leq C_\lambda \|u\|^s_\lambda \quad \textit{for all } u \in X_\lambda.$$

**Proof.** By $(\mathcal{V}_1)$ and $(\mathcal{V}_4)$ for $u \in X_\lambda$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} m(x)|u|^s dx &= \int_{|x|\geq R_0} m(x)|u|^s dx + \int_{|x|<R_0} m(x)|u|^s dx \\
&\leq C_0 \left( \int_{\mathbb{R}^N} |u|^s dx + \int_{\mathbb{R}^N} V_\lambda(x)^{\frac{1}{\alpha}} |u|^s dx \right) + \|m\|_{L^\infty(B_R)} \int_{\mathbb{R}^N} |u|^s dx \\
&\leq \left(C_0 + \|m\|_{L^\infty(B_R)}\right) \int_{\mathbb{R}^N} |u|^s dx + C_0 \left( \int_{\mathbb{R}^N} V_\lambda(x)|u|^p dx \right)^{\frac{1}{\alpha}} \times \left( \int_{\mathbb{R}^N} |u|^{\frac{\alpha s - p}{\alpha - 1}} dx \right)^{\frac{\alpha-1}{\alpha}} \\
&\leq C \left( \int_{\mathbb{R}^N} |u|^s dx + \left( \int_{\mathbb{R}^N} |u|^{\frac{\alpha s - p}{\alpha - 1}} dx \right)^{\frac{\alpha-1}{\alpha}} \|u\|_\lambda^{\frac{p}{\alpha}} \right).
\end{aligned} \tag{2.2}$$

Since $X_\lambda$ is continuously embedded into $W^{1,p}(\mathbb{R}^N)$ and $p \leq s \leq \frac{\alpha s - p}{\alpha - 1} < p^\star$,

$$\int_{\mathbb{R}^N} m(x)|u|^s dx \leq C_\lambda \left( \|u\|^s_\lambda + \|u\|_\lambda^{\frac{p}{\alpha}} \|u\|_\lambda^{\frac{\alpha s - p}{\alpha}} \right).$$

Hence

$$\int_{\mathbb{R}^N} m(x)|u|^s dx \leq C_\lambda \|u\|^s_\lambda. \quad \square$$

**Lemma 2.2.** *Assume that* $(\mathcal{V}_1)$, $(\mathcal{V}_2)$, $(\mathcal{V}_4)$ *and* $(f_1)$–$(f_3)$ *hold. Then for* $\lambda \geq 1$, *the functional* $J_\lambda$ *is well defined and of class* $C^1$ *on* $X_\lambda$. *Furthermore, for all* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *such that*

$$\int_{\mathbb{R}^N} F(x,u)dx \leq \varepsilon \|u\|^p_\lambda + C_\varepsilon \|u\|^q_\lambda \quad \textit{for all } u \in X_\lambda. \tag{2.3}$$

**Proof.** By $(f_1)$ and $(f_2)$, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$F(x,t) \leq m(x)(\varepsilon |t|^p + C_\varepsilon |t|^q). \tag{2.4}$$

Since $p, q \in [p, p^\sharp)$, (2.3) follows from Lemma 2.1. It is standard to see that $J_\lambda$ is of $C^1$ on $X_\lambda$. $\square$

**Lemma 2.3.** *For* $\lambda \geq 1$, *there exist* $r > 0$ *and* $u_0 \in X_\lambda$ *such that* $\|u_0\|_\lambda > r$ *and*

$$J_\lambda(u_0) < 0 = J_\lambda(0) < \inf_{\|u\|_\lambda = r} J_\lambda(u).$$

**Proof.** It follows from (2.3) that

$$J_\lambda(u) \geq \frac{1}{p}\|u\|_\lambda^p - \varepsilon\|u\|_\lambda^p - C_\varepsilon\|u\|_\lambda^q.$$

Choose $\varepsilon = \frac{1}{2p}$, thus

$$J_\lambda(u) \geq C_2\|u\|_\lambda^p - C_3\|u\|_\lambda^q.$$

Since $p < q$, we can find $r > 0$ such that $\inf_{\|u\|_\lambda = r} J_\lambda(u) \geq \rho > 0$.

Let $\phi_0 \in C_0^\infty(\mathbb{R}^N)$ such that $\|\phi_0\|_\lambda = 1$. By ($f_3$), for $t > 0$ we have

$$J_\lambda(t\phi_0) \leq \frac{1}{p}t^p - t^{\mu_0}\int_{\mathbb{R}^N}\gamma_0(x)|\phi_0|^{\mu_0}dx.$$

Therefore $J_\lambda(t\phi_0) \to -\infty$ as $t \to +\infty$, so choose $u_0 = t\phi_0$ with $t$ large enough. □

Set

$$\Gamma = \{\phi \in C([0,1], X_\lambda) / \phi(0) = 0 \text{ and } J_\lambda(\phi(1)) < 0\}.$$

By Lemma 2.3, we see that $\Gamma \neq \emptyset$. We take

$$c_\lambda = \inf_{\phi\in\Gamma}\max_{t\in[0,1]} J_\lambda(\phi(t)).$$

Using a version of the mountain pass theorem without (PS) condition, there exists a sequence $(u_n^\lambda) \subset X_\lambda$ such that

$$J(u_n^\lambda) \to c_\lambda \quad \text{and} \quad J'(u_n^\lambda) \to 0 \quad \text{in } X_\lambda' \text{ as } n \to \infty. \tag{2.5}$$

Moreover, by Lemma 2.3 we see that $c_\lambda > 0$.

**Lemma 2.4.** *Let $\lambda \geq 1$. Then,*

(i) *there exists a constant $\sigma > 0$ independent of $\lambda$ such that*

$$\limsup_{n\to\infty}\|u_n^\lambda\|_\lambda^p \leq \sigma c_\lambda.$$

(ii) *there exists a weak solution $u^\lambda$ of (1.1) such that a subsequence of $(u_n^\lambda)$ converges to $u^\lambda$ weakly in $X_\lambda$.*

**Proof.** (i) Using ($f_3$), for n large enough we have

$$\begin{aligned} C + \|u_n^\lambda\|_\lambda &\geq J_\lambda(u_n^\lambda) - \frac{1}{\mu}\langle J_\lambda'(u_n^\lambda), u_n^\lambda\rangle \\ &\geq \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n^\lambda\|_\lambda^p. \end{aligned}$$

This shows that $u_n^\lambda$ is bounded in $X_\lambda$, and the desired result follows from the following inequality

$$\left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n^\lambda\|_\lambda^p \leq J_\lambda(u_n^\lambda) - \frac{1}{\mu}\langle J_\lambda'(u_n^\lambda), u_n^\lambda\rangle \to c_\lambda,$$

(ii) Since $u_n^\lambda$ is bounded in $X_\lambda$, up to a subsequence, we may assume that

$$\begin{cases} u_n^\lambda \rightharpoonup u^\lambda & \text{weakly in } X_\lambda, \\ u_n^\lambda \to u^\lambda & \text{a.e. in } \mathbb{R}^N, \\ u_n^\lambda \to u^\lambda & \text{in } L_{loc}^s(\mathbb{R}^N),\ s \in [p, p^\star). \end{cases} \tag{2.6}$$

Let $R > 0$ and $0 \leq \psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \equiv 1$ on $B_R$. Then,

$$\langle J_\lambda'(u_n^\lambda) - J_\lambda'(u^\lambda), \psi(u_n^\lambda - u^\lambda)\rangle = o_n(1). \tag{2.7}$$

It is well known that the following inequality

$$(|\xi|^{t-2}\xi - |\eta|^{t-2}\eta)(\xi - \eta) > 0 \tag{2.8}$$

holds for any $t > 1$ and $\xi, \eta \in \mathbb{R}^N$ with $\xi \neq \eta$. Thus by using the fact that $\psi \equiv 1$ on $B_R$, we get

$$\begin{aligned}
&\int_{B_R} (|\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda - |\nabla u^\lambda|^{p-2}\nabla u^\lambda)\nabla(u_n^\lambda - u^\lambda)dx \\
&\le \int_{\mathbb{R}^N} \psi(x)(|\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda - |\nabla u^\lambda|^{p-2}\nabla u^\lambda)\nabla(u_n^\lambda - u^\lambda)dx \\
&= \langle J_\lambda'(u_n^\lambda) - J_\lambda'(u^\lambda), \psi(u_n^\lambda - u^\lambda)\rangle - \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda \nabla\psi(x)(u_n^\lambda - u^\lambda)dx \\
&\quad + \int_{\mathbb{R}^N} |\nabla u^\lambda|^{p-2}\nabla u^\lambda \nabla\psi(x)(u_n^\lambda - u^\lambda)dx - \lambda\int_{\mathbb{R}^N} V_\lambda(x)|u_n^\lambda|^{p-2}u_n^\lambda(u_n^\lambda - u^\lambda)\psi(x)dx \\
&\quad + \lambda\int_{\mathbb{R}^N} V_\lambda(x)|u^\lambda|^{p-2}u^\lambda(u_n^\lambda - u^\lambda)\psi(x)dx + \int_{\mathbb{R}^N}(f(x,u_n^\lambda) - f(x,u))(u_n^\lambda - u^\lambda)\psi(x)dx. \qquad (2.9)
\end{aligned}$$

By Hölder's inequality and using the fact that $u_n^\lambda$ is bounded, $\psi \in C_0^\infty(\mathbb{R}^N)$ and $u_n^\lambda \to u^\lambda$ in $L^p_{loc}(\mathbb{R}^N)$, it is easy to see that

$$\begin{cases}
\int_{\mathbb{R}^N} |\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda \nabla\psi(x)(u_n^\lambda - u^\lambda)dx \to 0, \\
\int_{\mathbb{R}^N} |\nabla u^\lambda|^{p-2}\nabla u^\lambda \nabla\psi(x)(u_n^\lambda - u^\lambda)dx \to 0, \\
\int_{\mathbb{R}^N} V_\lambda(x)|u_n^\lambda|^{p-2}u_n^\lambda(u_n^\lambda - u^\lambda)\psi(x)dx \to 0, \\
\int_{\mathbb{R}^N} V_\lambda(x)|u^\lambda|^{p-2}u^\lambda(u_n^\lambda - u^\lambda)\psi(x)dx \to 0.
\end{cases} \qquad (2.10)$$

By ($f_1$) and ($f_2$), for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \le m(x)(\varepsilon|t|^{p-1} + C_\varepsilon|t|^{q-1}). \qquad (2.11)$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(f(x,u_n^\lambda) - f(x,u^\lambda))(u_n^\lambda - u^\lambda)\psi(x)|dx \\
&\le |\psi|_\infty \int_{\mathrm{supp}\psi} m(x)\left[\varepsilon\left(|u_n^\lambda|^{p-1} + |u^\lambda|^{p-1}\right) + C_\varepsilon\left(|u_n^\lambda|^{q-1} + |u^\lambda|^{q-1}\right)\right]|u_n^\lambda - u^\lambda|dx \\
&\le \varepsilon|\psi|_\infty \int_{\mathbb{R}^N} m(x)\left(|u_n^\lambda|^p + |u^\lambda|^p + |u_n^\lambda|^{p-1}|u^\lambda| + |u^\lambda|^{p-1}|u_n^\lambda|\right)dx \\
&\quad + C_\varepsilon|\psi|_\infty\left(\int_{\mathrm{supp}\psi} m(x)|u_n^\lambda|^{q-1}|u_n^\lambda - u^\lambda|dx + \int_{\mathrm{supp}\psi} m(x)|u^\lambda|^{q-1}|u_n^\lambda - u^\lambda|dx\right). \qquad (2.12)
\end{aligned}$$

Using the fact that $u_n^\lambda$ is bounded in $X_\lambda$, it follows from Lemma 2.1 that

$$a_1 := \sup_n \int_{\mathbb{R}^N} m(x)|u_n^\lambda|^p dx < \infty. \qquad (2.13)$$

By Hölder's inequality and (2.13) we also have

$$a_2 := \sup_n \int_{\mathbb{R}^N} m(x)|u_n^\lambda|^{p-1}|u^\lambda|dx < \infty$$

and

$$a_3 := \sup_n \int_{\mathbb{R}^N} m(x)|u^\lambda|^{p-1}|u_n^\lambda|dx < \infty$$

It follows from (2.12) that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |(f(x,u_n^\lambda) - f(x,u^\lambda))(u_n^\lambda - u^\lambda)\psi(x)|dx \le \varepsilon|\psi|_\infty\left(a_1 + a_2 + a_3 + \int_{\mathbb{R}^N} m(x)|u^\lambda|^p\right) \\
&\quad + C_\varepsilon|\psi|_\infty\left[\left(\int_{\mathbb{R}^N} m(x)|u_n^\lambda|^q dx\right)^{\frac{q-1}{q}} + \left(\int_{\mathbb{R}^N} m(x)|u^\lambda|^q dx\right)^{\frac{q-1}{q}}\right]\left(\int_{\mathrm{supp}\psi} m(x)|u_n^\lambda - u^\lambda|^q dx\right)^{\frac{1}{q}}. \qquad (2.14)
\end{aligned}$$

Using again Lemma 2.1 we have

$$\sup_n \int_{\mathbb{R}^N} m(x)|u_n^\lambda|^q dx < \infty. \tag{2.15}$$

Since $u_n^\lambda \to u^\lambda$ in $L^q_{loc}(\mathbb{R}^N)$,

$$\int_{\mathrm{supp}\psi} m(x)|u_n^\lambda - u^\lambda|^q dx \le \|m\|_{L^\infty(\mathrm{supp}\psi)} \int_{\mathrm{supp}\psi} |u_n^\lambda - u^\lambda|^q dx \to 0. \tag{2.16}$$

Hence, it follows from (2.14)–(2.16) that

$$\int_{\mathbb{R}^N} (f(x, u_n^\lambda) - f(x, u^\lambda))(u_n^\lambda - u^\lambda)\psi(x) dx \to 0. \tag{2.17}$$

Combining (2.9), (2.10) and (2.17) we deduce that

$$\int_{B_R} (|\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda - |\nabla u^\lambda|^{p-2}\nabla u^\lambda)\nabla(u_n^\lambda - u^\lambda) dx \to 0. \tag{2.18}$$

Now we recall the following result.

**Lemma 2.5** (*Lemma 2.7 in [16]*). *Suppose that $p > 1$, $\Omega$ is an open set in $\mathbb{R}^N$ and $a(x, \xi) \in C^0(\Omega \times \mathbb{R}^N, \mathbb{R}^N)$ is such that*

$$\alpha_0|\xi|^p \le a(x, \xi).\xi$$
$$|a(x, \xi)| \le \alpha_1|\xi|^{p-1},$$

*for some $\alpha_0, \alpha_1 > 0$, and*

$$(a(x, \xi) - a(x, \eta))(\xi - \eta) > 0,$$

*for any $\xi, \eta \in \mathbb{R}^N$ with $\xi \ne \eta$.*

*Suppose that $u_n, u \in W^{1,p}(\Omega)$, $n = 1, 2, \dots$, then*

$$\lim_{n\to\infty} \int_\Omega [a(x, \nabla u_n) - a(x, \nabla u)] (\nabla u_n - \nabla u) dx = 0$$

*if and only if $\nabla u_n \to \nabla u$ in $L^p(\Omega)$.*

By Lemma 2.5, (2.8) and (2.18) we see that $\nabla u_n \to \nabla u$ in $L^p(B_R)$. Since $R > 0$ is arbitrary, we see that $\nabla u_n \to \nabla u$ a.e. in $\mathbb{R}^N$. Then, by Vitali's theorem, (2.6) and (2.11), it is easy to see that for every $\varphi \in C_0^\infty(\mathbb{R}^N)$ we have

$$\begin{cases} \int_{\mathbb{R}^N} |\nabla u_n^\lambda|^{p-2}\nabla u_n^\lambda \nabla\varphi(x) dx \to \int_{\mathbb{R}^N} |\nabla u^\lambda|^{p-2}\nabla u^\lambda \nabla\varphi(x) dx \\ \int_{\mathbb{R}^N} V_\lambda(x)|u_n^\lambda|^{p-2}u_n^\lambda \varphi(x) dx \to \int_{\mathbb{R}^N} V_\lambda(x)|u^\lambda|^{p-2}u^\lambda \varphi(x) dx \\ \int_{\mathbb{R}^N} f(x, u_n^\lambda)\varphi(x) dx \to \int_{\mathbb{R}^N} f(x, u^\lambda)\varphi(x) dx. \end{cases}$$

Therefore

$$\langle J_\lambda'(u^\lambda), \varphi\rangle = \lim_{n\to\infty} \langle J_\lambda'(u_n^\lambda), \varphi\rangle = 0,$$

hence $u^\lambda$ is a weak solution of (1.1). □

Now we show that $u^\lambda$ is nontrivial for $\lambda$ sufficiently large.

**Lemma 2.6.** *Suppose that the assumptions of either Theorem 1.1 or Theorem 1.2 hold. Then*

$$\lim_{\lambda\to\infty} \lambda^{\frac{N}{p} - \frac{q}{q-p}} c_\lambda = 0.$$

**Proof.** We use the technique presented in [8]. Note that

$$c_\lambda \le \inf_{u\in X_\lambda\setminus\{0\}} \max_{t\ge 0} J_\lambda(tu). \tag{2.19}$$

Let $\delta \in \mathcal{F}$. Then

$$c_\lambda \le \inf_{u\in X_\lambda\setminus\{0\}} \max_{t\ge 0} \Phi_\lambda(tu), \tag{2.20}$$

where

$$\Phi_\lambda(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx - \int_{\mathbb{R}^N} \gamma(x)|u|^\delta dx.$$

By a direct calculation, we see that

$$\max_{t\geq 0} \Phi_\lambda(tu) = \frac{\delta - p}{p\delta^{\frac{\delta}{\delta-p}}}\left(\frac{\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx}{\left(\int_{\mathbb{R}^N} \gamma(x)|u|^\delta dx\right)^{\frac{p}{\delta}}}\right)^{\frac{\delta}{\delta-p}}. \tag{2.21}$$

Set

$$g_\delta(\lambda) = \inf_{u\in X_\lambda\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + \lambda V_\lambda(x)|u|^p dx}{\left(\int_{\mathbb{R}^N} \gamma(x)|u|^\delta dx\right)^{\frac{p}{\delta}}}.$$

From (2.20) and (2.21) we deduce that

$$c_\lambda \leq C\left(g_\delta(\lambda)\right)^{\frac{\delta}{\delta-p}}. \tag{2.22}$$

Using the change of variables $y = \lambda^{\frac{1}{p}} x$ we get

$$\begin{aligned} g_\delta(\lambda) &= \lambda^{1+\frac{N}{\delta}-\frac{N}{p}} \inf_{u\in X_\lambda\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + V_\lambda(\lambda^{-\frac{1}{p}}y)|u|^p dy}{\left(\int_{\mathbb{R}^N} \gamma(\lambda^{-\frac{1}{p}}y)|u|^\delta dy\right)^{\frac{p}{\delta}}} \\ &= \lambda^{1-\frac{N(\delta-p)}{p\delta}} h_\delta(\lambda), \end{aligned} \tag{2.23}$$

where

$$h_\delta(\lambda) = \inf_{u\in X_\lambda\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + V_\lambda(\lambda^{-\frac{1}{p}}y)|u|^p dy}{\left(\int_{\mathbb{R}^N} \gamma(\lambda^{-\frac{1}{p}}y)|u|^\delta dy\right)^{\frac{p}{\delta}}}.$$

Suppose that $q \in \mathcal{F}$ (Theorem 1.1). We have the following claim.

**Claim 2.1.**

$$\lim_{\lambda\to\infty} h_q(\lambda) = 0.$$

**Proof of Claim.** Set

$$E_\lambda = \left\{u \in X_\lambda \Big/ \int_{\mathbb{R}^N} \gamma(\lambda^{-\frac{1}{p}}y)|u|^q dy = 1\right\}.$$

Then

$$h_q(\lambda) = \inf_{u\in E_\lambda} \int_{\mathbb{R}^N} |\nabla u|^p + V_\lambda(\lambda^{-\frac{1}{p}}y)|u|^p dy.$$

Suppose by contradiction that there exists a sequence $\lambda_m \to \infty$ as $m \to \infty$, such that

$$h_q(\lambda_m) \geq d_0 > 0. \tag{2.24}$$

Choose $u_n \in C_0^\infty(\mathbb{R}^N)$ such that

$$\lim_{n\to\infty} \|\nabla u_n\|_{L^p(\mathbb{R}^N)} = 0 \quad \text{and} \quad \|u_n\|_{L^q(\mathbb{R}^N)} = 1. \tag{2.25}$$

Let

$$v_{n,m} := \frac{u_n}{\left(\int_{\mathbb{R}^N} \gamma(\lambda_m^{-\frac{1}{p}}y)|u_n|^q dy\right)^{\frac{1}{q}}}.$$

Clearly $v_{n,m} \in E_{\lambda_m}$. Using the fact that $u_n \in C_0^\infty(\mathbb{R}^N)$ and $\gamma(x) \in C(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \gamma(\lambda_m^{-\frac{1}{p}} y)|u_n|^q dy \ge \gamma(\lambda_m^{-\frac{1}{p}} y_0) \int_{\mathbb{R}^N} |u_n|^q dy = \gamma(\lambda_m^{-\frac{1}{p}} y_0),$$

where $\gamma(\lambda_m^{-\frac{1}{p}} y_0) = \min_{y\in \mathrm{supp} u_n} \gamma(\lambda_m^{-\frac{1}{p}} y)$.

Since $\gamma(\lambda_m^{-\frac{1}{p}} y_0) \to \gamma(0)$ as $m \to \infty$, it follows that for every $n$ there exists $m_n$ such that for $m > m_n$

$$\int_{\mathbb{R}^N} \gamma(\lambda_m^{-\frac{1}{p}} y)|u_n|^q dy > \frac{\gamma(0)}{2}.$$

So, in view of (2.25) we can find $n_0$ such that for $m > m_{n_0}$

$$\int_{\mathbb{R}^N} |\nabla v_{n_0,m}|^p dy < \frac{d_0}{2}.$$

Hence by using (2.24) we obtain

$$\int_{\mathbb{R}^N} V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y)|v_{n_0,m}|^p dy \ge \frac{d_0}{2} \quad \text{for } m > m_{n_0}. \tag{2.26}$$

By $(\mathcal{V}_3)$ (Theorem 1.1), for $m > m_{n_0}$ we have

$$\begin{aligned}\int_{\mathbb{R}^N} V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y)|v_{n_0,m}|^p dy &\le V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y_1) \int_{\mathbb{R}^N} |v_{n_0,m}|^p dy \\ &\le V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y_1) \left(\frac{\gamma(0)}{2}\right)^{-\frac{p}{q}} \int_{\mathbb{R}^N} |u_{n_0}|^p dy \to 0 \quad \text{as } m \to \infty,\end{aligned}$$

where $V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y_1) = \max_{y\in \mathrm{supp} u_{n_0}} V_{\lambda_m}(\lambda_m^{-\frac{1}{p}} y)$. This contradicts with (2.26), and the claim follows. □

From (2.22) and (2.23) we get

$$c_\lambda \le C\lambda^{\frac{q}{q-p}-\frac{N}{p}} (h_q(\lambda))^{\frac{q}{q-p}},$$

in view of Claim 2.1 we obtain

$$\lim_{\lambda\to\infty} \lambda^{\frac{N}{p}-\frac{q}{q-p}} c_\lambda = 0.$$

Now, suppose that $(\mathcal{V}_5)$ holds (Theorem 1.2). Then

$$\begin{aligned} h_{\delta_0}(\lambda) &\le \inf_{u\in W_0^{1,p}(B_\varrho)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + C_1\left(\lambda^{-\frac{\kappa}{p}}|y|^\kappa + \lambda^{-\frac{\kappa}{\kappa+p}}\right)|u|^p dy}{\left(\int_{\mathbb{R}^N} \gamma(\lambda^{-\frac{1}{p}} y)|u|^{\delta_0} dy\right)^{\frac{p}{\delta_0}}} \\ &\le C \inf_{u\in W_0^{1,p}(B_\varrho)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + \left(\lambda^{-\frac{\kappa}{p}}|y|^\kappa + \lambda^{-\frac{\kappa}{\kappa+p}}\right)|u|^p dy}{\left(\int_{\mathbb{R}^N} |u|^{\delta_0} dy\right)^{\frac{p}{\delta_0}}},\end{aligned}$$

where $\varrho = \eta_0 \lambda^{\frac{\kappa}{p(\kappa+p)}}$. By making the change of variables $z = \lambda^{-\frac{\kappa}{p(\kappa+p)}} y$ we obtain

$$\begin{aligned} h_{\delta_0}(\lambda) &\le C\lambda^{\frac{-\kappa}{\kappa+p}\left(1-\frac{N}{p}+\frac{N}{\delta_0}\right)} \inf_{u\in W_0^{1,p}(B_{\eta_0})\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + (|z|^\kappa + 1)|u|^p dz}{\left(\int_{\mathbb{R}^N} |u|^{\delta_0} dz\right)^{\frac{p}{\delta_0}}} \\ &\le C\lambda^{\frac{-\kappa}{\kappa+p}\left(1-\frac{N(\delta_0-p)}{p\delta_0}\right)} \inf_{u\in W_0^{1,p}(B_{\eta_0})\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^p + \left(\eta_0^\kappa + 1\right)|u|^p dz}{\left(\int_{\mathbb{R}^N} |u|^{\delta_0} dz\right)^{\frac{p}{\delta_0}}} \\ &\le C\lambda^{\frac{-\kappa}{\kappa+p}\left(1-\frac{N(\delta_0-p)}{p\delta_0}\right)}.\end{aligned}$$

This and (2.23) imply

$$g_{\delta_0}(\lambda) \le C\lambda^{\frac{p}{\kappa+p}\left(1-\frac{N(\delta_0-p)}{p\delta_0}\right)}.$$

Hence, it follows from (2.22) that

$$\lambda^{\frac{N}{p}-\frac{q}{q-p}}c_\lambda \le C\lambda^{\frac{p}{\kappa+p}\left(\frac{\delta_0}{\delta_0-p}-\frac{N}{p}\right)-\left(\frac{q}{q-p}-\frac{N}{p}\right)}.$$

Consequently by $(\mathcal{V}_5)$,

$$\lim_{\lambda\to\infty}\lambda^{\frac{N}{p}-\frac{q}{q-p}}c_\lambda = 0.$$

The proof of Lemma 2.6 is complete. $\square$

**Lemma 2.7.** *For every $\lambda \ge 1$ there exists $R(\lambda) \ge R_0$, with $R_0$ given by $(\mathcal{V}_4)$, such that*

$$\limsup_{n\to\infty}\|u_n^\lambda\|^p_{L^p(\mathbb{R}^N\setminus B_{R(\lambda)})} \le \overline{\sigma}\frac{c_\lambda}{\lambda},$$

*where $\overline{\sigma} > 0$ is a constant independent of $\lambda$.*

**Proof.** In view of Lemma 2.4(i), for $R > 0$ we have

$$\begin{aligned}
\sigma c_\lambda + o_n(1) &\ge \int_{\mathbb{R}^N}|\nabla u_n^\lambda|^p + \lambda V_\lambda(x)|u_n^\lambda|^p dx\\
&\ge \int_{\mathbb{R}^N}|\nabla u_n^\lambda|^p dx + \lambda M\int_{\mathbb{R}^N\setminus\Omega_{M,\lambda}}|u_n^\lambda|^p dx\\
&\ge \int_{\mathbb{R}^N}|\nabla u_n^\lambda|^p dx + \lambda M\int_{\mathbb{R}^N}|u_n^\lambda|^p dx - \lambda M\int_{\Omega_{M,\lambda}\cap B_R}|u_n^\lambda|^p dx - \lambda M\int_{\Omega_{M,\lambda}\setminus B_R}|u_n^\lambda|^p dx\\
&\ge \int_{\mathbb{R}^N}|\nabla u_n^\lambda|^p dx + \lambda M\int_{\mathbb{R}^N\setminus B_R}|u_n^\lambda|^p dx - \lambda M\int_{\Omega_{M,\lambda}\setminus B_R}|u_n^\lambda|^p dx.
\end{aligned}\tag{2.27}$$

On the other hand, by Hölder and Sobolev inequalities

$$\begin{aligned}
\int_{\Omega_{M,\lambda}\setminus B_R}|u_n^\lambda|^p dx &\le |\Omega_{M,\lambda}\setminus B_R|^{1-\frac{p}{p^\star}}\|u_n^\lambda\|^p_{L^{p^\star}(\mathbb{R}^N)}\\
&\le C|\Omega_{M,\lambda}\setminus B_R|^{1-\frac{p}{p^\star}}\|\nabla u_n^\lambda\|^p_{L^p(\mathbb{R}^N)},
\end{aligned}$$

so, we can find $R = R(\lambda) \ge R_0$ such that

$$\int_{\Omega_{M,\lambda}\setminus B_R}|u_n^\lambda|^p dx \le \frac{1}{2\lambda M}\|\nabla u_n^\lambda\|^p_{L^p(\mathbb{R}^N)}.$$

From (2.27) we conclude

$$\begin{aligned}
\sigma c_\lambda + o_n(1) &\ge \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n^\lambda|^p dx + \lambda M\int_{\mathbb{R}^N\setminus B_R}|u_n^\lambda|^p dx\\
&\ge \lambda M\int_{\mathbb{R}^N\setminus B_R}|u_n^\lambda|^p dx,
\end{aligned}$$

and the desired result follows. The proof of Lemma 2.7 is complete. $\square$

Now, we turn to show that $u^\lambda \ne 0$ for $\lambda$ sufficiently large. Suppose by contradiction that there exists a sequence $\lambda_m \to \infty$ as $m \to \infty$, such that $u^{\lambda_m} \equiv 0$. Then

$$\begin{aligned}
c_{\lambda_m} &= \lim_{n\to\infty}\left(J_{\lambda_m}(u_n^{\lambda_m}) - \frac{1}{p}\langle J'_{\lambda_m}(u_n^{\lambda_m}), u_n^{\lambda_m}\rangle\right)\\
&= \lim_{n\to\infty}\left(\frac{1}{p}\int_{\mathbb{R}^N}u_n^{\lambda_m}f(x,u_n^{\lambda_m})dx - \int_{\mathbb{R}^N}F(x,u_n^{\lambda_m})dx\right)\\
&\le \frac{1}{p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}u_n^{\lambda_m}f(x,u_n^{\lambda_m})dx\\
&\le \frac{1}{p}\liminf_{n\to\infty}\int_{\mathbb{R}^N}\varepsilon m(x)|u_n^{\lambda_m}|^p + C_\varepsilon m(x)|u_n^{\lambda_m}|^q dx
\end{aligned}$$

$$\leq \|m\|_{L^\infty(B_R)} \liminf_{n\to\infty} \int_{B_R} \varepsilon |u_n^{\lambda_m}|^p + C_\varepsilon |u_n^{\lambda_m}|^q dx$$
$$+ \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} \varepsilon m(x)|u_n^{\lambda_m}|^p + C_\varepsilon m(x)|u_n^{\lambda_m}|^q dx. \tag{2.28}$$

Since $u_n^{\lambda_m} \rightharpoonup u^{\lambda_m} \equiv 0$ in $X_{\lambda_m}$, $u_n^{\lambda_m} \to 0$ in $L^s_{loc}(\mathbb{R}^N)$ for $s \in \{p, q\}$. It follows from (2.28) that

$$c_{\lambda_m} \leq \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} \varepsilon m(x)|u_n^{\lambda_m}|^p + C_\varepsilon m(x)|u_n^{\lambda_m}|^q dx$$
$$\leq \varepsilon \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} m(x)|u_n^{\lambda_m}|^p dx + C_\varepsilon \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} m(x)|u_n^{\lambda_m}|^q dx. \tag{2.29}$$

By $(\mathcal{V}_4)$ and the Gagliardo–Nirenberg inequality, for $s \in \{p, q\}$ and $R = R(\lambda_m)$, we have

$$\begin{aligned}\int_{\mathbb{R}^N\setminus B_R} m(x)|u_n^{\lambda_m}|^s dx &\leq C_0 \|u_n^{\lambda_m}\|^s_{L^s(\mathbb{R}^N\setminus B_R)} + C_0 \int_{\mathbb{R}^N\setminus B_R} V_{\lambda_m}(x)^{\frac{1}{\alpha}} |u_n^{\lambda_m}|^s dx \\ &\leq C\|u_n^{\lambda_m}\|^{(1-\theta)s}_{L^p(\mathbb{R}^N\setminus B_R)} \|\nabla u_n^{\lambda_m}\|^{\theta s}_{L^p(\mathbb{R}^N\setminus B_R)} \\ &\quad + C_0 \left(\int_{\mathbb{R}^N\setminus B_R} V_{\lambda_m}(x)|u_n^{\lambda_m}|^p dx\right)^{\frac{1}{\alpha}} \times \left(\int_{\mathbb{R}^N\setminus B_R} |u_n^{\lambda_m}|^{\frac{\alpha s-p}{\alpha-1}} dx\right)^{\frac{\alpha-1}{\alpha}} \\ &\leq C\|u_n^{\lambda_m}\|^{(1-\theta)s}_{L^p(\mathbb{R}^N\setminus B_R)} \|\nabla u_n^{\lambda_m}\|^{\theta s}_{L^p(\mathbb{R}^N\setminus B_R)} \\ &\quad + C\|u_n^{\lambda_m}\|^{(1-\bar\theta)\frac{\alpha s-p}{\alpha}}_{L^p(\mathbb{R}^N\setminus B_R)} \|\nabla u_n^{\lambda_m}\|^{\bar\theta\frac{\alpha s-p}{\alpha}}_{L^p(\mathbb{R}^N\setminus B_R)} \left(\int_{\mathbb{R}^N\setminus B_R} V_{\lambda_m}(x)|u_n^{\lambda_m}|^p dx\right)^{\frac{1}{\alpha}}\end{aligned} \tag{2.30}$$

where $\theta = \frac{N(s-p)}{ps}$ and $\bar\theta = \frac{N\alpha(s-p)}{p(\alpha s-p)}$. Using Lemma 2.4(i) and Lemma 2.7, we see that

$$\limsup_{n\to\infty} \|\nabla u_n^{\lambda_m}\|^p_{L^p(\mathbb{R}^N\setminus B_R)} \leq \sigma c_{\lambda_m}, \qquad \limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} V_{\lambda_m}(x)|u_n^{\lambda_m}|^p dx \leq \sigma \frac{c_{\lambda_m}}{\lambda_m}$$

and

$$\limsup_{n\to\infty} \|u_n^{\lambda_m}\|^p_{L^p(\mathbb{R}^N\setminus B_R)} \leq \overline{\sigma}\frac{c_{\lambda_m}}{\lambda_m}.$$

By (2.30) we obtain

$$\begin{aligned}\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus B_R} m(x)|u_n^{\lambda_m}|^s dx &\leq C\left[\left(\frac{c_{\lambda_m}}{\lambda_m}\right)^{\frac{(1-\theta)s}{p}} c_{\lambda_m}^{\frac{\theta s}{p}} + \left(\frac{c_{\lambda_m}}{\lambda_m}\right)^{\frac{(1-\bar\theta)(\alpha s-p)}{\alpha p}} c_{\lambda_m}^{\frac{\bar\theta(\alpha s-p)}{\alpha p}} \left(\frac{c_{\lambda_m}}{\lambda_m}\right)^{\frac{1}{\alpha}}\right] \\ &\leq C\lambda_m^{-\frac{(1-\theta)s}{p}} c_{\lambda_m}^{\frac{s}{p}}.\end{aligned} \tag{2.31}$$

So, it follows from (2.29) that

$$c_{\lambda_m} \leq C\left(\varepsilon\lambda_m^{-1}c_{\lambda_m} + C_\varepsilon \lambda_m^{\frac{N(q-p)-pq}{p^2}} c_{\lambda_m}^{\frac{q}{p}}\right),$$

and hence

$$(1 - C\varepsilon)c_{\lambda_m} \leq CC_\varepsilon \lambda_m^{\frac{N(q-p)-pq}{p^2}} c_{\lambda_m}^{\frac{q}{p}}. \tag{2.32}$$

Choose $\varepsilon$ sufficiently small in (2.32), we get

$$c_{\lambda_m} \leq C\lambda_m^{\frac{N(q-p)-pq}{p^2}} c_{\lambda_m}^{\frac{q}{p}},$$

thus

$$0 < C \leq \lambda_m^{\frac{N}{p}-\frac{q}{q-p}} c_{\lambda_m},$$

and consequently

$$\limsup_{m\to\infty} \lambda_m^{\frac{N}{p}-\frac{q}{q-p}} c_{\lambda_m} > 0.$$

This contradicts with Lemma 2.6, and the proof of Theorems 1.1 and 1.2 is complete.

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