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# Hyperasymptotics and hyperterminants: Exceptional cases

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ABSTRACT

A new method is introduced for the computation of hyperterminants. It is based on recurrence relations, and can also be used to compute the parameter derivatives of the hyperterminants. These parameter derivatives are needed in hyperasymptotic expansions in exceptional cases. Numerical illustrations and an application are included.



## 1. Hyperterminants

In the last two decades hyperasymptotic expansions were constructed for solutions of differential equations and difference equations, and for integrals with saddles. See [1–12]. In this way exponentially small phenomena were incorporated in the expansions, and it gave a powerful method to compute the so-called Stokes multipliers, or connection coefficients, to arbitrary precision [8]. Hyperasymptotic expansions also incorporate the higher-order Stokes phenomenon, which seems to play an important role in some partial differential equations, [13,6].

Hyperasymptotic expansions are in terms of hyperterminants. In [14] the hyperterminants are defined and a new integral representation is used to obtain convergent expansions for the hyperterminants in series of confluent hypergeometric functions. For the coefficients in these expansions a recursive scheme is given. These expansions can be used to compute the hyperterminants to any given accuracy.

In the papers mentioned above, the asymptotic approximations are of the form $w_j(z) \sim \mathrm{e}^{\lambda_j z} z^{\mu_j}, j = 1, \ldots, n$, as $|z| \to \infty$. It is usually assumed that $\lambda_j \neq \lambda_k$, whenever $j \neq k$. In the case that there are $j, k$ such that $j \neq k$, $\lambda_j = \lambda_k$ and $\mu_j - \mu_k$ is an integer, extra logarithmic factors, $\ln z$, appear in the expansions, and new methods are needed to compute the corresponding hyperterminants. For examples see [1,3] and the main application in this paper. Note that $\frac{\mathrm{d}}{\mathrm{d}\mu} z^{\mu} = \ln(z) z^{\mu}$. Hence, the new logarithmic factor can be seen as the result of a $\mu$-derivative of the original expansion.

In this paper we construct an alternative method based on recurrence relations for the computation of the hyperterminants. As is shown in [15], the computation of parameter derivatives of solutions of recurrence relations is not a big problem. Taking a parameter derivative of a linear recurrence relation does not change the shape of the recurrence relation itself. Hence, if it is possible to use the recurrence relation to compute its solutions numerically, then it is also possible to use the recurrence relation to compute the parameter derivatives of its solutions.

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The definition of the hyperterminants is

$$
\begin{aligned}
&F^{(0)}(z) = 1\\
&F^{(1)}\left(z;\ \begin{matrix} M_0 \\ \sigma_0 \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \frac{e^{\sigma_0 t_0} t_0^{M_0-1}}{z-t_0}\, dt_0\\
&F^{(\ell+1)}\left(z;\ \begin{matrix} M_0,\ldots,M_\ell \\ \sigma_0,\ldots,\sigma_\ell \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_\ell]} \frac{e^{\sigma_0 t_0+\cdots+\sigma_\ell t_\ell} t_0^{M_0-1} \cdots t_\ell^{M_\ell-1}}{(z-t_0)(t_0-t_1)\cdots(t_{\ell-1}-t_\ell)}\, dt_\ell \cdots dt_0,
\end{aligned}
\tag{1.1}
$$

where we use the notation $\theta_j = \operatorname{ph}\sigma_j$ and $\int^{[\eta]} = \int^{\infty e^{i\eta}}$. In [14] we also give an alternative integral representation. From integral representation (1.1) it is obvious that the hyperterminants are multi-valued functions with respect to $z$, but also with respect to $\sigma_j$. Connection relations with respect to all these variables are given in [14].

In exceptional cases (see for example [1] and [3]) extra factors $\left(\ln t_j\right)^n$ appear in the integrand, and these new functions can be seen as parameter derivatives of the original hyperterminants:

$$
\frac{\partial^n}{\partial M_j^n} F^{(\ell+1)}\left(z;\ \begin{matrix} M_0,\ldots,M_\ell \\ \sigma_0,\ldots,\sigma_\ell \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_\ell]} \frac{e^{\sigma_0 t_0+\cdots+\sigma_\ell t_\ell} t_0^{M_0-1} \cdots t_\ell^{M_\ell-1} \left(\ln t_j\right)^n}{(z-t_0)(t_0-t_1)\cdots(t_{\ell-1}-t_\ell)}\, dt_\ell \cdots dt_0. \tag{1.2}
$$

We could construct recurrence relations with respect to each of the $M_j$ parameters, but in applications one mainly needs recurrence relations with respect to the final $M_j$ parameter. In this paper we will use linear first-order recurrence relations with respect to the $M_\ell$ parameter. Our method requires that this parameter is not an integer. Since it is always possible to interchange the $M_j$ parameters, the method that we give in this paper will always work, except when all the $M_j$ are integers.

In section $\ell+1$, $\ell = 1, 2, 3$, we discuss the computation of the level $\ell$ hyperterminant. Each of these sections is split in two parts: First we deal with the case that the variable $z = 0$, and then we use these results and deal with the case $z \neq 0$. In applications $|z|$ is large, but for the computation of the Stokes multipliers we will need hyperterminants with $z = 0$. In Section 3 we also give a numerical illustration.

Finally, in Section 5 we apply these results and discuss the hyperasymptotics of a linear third-order differential equation in which logarithmic factors appear.

## 2. Level 1

We will assume that $M$ is not an integer and $z \neq 0$. The definition of the first hyperterminant reads

$$
\begin{aligned}
F^{(1)}\left(z;\ \begin{matrix} M \\ \sigma \end{matrix}\right) &= \int_0^{[\pi-\theta]} \frac{e^{\sigma t} t^{M-1}}{z-t}\, dt\\
&= e^{M\pi i}\sigma^{1-M} \int_0^{\infty} \frac{e^{-\tau}\tau^{M-1}}{z\sigma+\tau}\, d\tau = e^{M\pi i+\sigma z} z^{M-1}\Gamma(M)\Gamma(1-M,\sigma z),
\end{aligned}
\tag{2.1}
$$

when $|\operatorname{ph}(\sigma z)| < \pi$, where $\Gamma(a,z)$ is the incomplete gamma function (see Section 11.2 in [16]). The integrals in (2.1) converge for $\Re M > 0$. We use analytical continuation via the recurrence relation below to define this function for $\Re M \leqslant 0$. It follows that

$$
F^{(1)}\left(0;\ \begin{matrix} M \\ \sigma \end{matrix}\right) = e^{M\pi i}\sigma^{1-M}\Gamma(M-1). \tag{2.2}
$$

Hence,

$$
\frac{\partial}{\partial M} F^{(1)}\left(0;\ \begin{matrix} M \\ \sigma \end{matrix}\right) = \left(\pi i - \ln(\sigma) + \psi(M-1)\right) F^{(1)}\left(0;\ \begin{matrix} M \\ \sigma \end{matrix}\right), \tag{2.3}
$$

where $\psi(z)$ is the logarithmic derivative of the gamma function (see Section 3.4 in [16]).

For $r = 0, 1, 2, \ldots$, let

$$
y_r = F^{(1)}\left(z;\ \begin{matrix} M+r \\ \sigma \end{matrix}\right) \quad \text{and} \quad y_r' = \frac{\partial y_r}{\partial M}, \tag{2.4}
$$

then we have the recurrence relation

$$
y_{r+1} - z y_r = F^{(1)}\left(0;\ \begin{matrix} M+r+1 \\ \sigma \end{matrix}\right), \tag{2.5}
$$

with normalising condition

$$
\sum_{r=0}^{\infty} \frac{(-\sigma)^r y_r}{r!} = F^{(1)}\left(z;\ \begin{matrix} M \\ 0\sigma \end{matrix}\right) = e^{M\pi i} z^{M-1}\Gamma(M)\Gamma(1-M) = \frac{\pi e^{M\pi i} z^{M-1}}{\sin M\pi}, \quad \Re M < 1. \tag{2.6}
$$

These two results follow from the first integral representation in (2.1), where we need $0 < \Re M < 1$ for the proof of (2.6), and use analytic continuation to extend the result to $\Re M < 1$. Note that in the definitions (1.1) the phase of the

$\sigma$-variables determines the direction of integration. Since the hyperterminants are multi-valued functions with respect to the $\sigma$-variables we write $0\sigma$ in (2.6) to indicate the direction of integration.

From the second integral representation in (2.1) it follows that $y_r = \sigma^{-r}\Gamma(M+r)\mathcal{O}(1)$ as $r \to \infty$. The complementary solutions of recurrence relation (2.5) are $Cz^r$. Hence, the $y_r$ are dominant solutions and the use of (2.5) to evaluate the $y_r$ in the forward direction is numerically stable.

The computation of the $y_0$ is no problem at all and using (2.5) is useful when many $y_r$ are needed. However, we can compute the $y_r$ also directly from (2.5) and (2.6) without the correct initial value $y_0$ as follows: Let $\tilde{y}_r$ satisfy (2.5) with initial value $\tilde{y}_0 = 0$. Then there exists a constant $C$ such that $y_r = \tilde{y}_r + Cz^r$. Combining this relation with (2.6) we obtain that

$$\sum_{r=0}^{\infty} \frac{(-\sigma)^r \tilde{y}_r}{r!} + C\mathrm{e}^{-\sigma z} = \frac{\pi \mathrm{e}^{M\pi \mathrm{i}} z^{M-1}}{\sin M\pi}. \tag{2.7}$$

Hence,

$$C = \frac{\pi \mathrm{e}^{M\pi \mathrm{i}+\sigma z} z^{M-1}}{\sin M\pi} - \mathrm{e}^{\sigma z} \sum_{r=0}^{\infty} \frac{(-\sigma)^r \tilde{y}_r}{r!}. \tag{2.8}$$

By decreasing $\Re M$ the convergence speed of the infinite series in (2.8) can be increased substantially. In practice, one probably wants $\Re M \leqslant -5$. Note that since we have (2.5) we can replace $M$ by $\tilde{M} = M - \tilde{r}$, where $\tilde{r}$ is a positive integer, use the methods described above to compute the $y_0$ corresponding to $\tilde{M}$, and then use (2.5) to compute $y_{\tilde{r}}$.

Combining (2.3) with the $M$-derivatives of (2.5) and (2.6) we obtain that

$$y'_{r+1} - zy'_r = (\pi \mathrm{i} - \ln(\sigma) + \psi(M+r))\, F^{(1)}\left(0; \begin{matrix} M+r+1 \\ \sigma \end{matrix}\right), \tag{2.9}$$

with normalising condition

$$\sum_{r=0}^{\infty} \frac{(-\sigma)^r y'_r}{r!} = \left(\pi \mathrm{i} + \ln z - \frac{\pi \cos M\pi}{\sin M\pi}\right) \frac{\pi \mathrm{e}^{M\pi \mathrm{i}} z^{M-1}}{\sin M\pi}, \quad \Re M < 1. \tag{2.10}$$

As in the case of the $y_r$ the parameter derivatives $y'_r$ can be computed directly from these two results.

Note that the restriction that $M$ is not an integer is critical here, and that when we let $M$ approach an integer, the results will become useless.

## 3. Level 2

We will assume that $M_1$ is not an integer. First we will deal with the special values at $z = 0$, which will be needed when we create the recurrence relation for the case $z \neq 0$. In integral representation (1.1) with $\ell = 1$ take $t_1 = t_0\tau$

$$\begin{aligned} F^{(2)}\left(0; \begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \sigma_1 \end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \int_0^{[\theta_0-\theta_1]} \frac{\mathrm{e}^{t_0(\sigma_0+\sigma_1\tau)} t_0^{M_0+M_1-2} \tau^{M_1-1}}{\tau - 1}\, \mathrm{d}\tau\, \mathrm{d}t_0 \\ &= \mathrm{e}^{(M_0+M_1-1)\pi \mathrm{i}}\, \Gamma(M_0+M_1-1) \int_0^{[\theta_0-\theta_1]} \frac{\tau^{M_1-1}}{(\sigma_0+\sigma_1\tau)^{M_0+M_1-1}(\tau-1)}\, \mathrm{d}\tau. \end{aligned} \tag{3.1}$$

In this section and in applications it is more natural to add 1 to the $M_0$ parameter in the case $z = 0$. The final integral in (3.1) is of hypergeometric type and can be identified (via 3.6(2) in [17]) as

$$F^{(2)}\left(0; \begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \sigma_1 \end{matrix}\right) = \frac{\mathrm{e}^{(M_0+M_1)\pi \mathrm{i}}\, \Gamma(M_0)\Gamma(M_1)}{\sigma_0^{M_0-1}\sigma_1^{M_1}(M_0+M_1-1)}\, {}_2F_1\left(\begin{matrix} 1, M_1 \\ M_0+M_1 \end{matrix}; 1 + \frac{\sigma_0}{\sigma_1}\right). \tag{3.2}$$

Although this identification could be used to evaluate the left-hand side and its parameter derivatives, we will also give a method that is based on recurrence relations which can be generalised to higher levels.

Use the first line of (3.1) and let

$$v_r = F^{(2)}\left(0; \begin{matrix} M_0+1, & M_1+r \\ \sigma_0, & \sigma_1 \end{matrix}\right) = \int_0^{[\pi-\theta_0]} \int_0^{[\theta_0-\theta_1]} \frac{\mathrm{e}^{t_0(\sigma_0+\sigma_1\tau)} t_0^{M_0+M_1+r-2} \tau^{M_1+r-1}}{\tau - 1}\, \mathrm{d}\tau\, \mathrm{d}t_0. \tag{3.3}$$

Using integration by parts we obtain the recurrence relation

$$(\sigma_0+\sigma_1)\, v_{r+1} + (M_0+M_1+r-1)\, v_r = \sigma_0 F^{(1)}\left(0; \begin{matrix} M_0+1 \\ \sigma_0 \end{matrix}\right) F^{(1)}\left(0; \begin{matrix} M_1+r+1 \\ \sigma_1 \end{matrix}\right), \tag{3.4}$$

with the complementary solution

$$\frac{(M_0+M_1-1)_r}{(-\sigma_0-\sigma_1)^r}, \tag{3.5}$$

and normalising condition

$$\sum_{r=0}^{\infty} \frac{(\tilde{\sigma}_1-\sigma_1)^r\, v_r}{r!} = F^{(2)}\left(0;\ \begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \tilde{\sigma}_1 \end{matrix}\right). \tag{3.6}$$

Note that in the case that $\sigma_1 = 0$ the double integral in (3.1) de-couples and we obtain as a special case the normalising condition

$$\begin{aligned}\sum_{r=0}^{\infty} \frac{(-\sigma_1)^r\, v_r}{r!} &= F^{(1)}\left(0;\ \begin{matrix} M_0+M_1 \\ \sigma_0 \end{matrix}\right) \int_0^{[\theta_0-\theta_1]} \frac{\tau^{M_1-1}}{1-\tau}\,\mathrm{d}\tau \\ &= F^{(1)}\left(0;\ \begin{matrix} M_0+M_1 \\ \sigma_0 \end{matrix}\right) \frac{\pi\,\mathrm{e}^{\mp M_1\pi\mathrm{i}}}{\sin M_1\pi}, \quad \theta_0 \lessgtr \theta_1,\end{aligned} \tag{3.7}$$

where we take the $-$ sign when $\theta_0 < \theta_1$ and $+$ sign when $\theta_0 > \theta_1$. For the intermediate result in (3.7) we need the restriction $0 < \Re M_1 < 1$ and the final result holds for $\Re M_1 < 1$.

Now let us discuss the convergence of the infinite series in (3.6) and (3.7). From, for example, (3.2) it follows that $v_r = \sigma_1^{-r}\Gamma(M_1+r-1)\mathcal{O}(1)$ as $r \to \infty$. Assuming that $\Re M_1 < 1$ it follows that the infinite series (3.6) is absolutely convergent as long as $|\tilde{\sigma}_1-\sigma_1| \leqslant |\sigma_1|$ and infinite series (3.7) is absolutely convergent.

Comparing the asymptotic behaviour of $v_r$ with the complementary solution in (3.5) it follows that in the case that $|\sigma_1| < |\sigma_0+\sigma_1|$ our function $v_r$ is a dominant solution of recurrence relation (3.4) and in the case that $|\sigma_1| > |\sigma_0+\sigma_1|$ it is the recessive solution. There are three cases that we should consider: (1) $|1+(\sigma_0/\sigma_1)| < 1-\varepsilon$, (2) $|1+(\sigma_0/\sigma_1)| > 1+\varepsilon$, and (3) $||1+(\sigma_0/\sigma_1)|-1| \leqslant \varepsilon$. In practice, this $\varepsilon$ is positive, but not very small, say $\varepsilon = 0.3$.

In case (1) $v_r$ is the recessive solution. Take $N$ large, $\tilde{v}_N = 0$, and compute the other $\tilde{v}_r$ via backward recursion in (3.4). Then there is a constant $C$ such that

$$v_r = \tilde{v}_r + C\frac{(M_0+M_1-1)_r}{(-\sigma_0-\sigma_1)^r}, \tag{3.8}$$

for $r = 0, \ldots, N$. Since $v_r$ is the recessive solution and $\tilde{v}_N = 0$ it follows that $C$ is approximately zero, and $v_r \approx \tilde{v}_r$ for $r \ll N$.

In case (2) $v_r$ is a dominant solution, and we can copy the method of Section 2: Let $\tilde{v}_r$ satisfy (3.4) with initial value $\tilde{v}_0 = 0$. Then there exists a constant $C$ such that (3.8) holds for all $r$. We compute the constant $C$ via the normalising condition (3.7), that is, via the identity

$$C\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^{1-M_0-M_1} + \sum_{r=0}^{\infty} \frac{(-\sigma_1)^r\,\tilde{v}_r}{r!} = F^{(1)}\left(0;\ \begin{matrix} M_0+M_1 \\ \sigma_0 \end{matrix}\right) \frac{\pi\,\mathrm{e}^{\mp M_1\pi\mathrm{i}}}{\sin M_1\pi}, \quad \theta_0 \lessgtr \theta_1. \tag{3.9}$$

Finally, in case (3) we 'walk' in the $\sigma_1$-space, to end up with one of the other two cases. There is some freedom in this case. We choose $\tilde{\sigma}_1$ relatively close to $\sigma_1$ such that $|1+(\sigma_0/\tilde{\sigma}_1)|$ is either larger or smaller than $|1+(\sigma_0/\sigma_1)|$. The $\tilde{v}_r$ are computed in the backward or forward direction, depending on whether $|1+(\sigma_0/\sigma_1)| \lessgtr 1$. The corresponding constant $C$ in (3.8) follows from normalising condition (3.6), thus

$$C\left(\frac{\sigma_0+\tilde{\sigma}_1}{\sigma_0+\sigma_1}\right)^{1-M_0-M_1} + \sum_{r=0}^{\infty} \frac{(\tilde{\sigma}_1-\sigma_1)^r\,\tilde{v}_r}{r!} = F^{(2)}\left(0;\ \begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \tilde{\sigma}_1 \end{matrix}\right). \tag{3.10}$$

Note that the right-hand side of (3.10) is similar to $v_0$. The right-hand side of (3.10) should be easier to compute than $v_0$. Ideally, it is already of case (1) or (2) and we can use the methods given above to evaluate it. It might take several steps in the $\sigma_1$-space, but we will end up with a $\tilde{\sigma}_1$ such that we are in case (1) or (2), that is, such that $||1+(\sigma_0/\tilde{\sigma}_1)|-1| > \varepsilon$.

In implementations of this method one should, of course, use recursive procedures. In the $\sigma_1$-space one should 'walk' in the direction of either $\tilde{\sigma}_1 = -\sigma_0$, such that $1+(\sigma_0/\tilde{\sigma}_1) = 0$, or, say, $\tilde{\sigma}_1 = \sigma_0$, such that $1+(\sigma_0/\tilde{\sigma}_1) = 2$. Some care has to be taken: the hyperterminants are multi-valued functions with respect to $\sigma_1$, and in the $\sigma_1$-space they have the branch-cut $\{x\sigma_0 \mid x > 0\}$. Hence, it is better not to cross this half-line.

The method described above is, again, ideal for the computation of the parameter derivatives. When we take $v_r' = \partial v_r/\partial M_0$ then we obtain from (3.4) for $v_r'$ the recurrence relation

$$(\sigma_0+\sigma_1)\,v_{r+1}' + (M_0+M_1+r-1)\,v_r' = -v_r + \sigma_0\frac{\partial}{\partial M_0}F^{(1)}\left(0;\ \begin{matrix} M_0+1 \\ \sigma_0 \end{matrix}\right)F^{(1)}\left(0;\ \begin{matrix} M_1+r+1 \\ \sigma_1 \end{matrix}\right), \tag{3.11}$$

with as normalising conditions the $M_0$-derivatives of (3.6) and (3.7). Note that the left-hand sides of (3.4) and (3.11) are the same. Hence, the method given above can also be used to compute the $v_r'$, assuming that we already know the $v_r$. Similarly for the $M_1$-derivative of the $v_r$.

One identity that might be useful in implementing this method is

$$F^{(2)}\left(0;\begin{matrix}M_0+1, & M_1\\ \sigma_0, & \sigma_1\end{matrix}\right)+F^{(2)}\left(0;\begin{matrix}M_1+1, & M_0\\ \sigma_1, & \sigma_0\end{matrix}\right)=0. \tag{3.12}$$

Now we are able to compute the level 2 hyperterminants with $z=0$, we will use these results to compute the level 2 hyperterminants with $z\neq 0$. Let

$$y_r=F^{(2)}\left(z;\begin{matrix}M_0, & M_1+r\\ \sigma_0, & \sigma_1\end{matrix}\right). \tag{3.13}$$

Substituting $t_1=z-(z-t_0)-(t_0-t_1)$ into definition (1.1) we obtain the recurrence relation

$$y_{r+1}-zy_r=v_r+F^{(1)}\left(z;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right)F^{(1)}\left(0;\begin{matrix}M_1+r+1\\ \sigma_1\end{matrix}\right), \tag{3.14}$$

with normalising condition

$$\begin{aligned}\sum_{r=0}^{\infty}\frac{(-\sigma_1)^r\,y_r}{r!}&=F^{(2)}\left(z;\begin{matrix}M_0, & M_1\\ \sigma_0, & 0\sigma_1\end{matrix}\right)=F^{(1)}\left(z;\begin{matrix}M_0+M_1-1\\ \sigma_0\end{matrix}\right)\int_0^{[\theta_0-\theta_1]}\frac{\tau^{M_1-1}}{1-\tau}\,\mathrm{d}\tau\\&=F^{(1)}\left(z;\begin{matrix}M_0+M_1-1\\ \sigma_0\end{matrix}\right)\frac{\pi\,\mathrm{e}^{\mp M_1\pi\mathrm{i}}}{\sin M_1\pi},\quad\theta_0\lessgtr\theta_1,\end{aligned} \tag{3.15}$$

where in the integral representation (1.1) with $\ell=1$ of the $F^{(2)}$ function we have used the substitution $t_1=t_0\tau$. The final result is valid for $\Re M_1<1$.

As in Section 2 we can use these results to compute the dominant solution $y_r$ of recurrence relation (3.14): Let $\tilde{y}_r$ satisfy (3.14) with initial value $\tilde{y}_0=0$. Then there exists a constant $C$ such that $y_r=\tilde{y}_r+Cz^r$. Combining this relation with (3.15) we can compute $C$ via the identity

$$\sum_{r=0}^{\infty}\frac{(-\sigma_1)^r\,\tilde{y}_r}{r!}+C\mathrm{e}^{-\sigma_1 z}=F^{(1)}\left(z;\begin{matrix}M_0+M_1-1\\ \sigma_0\end{matrix}\right)\frac{\pi\,\mathrm{e}^{\mp M_1\pi\mathrm{i}}}{\sin M_1\pi},\quad\theta_0\lessgtr\theta_1. \tag{3.16}$$

By decreasing $\Re M_1$ the convergence speed of the infinite series in (3.16) can be increased substantially.

The results above can be used to compute the $M_j$-derivative of $y_r$. We omit the details since they are obvious.

Note, again, that the restriction that $M_1$ is not an integer is critical here, and that when we let $M_1$ approach an integer, the results will become useless. Identity (3.12) and

$$F^{(2)}\left(z;\begin{matrix}M_0, & M_1\\ \sigma_0, & \sigma_1\end{matrix}\right)+F^{(2)}\left(z;\begin{matrix}M_1, & M_0\\ \sigma_1, & \sigma_0\end{matrix}\right)=F^{(1)}\left(z;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right)F^{(1)}\left(z;\begin{matrix}M_1\\ \sigma_1\end{matrix}\right), \tag{3.17}$$

might be useful when $M_1$ is an integer and $M_0$ is not, and when the computation of the right-hand side of (3.17) is no problem. Identity (3.17) follows from the observation that

$$\frac{1}{(z-t_0)(t_0-t_1)}+\frac{1}{(z-t_1)(t_1-t_0)}=\frac{1}{(z-t_0)(z-t_1)}.$$

**Example.** We will use the methods above with $M_0=11/2$ and $M_1=-17/4$. The 'exact' values are computed via (3.2), in the case $z=0$, and via the alternative methods in [14] in the case $z\neq 0$.

When $\sigma_0=1+\mathrm{i}/10$ and $\sigma_1=-1+\mathrm{i}/2$ then $|1+(\sigma_0/\sigma_1)|=\sqrt{36/125}$. Hence, we deal with case (1). Take $N=10$ and $\tilde{v}_N=0$. Compute the other $\tilde{v}_r$ via (3.4). Then

$$\tilde{v}_0=10.355613+18.676504\mathrm{i}\quad\text{and}\quad F^{(2)}\left(0;\begin{matrix}M_0+1, & M_1\\ \sigma_0, & \sigma_1\end{matrix}\right)=10.355606+18.676501\mathrm{i}. \tag{3.18}$$

Hence, even with this relatively small $N$ we already obtain 6 correct digits.

Next take $\sigma_0=1+\mathrm{i}/10$ and $\sigma_1=(1+\mathrm{i})/2$ then $|1+(\sigma_0/\sigma_1)|=\sqrt{261/50}$. Hence, we deal with case (2). Take $N=15$ and $\tilde{v}_0=0$. Compute the other $\tilde{v}_r$ via (3.4). Then

$$C\approx\left(\frac{\sigma_0}{\sigma_0+\sigma_1}\right)^{M_0+M_1-1}\left(F^{(1)}\left(0;\begin{matrix}M_0+M_1\\ \sigma_0\end{matrix}\right)\frac{\pi\,\mathrm{e}^{-M_1\pi\mathrm{i}}}{\sin M_1\pi}-\sum_{r=0}^{N}\frac{(-\sigma_1)^r\,\tilde{v}_r}{r!}\right). \tag{3.19}$$

The result is

$$C=1.282848+14.116590\mathrm{i}\quad\text{and}\quad F^{(2)}\left(0;\begin{matrix}M_0+1, & M_1\\ \sigma_0, & \sigma_1\end{matrix}\right)=1.282848+14.116594\mathrm{i}. \tag{3.20}$$

The first 7 digits are correct.

Finally $\sigma_0 = 1 + \mathrm{i}/10$ and $\sigma_1 = -11/20 + \mathrm{i}/2$ then $|1 + (\sigma_0/\sigma_1)| = \sqrt{225/221}$. Hence, we deal with case (3). Take $\tilde{\sigma}_1 = -1 + \mathrm{i}/2$ (see the first example), $N = 10$ and $\tilde{v}_N = 0$. Compute the other $\tilde{v}_r$ via (3.4). Then

$$C \approx \left(\frac{\sigma_0+\tilde{\sigma}_1}{\sigma_0+\sigma_1}\right)^{M_0+M_1-1}\left(F^{(2)}\left(0;\begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \tilde{\sigma}_1\end{matrix}\right) - \sum_{r=0}^{N}\frac{(\tilde{\sigma}_1-\sigma_1)^r\,\tilde{v}_r}{r!}\right), \tag{3.21}$$

where we know from the first example that $F^{(2)}(0;\ldots) \approx 10.355613 + 18.676504\mathrm{i}$. The result is

$$C + \tilde{v}_0 = 4.289404 + 16.706475\mathrm{i} \quad \text{and} \quad F^{(2)}\left(0;\begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \sigma_1\end{matrix}\right) = 4.289397 + 16.706471\mathrm{i}. \tag{3.22}$$

Again, the first 6 digits are correct.

We return to the first example and take $z = 5/2$, $\sigma_0 = 1 + \mathrm{i}/10$ and $\sigma_1 = -1 + \mathrm{i}/2$. Let $N = 15$, $v_N = 0$ and $\tilde{y}_0 = 0$. Compute the other $v_r$ via (3.4) and use these results in (3.14) to compute the other $\tilde{y}_r$. Then

$$C \approx \mathrm{e}^{\sigma_1 z}\left(F^{(1)}\left(z;\begin{matrix} M_0+M_1-1 \\ \sigma_0\end{matrix}\right)\frac{\pi\,\mathrm{e}^{-M_1\pi\mathrm{i}}}{\sin M_1\pi} - \sum_{r=0}^{N}\frac{(-\sigma_1)^r\,\tilde{y}_r}{r!}\right). \tag{3.23}$$

The result is

$$C = -2.2796687 - 7.0256327\mathrm{i} \quad \text{and} \quad F^{(2)}\left(z;\begin{matrix} M_0, & M_1 \\ \sigma_0, & \sigma_1\end{matrix}\right) = -2.2796691 - 7.0256332\mathrm{i}. \tag{3.24}$$

The first 7 digits are correct.

## 4. Level 3

As in the previous section we will assume that the final $M_j$-parameter, in this case $M_2$, is not an integer, and we will deal first with the special values at $z = 0$. In integral representation (1.1) with $\ell = 2$ take $t_1 = t_0\tau_1$ and $t_2 = t_0\tau_2$

$$F^{(3)}\left(0;\begin{matrix} M_0+1, & M_1, & M_2 \\ \sigma_0, & \sigma_1, & \sigma_2\end{matrix}\right) = \int_0^{[\pi-\theta_0]}\int_0^{[\theta_0-\theta_1]}\int_0^{[\theta_0-\theta_2]}\frac{\mathrm{e}^{t_0(\sigma_0+\sigma_1\tau_1+\sigma_2\tau_2)}t_0^{M_0+M_1+M_2-3}\tau_1^{M_1-1}\tau_2^{M_2-1}}{(\tau_1-1)(\tau_1-\tau_2)}\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1\,\mathrm{d}t_0. \tag{4.1}$$

Integration by parts will give us a recurrence relation with respect to the $M_0$-parameter. Combining that result with the identity

$$F^{(3)}\left(0;\begin{matrix} M_0+1, & M_1, & M_2 \\ \sigma_0, & \sigma_1, & \sigma_2\end{matrix}\right) = F^{(3)}\left(0;\begin{matrix} M_2+1, & M_1, & M_0 \\ \sigma_2, & \sigma_1, & \sigma_0\end{matrix}\right), \tag{4.2}$$

leads to the recurrence relation

$$\begin{aligned}&(\sigma_0+\sigma_1+\sigma_2)\,v_{r+1} + (M_0+M_1+M_2+r-2)\,v_r \\ &\quad = (\sigma_0+\sigma_1)F^{(2)}\left(0;\begin{matrix} M_0+1, & M_1 \\ \sigma_0, & \sigma_1\end{matrix}\right)F^{(1)}\left(0;\begin{matrix} M_2+r+1 \\ \sigma_2\end{matrix}\right) + \sigma_0F^{(1)}\left(0;\begin{matrix} M_0+1 \\ \sigma_0\end{matrix}\right)F^{(2)}\left(0;\begin{matrix} M_1+1, & M_2+r \\ \sigma_1, & \sigma_2\end{matrix}\right),\end{aligned} \tag{4.3}$$

where

$$v_r = F^{(3)}\left(0;\begin{matrix} M_0+1, & M_1, & M_2+r \\ \sigma_0, & \sigma_1, & \sigma_2\end{matrix}\right). \tag{4.4}$$

The normalising condition

$$\sum_{r=0}^{\infty}\frac{(\tilde{\sigma}_2-\sigma_2)^r\,v_r}{r!} = F^{(3)}\left(0;\begin{matrix} M_0+1, & M_1, & M_2 \\ \sigma_0, & \sigma_1, & \tilde{\sigma}_2\end{matrix}\right), \tag{4.5}$$

has the special case

$$\sum_{r=0}^{\infty}\frac{(-\sigma_2)^r\,v_r}{r!} = F^{(2)}\left(0;\begin{matrix} M_0+1, & M_1+M_2-1 \\ \sigma_0, & \sigma_1\end{matrix}\right)\frac{\pi\,\mathrm{e}^{\mp M_2\pi\mathrm{i}}}{\sin M_2\pi}, \quad \theta_1 \leqslant \theta_2. \tag{4.6}$$

The proof of this result is almost a copy of the proof of (3.7).

Let $\tilde{v}_r$ be another solution of recurrence relation (4.3) then there exists a constant $C$ such that

$$v_r = \tilde{v}_r + C\frac{(M_0+M_1+M_2-2)_r}{(-\sigma_0-\sigma_1-\sigma_2)^r}. \tag{4.7}$$

As in Section 3, we have the estimate $v_r = \sigma_2^{-r}\Gamma(M_2+r-1)\mathcal{O}(1)$ as $r \to \infty$.

There are again three cases that we should consider: (1) $|1+(\sigma_0+\sigma_1)/\sigma_2| < 1-\varepsilon$, (2) $|1+(\sigma_0+\sigma_1)/\sigma_2| > 1+\varepsilon$, and (3) $||1+(\sigma_0+\sigma_1)/\sigma_2|-1| \leqslant \varepsilon$. The following details are very similar to the ones in Section 3 and we give the main results.

In case (1) $v_r$ is the recessive solution. Take $N$ large, $\tilde{v}_N = 0$, and compute the other $\tilde{v}_r$ via backward recursion in (4.3). Then $v_r \approx \tilde{v}_r$ for $r \ll N$.

In case (2) $v_r$ is a dominant solution. Let $\tilde{v}_r$ satisfy (4.3) with initial value $\tilde{v}_0 = 0$. Then there exists a constant $C$ such that (4.7) holds for all $r$. We compute the constant $C$ via the normalising condition (4.6), that is, via the identity

$$C\left(\frac{\sigma_0+\sigma_1}{\sigma_0+\sigma_1+\sigma_2}\right)^{2-M_0-M_1-M_2} + \sum_{r=0}^{\infty}\frac{(-\sigma_2)^r\,\tilde{v}_r}{r!} = F^{(2)}\left(0;\begin{array}{cc} M_0+1, & M_1+M_2-1\\ \sigma_0, & \sigma_1\end{array}\right)\frac{\pi\,\mathrm{e}^{\mp M_2\pi \mathrm{i}}}{\sin M_2\pi},\quad \theta_1\leqslant\theta_2. \tag{4.8}$$

Finally, in case (3) we 'walk' in the $\sigma_2$-space, to end up with one of the other two cases. We choose $\tilde{\sigma}_2$ relatively close to $\sigma_2$ such that $|1+(\sigma_0+\sigma_1)/\tilde{\sigma}_2|$ is either larger or smaller than $|1+(\sigma_0+\sigma_1)/\sigma_2|$. The $\tilde{v}_r$ are computed in the backward or forward direction, depending on whether $|1+(\sigma_0+\sigma_1)/\sigma_2|\leqslant 1$. The corresponding constant $C$ in (4.7) follows from normalising condition (4.5), thus

$$C\left(\frac{\sigma_0+\sigma_1+\tilde{\sigma}_2}{\sigma_0+\sigma_1+\sigma_2}\right)^{2-M_0-M_1-M_2} + \sum_{r=0}^{\infty}\frac{(\tilde{\sigma}_2-\sigma_2)^r\,\tilde{v}_r}{r!} = F^{(3)}\left(0;\begin{array}{ccc} M_0+1, & M_1, & M_2\\ \sigma_0, & \sigma_1, & \tilde{\sigma}_2\end{array}\right). \tag{4.9}$$

It might take several steps in the $\sigma_2$-space, but we will end up with a $\tilde{\sigma}_2$ such that we are in case (1) or (2).

Some care has to be taken: the hyperterminants are multi-valued functions with respect to $\sigma_2$, and in the $\sigma_2$-space they have the branch-cut $\{x\sigma_1 \mid x>0\}$. In the case that one wants to cross this line, one has to use the connection formulae (2.7) in [14].

Now we are able to compute the level 3 hyperterminants with $z=0$, we will use these results to compute the level 3 hyperterminants with $z\neq 0$. Let

$$y_r = F^{(3)}\left(z;\begin{array}{ccc} M_0, & M_1, & M_2+r\\ \sigma_0, & \sigma_1, & \sigma_2\end{array}\right). \tag{4.10}$$

These functions are solutions of the recurrence relation (see (2.8) in [14])

$$y_{r+1}-zy_r = v_r + F^{(1)}\left(z;\begin{array}{c}M_0\\ \sigma_0\end{array}\right)F^{(2)}\left(0;\begin{array}{cc}M_1+1, & M_2+r\\ \sigma_1, & \sigma_2\end{array}\right) + F^{(2)}\left(z;\begin{array}{c}M_0, M_1\\ \sigma_0, \sigma_1\end{array}\right)F^{(1)}\left(0;\begin{array}{c}M_2+r+1\\ \sigma_2\end{array}\right), \tag{4.11}$$

with normalising condition

$$\sum_{r=0}^{\infty}\frac{(-\sigma_2)^r\,y_r}{r!} = F^{(3)}\left(z;\begin{array}{ccc}M_0, & M_1, & M_2\\ \sigma_0, & \sigma_1, & 0\sigma_2\end{array}\right) = F^{(2)}\left(z;\begin{array}{cc}M_0, & M_1+M_2-1\\ \sigma_0, & \sigma_1\end{array}\right)\frac{\pi\,\mathrm{e}^{\mp M_2\pi\mathrm{i}}}{\sin M_2\pi}, \tag{4.12}$$

$\theta_1\leqslant\theta_2$, where $\Re M_2 < 1$.

Let $\tilde{y}_r$ satisfy (4.11) with initial value $\tilde{y}_0 = 0$. Then there exists a constant $C$ such that $y_r = \tilde{y}_r + Cz^r$. Combining this relation with (4.12) we can compute $C$ via the identity

$$\sum_{r=0}^{\infty}\frac{(-\sigma_2)^r\,\tilde{y}_r}{r!} + C\mathrm{e}^{-\sigma_2 z} = F^{(2)}\left(z;\begin{array}{cc}M_0, & M_1+M_2-1\\ \sigma_0, & \sigma_1\end{array}\right)\frac{\pi\,\mathrm{e}^{\mp M_2\pi\mathrm{i}}}{\sin M_2\pi},\quad \theta_1\leqslant\theta_2. \tag{4.13}$$

By decreasing $\Re M_2$ the convergence speed of the infinite series in (4.13) can be increased substantially.

The results in this section can be used to compute the $M_j$-derivative of $v_r$ and $y_r$. We omit the details since they are obvious.

Note, again, that the restriction that $M_2$ is not an integer is critical here, and that when we let $M_2$ approach an integer, the results will become useless. Identities like (4.2) and

$$\begin{aligned} F^{(3)}\left(z;\begin{array}{c}M_0, M_1, M_2\\ \sigma_0, \sigma_1, \sigma_2\end{array}\right) - F^{(3)}\left(z;\begin{array}{c}M_2, M_1, M_0\\ \sigma_2, \sigma_1, \sigma_0\end{array}\right) &= F^{(1)}\left(z;\begin{array}{c}M_2\\ \sigma_2\end{array}\right)F^{(2)}\left(z;\begin{array}{c}M_0, M_1\\ \sigma_0, \sigma_1\end{array}\right)\\ &\quad - F^{(1)}\left(z;\begin{array}{c}M_0\\ \sigma_0\end{array}\right)F^{(2)}\left(z;\begin{array}{c}M_2, M_1\\ \sigma_2, \sigma_1\end{array}\right), \end{aligned} \tag{4.14}$$

might be useful when $M_2$ is an integer. Many of these identities exist, but for the case in which all $M_j$ are integers other methods have to be constructed.

## 5. An application

Hyperasymptotic expansions and the computation of Stokes multipliers for linear differential equations are discussed in [8]. We will use these results in this section. The asymptotic approximations in that paper are of the form $w_j(z)\sim \mathrm{e}^{-\lambda_j z}z^{\mu_j}$, $j=1,\ldots,n$. In [8] it is assumed that $\lambda_j\neq\lambda_k$ whenever $j\neq k$. In the case that, say, $\lambda_2=\lambda_3$ and $\mu_2-\mu_3$ is not an integer, the results in [8] are still valid, and the Stokes multipliers $K_{23}=0$. When $\lambda_2=\lambda_3$ and $\mu_2-\mu_3$ is an integer an extra factor $\ln z$ appears. See (5.2). Since $\frac{\mathrm{d}}{\mathrm{d}\mu_3}z^{\mu_3} = \ln(z)z^{\mu_3}$ we can still use the results in [8], but we have to take parametric derivatives in the hyperterminants whenever $\mu_3$ appears. However, this does not change the shape of the expansions, the optimal number of terms, and the error estimates. For these details the reader is referred to [8].

As an example we will study the solutions of the third-order linear differential equation

$$w'''(z) + \left(1 + \frac{1}{4z}\right) w''(z) - \left(\frac{1}{2z} + \frac{13}{16z^2}\right) w'(z) - \frac{7}{16z^2} w(z) = 0, \tag{5.1}$$

which has as formal series solutions

$$\begin{aligned} \hat{w}_1(z) &= \mathrm{e}^{-z} \sum_{s=0}^{\infty} a_s z^{-s-\frac{3}{4}}, \\ \hat{w}_2(z) &= \sum_{s=0}^{\infty} b_s z^{-s-\frac{1}{4}}, \\ \hat{w}_3(z) &= \hat{w}_2(z) \ln(z) + \sum_{s=-2}^{\infty} c_s z^{-s-\frac{1}{4}}, \end{aligned} \tag{5.2}$$

where we take $a_0 = b_0 = 1$ and $c_0 = 0$. It follows that $c_{-1} = -\frac{128}{27}$, $c_{-2} = \frac{8192}{2457}$ and for the other coefficients we have the recurrence relations

$$\begin{aligned} s a_s &= \frac{1}{2}\left(1 - 4s^2\right) a_{s-1} - \left(s^2 + \frac{1}{4}s - \frac{15}{16}\right)\left(s - \frac{5}{4}\right) a_{s-2}, \\ s(s+2) b_s &= \left(s^2 + \frac{5}{4}s - \frac{9}{16}\right)\left(s - \frac{3}{4}\right) b_{s-1}, \\ s(s+2) c_s &= \left(s^2 + \frac{5}{4}s - \frac{9}{16}\right)\left(s - \frac{3}{4}\right) c_{s-1} + 2(s+1) b_s - \left(3s^2 + s - \frac{3}{2}\right) b_{s-1}, \end{aligned} \tag{5.3}$$

$s = 1, 2, 3, \ldots$. By specifying sectors of validity we define unique solutions $w_j(z)$: Let $w_1(z) \sim \hat{w}_1(z)$ as $|z| \to \infty$ in the sector $|\mathrm{ph}\,(z)| < \frac{3}{2}\pi$, and $w_2(z) \sim \hat{w}_2(z)$, $w_3(z) \sim \hat{w}_3(z)$ as $|z| \to \infty$ in the sector $-\frac{1}{2}\pi < \mathrm{ph}\,(z) < \frac{5}{2}\pi$. We also have the connection relation

$$w_1(z) = \mathrm{i} w_1\left(\mathrm{e}^{-2\pi \mathrm{i}} z\right) + K_{21} w_2(z) + K_{31} w_3(z), \tag{5.4}$$

where the constants $K_{21}$ and $K_{31}$ are the Stokes multipliers.

In the case that $\hat{w}_3(z) = z^{M_0} \hat{w}_2(z) + \sum_{s=-2}^{\infty} c_s z^{-s-\frac{1}{4}}$ we can obtain via the results in [8] asymptotic expansions for the late coefficients $a_n$, as $n \to \infty$, and hyperasymptotic expansions for $w_1(z)$. Taking the $M_0$-derivative of these results and then $M_0 = 0$ we obtain the following results.

The Stokes multipliers in (5.4) can be computed via the asymptotics of the late coefficients:

$$a_n \sim -\frac{K_{21}}{2\pi \mathrm{i}} \sum_{s=0}^{N-1} b_s F^{(1)}\left(0; \begin{matrix} n-s+\frac{3}{2} \\ 1 \end{matrix}\right) - \frac{K_{31}}{2\pi \mathrm{i}} \sum_{s=-2}^{N-1} c_s F^{(1)}\left(0; \begin{matrix} n-s+\frac{3}{2} \\ 1 \end{matrix}\right) - \frac{K_{31}}{2\pi \mathrm{i}} \sum_{s=0}^{N-1} b_s \frac{\partial}{\partial M_0} F^{(1)}\left(0; \begin{matrix} n-s+\frac{3}{2} \\ 1 \end{matrix}\right), \tag{5.5}$$

as $n \to \infty$. Note the $M_0$-derivative on the right-hand side of (5.5), which is a direct consequence of the logarithm in (5.2). The hyperterminants in the first two series on the right-hand side of (5.5) are, according to (2.2), gamma functions, showing that the coefficients $a_n$ grow like a factorial.

Note also that the series on the right-hand side of (5.5) are divergent. The optimal number of terms in approximation (5.5) is $N \approx n/2$. Since the coefficients $a_s$, $b_s$ and $c_s$ can be computed via (5.3) the only unknowns in (5.5) are the Stokes multipliers. Taking for example $n = 19$, $n = 20$ and $N = 10$ we obtain two equations with two unknowns, with solutions

$$K_{21} = 2.043754662 - 1.373747812\mathrm{i}, \qquad K_{31} = 0.4372775096. \tag{5.6}$$

These results are needed in the level one hyperasymptotic expansion

$$\begin{aligned} \mathrm{e}^{z} w_1(z) = {} & \sum_{s=0}^{2N-1} a_s z^{-s-\frac{3}{4}} + z^{\frac{1}{4}-2N} \left\{ \frac{K_{21}}{2\pi \mathrm{i}} \sum_{s=0}^{N-1} b_s F^{(1)}\left(z; \begin{matrix} 2N-s+\frac{1}{2} \\ 1 \end{matrix}\right) + \frac{K_{31}}{2\pi \mathrm{i}} \sum_{s=-2}^{N-1} c_s F^{(1)}\left(z; \begin{matrix} 2N-s+\frac{1}{2} \\ 1 \end{matrix}\right) \right. \\ & \left. + \frac{K_{31}}{2\pi \mathrm{i}} \sum_{s=0}^{N-1} b_s \frac{\partial}{\partial M_0} F^{(1)}\left(z; \begin{matrix} 2N-s+\frac{1}{2} \\ 1 \end{matrix}\right) \right\} + \mathcal{O}\left(\mathrm{e}^{-2|z|} z^{\frac{5}{4}}\right), \end{aligned} \tag{5.7}$$

as $|z| \to \infty$ in the sector $|\mathrm{ph}\,(z)| \leqslant \pi$. Again, this result follows from [8]. The first series on the right-hand side is an 'optimal'-truncated asymptotic expansion, and the re-expansions are in terms of hyperterminants.

For the numerical illustration we take $z = 10\mathrm{i}$. In that case the optimal $N = 10$ on the right-hand side of (5.7), and the Stokes multipliers in (5.6) are computed up to the required precision: 7 digits for $K_{21}$ and 10 digits for $K_{31}$. With these values for $N$ and the Stokes multipliers we obtain the approximation

$$w_1(z) \approx 0.007254669078 + 0.172904229415\mathrm{i}. \tag{5.8}$$

By taking $z = 40\mathrm{i}$ and 40 terms in asymptotic expansion $\hat{w}_1(z)$ we can approximate $w_1(z)$ and its derivative up to a much higher precision at $z = 40\mathrm{i}$. These results can be used in a direct numerical integration (see for example [18]) of the differential equation (5.1) in which we walk from $z = 40i$ to $z = 10i$. The result shows that all 12 digits in approximation (5.8) are correct.

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