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An unconditional existence result for the quasi-variational elastohydrodynamic free boundary value problem

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Abstract

In this paper a two-dimensional quasi-variational inequality arising in elastohydrodynamic lubrication is studied for non-constant viscosity. So far, existence results for such piezo–viscous problems require an L^∞ property for an auxiliary problem. For the usual pressure–viscosity relations, this property needs small data assumptions which are not observed in experimental conditions. In the present work, such small data assumptions are proved unnecessary for existence results. Besides well-established monotonicity behavior for the viscosity–pressure relation, the only condition used here is on the asymptotic behavior for this law as the pressure tends to infinity. If the procedure used here, namely the introduction of a reduced pressure by Grubin transform followed by a regularization procedure, appears somewhat classical, the way in which an upper bound is obtained is completely new.

Keywords: Free boundary problem; Lubrication; Variational inequality; Elastohydrodynamic

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1. Introduction

A basic problem in the theory of hydrodynamic lubrication is the determination of the pressure in a thin film of lubricant when the fluid is surrounded by two close surfaces in relative motion. One of these surfaces is usually a domain Ω of the $(x, y, 0)$ plane and the second one is the graph $(x, y, h(x, y))$ of a strictly non-negative function $h(x, y)$. It is well known [5,14,28] that if $h(x, y)$ is small (with respect of the dimensions of Ω), then the pressure p does not rely on the variable z. Then $p(x, y)$ obeys the Reynolds equation

$$Q(p) \equiv -\operatorname{div}\left(\frac{1}{\mu}h^3\nabla p\right) + \frac{\lambda\partial h}{\partial x} = 0, \tag{1}$$

where $\lambda = u\lambda_0$ and λ_0 is some physical constant, u is the relative tangential velocity between the two surfaces, assumed to be primarily along the x direction, and μ is the viscosity of the fluid.

Most of the time, Eq. (1) is only valid on an unknown part Ω^+ of the domain Ω where the pressure is greater than a critical value p_c. On the complementary part, $\Omega - \Omega^+$, the pressure is equal to p_c. This free boundary phenomenon is called cavitation. Although not completely satisfactory from a mechanical point of view, see [4], the following variational inequality model is often proposed to give a mathematical description of the problem and will be retained in this paper (with $p_c = 0$).

Find p in $H = \{\phi \text{ in } W_0^{1,2}(\Omega),\ \phi \geqslant 0 \text{ a.e.}\}$ such that:

$$Q(p) \geqslant 0, \quad p \geqslant 0, \quad pQ(p) = 0. \tag{2}$$

Implicit assumptions in this kind of problem are that h is a given known function of (x, y) and μ is a real, positive constant. However, for large values of the pressure, these two conditions are not fulfilled. First, the surrounding surfaces can be deformed due to the pressure, so that h is no longer a datum, but is one of the unknowns of the problem. Furthermore, chemical and physical modifications take place in the lubricant so that the viscosity is no longer constant and becomes also a function of the pressure.

For devices like ball bearings, the deformation induced by the hydrodynamic pressure p can be described from an initial given smooth gap $f(x, y)$ between the two surfaces by

$$h(p)(X) = f(X) + \iint_\Omega k(Z)p(X - Z)dZ \text{ with } f(x, y) = h_0 + x^2 + y^2. \tag{3}$$

Here $h_0 > 0$ is a given constant and k is a kernel given by

$$k(Z) = \frac{k_0}{\sqrt{z_1^2 + z_2^2}},$$

where k_0 is a given nonnegative constant.

For given constant μ, the non-local nonlinear problem (2) in which h is given by (3) is called the iso-viscous elastohydrodynamic problem and has been studied in [18,23]. Existence results have been obtained by fixed-point methods or by using a pseudo-monotone variational inequality, while uniqueness has required data (such as λ) to be small.

When the viscosity is allowed to depend on the pressure, the problem is said to be piezoviscous. Taking piezoviscosity $\mu = \mu(p)$ into account dramatically increases the difficulty of problem (2).

It has been known at least since the early 1900s that the film thickness predicted by the constant viscosity model can be about 100 times smaller than what is observed in experiments, see [21,13, p. 355] [28, p. 269].

The usual practice, both for mathematical study and for numerical computations, is to work with an auxiliary quantity called the reduced pressure instead of the pressure itself. The reduced pressure is obtained from the pressure by performing a transform (sometimes called the Grübin transform)

$$v = \int_0^{p(x)} \frac{1}{\mu(s)} ds. \tag{4}$$

One can then rewrite the partial differential equation (2) in terms of the new unknown function v. It turns out that this equation is more manageable than (2).

The idea is then to get an existence theorem for this reduced pressure equation. There are several existence results for the piezo-viscous two-dimensional problem (see [6,8,18,20,23,30] and the references therein). However, in dealing with the usual physical piezo-viscous law $(\mu(s) = \mu_0 e^{\alpha s})$, they all require that the data of the problem be small. The precise statement (see equation (3.15) of [23]) is too technical to be given here, but suffice it to say that the requirement of small data is equivalent to saying that the relative motion of the two surfaces is required to be at small speed. This restriction on the speed seems more one of expediency rather than of physical or mathematical necessity. This requirement for small data is used to establish a crucial upper bound for the reduced pressure. One of the open questions listed by A. Friedman in [17] is to prove existence without the *small data* assumptions of [18,23].

The goal of this paper is to prove that such small data assumptions are not always necessary for some realistic piezo-viscosity law. The plan is as follows: A precise statement of the problem and recollection of classical results appears in Section 2. Under minimal assumptions of the piezo-viscous law μ, in Section 3, using a regularization procedure, we start by showing that the pressure is bounded in L^1. This bound has a physical interpretation that the total force exerted on the bodies is finite. We will also establish an L^∞ estimate for the reduced pressure. If the regularized problem is somewhat classical, the upper bounds obtained are completely new and contrary to previous results: Ours do not require any small data assumptions. Then, in Section 4, assuming a particular, although realistic, pressure–viscosity law, we gain L^p estimates for the pressure. We finally show that these estimates yield sufficient regularity results to obtain a solution for the reduced problem. Theorem 3.1 and the regularity results of Section 4 make it easy to see how to obtain a solution to the original problem.

Our approach is partly inspired by the approach of Bellout in [7]. However, Bellout [7] dealt only with the one-space dimension case. There is a major difference in that in one dimension, $W^{1,2}$ functions are continuous, while this is not true in two-space dimension. This creates major complications here and several new ideas are required to overcome them.

For additional results on mathematical problems arising in lubrication theory see [10,11,16,19] and the references therein.

2. Statement of the problem

Let $f(x, y)$ be a given continuous, bounded, strictly non-negative function and λ a real constant, and let a pressure–viscosity relation be defined by way of a function μ such that:

$$\mu(p) \quad \text{is a continuous, increasing function,} \tag{5}$$

$$\mu(0) \quad = \mu_0 > 0. \tag{6}$$

The physical problem is to find p in H such that for any $\phi \in H$:

$$\iint_\Omega \left(\frac{h(p)^3}{\mu(p)} \nabla p \cdot \nabla(\phi - p) + \lambda \frac{\partial h}{\partial x}(\phi - p) \right) dx\, dy \geqslant 0, \tag{7}$$

where

$$h(p)(X) = f(X) + \iint_\Omega k(Z) p(X - Z)\, dZ \tag{8}$$

with

$$f = h_0 + x^2 + y^2 \text{ and } h_0 > 0. \tag{9}$$

Here k is a kernel given by

$$k(Z) = \frac{k_0}{\sqrt{z_1^2 + z_2^2}}, \tag{10}$$

where k_0 is a given nonnegative constant. To define the problem for the reduced pressure we introduce the function

$$a(s) = \int_0^s \frac{ds}{\mu(s)}. \tag{11}$$

Under assumptions (5), (6) on μ it follows that the function a is increasing and has an inverse γ such that

$$\gamma(a(s)) = a(\gamma(s)) = s. \tag{12}$$

The reduced pressure $v(x, y)$ is defined by

$$v(x, y) = a(p(x, y)) \quad \text{and} \quad p(x, y) = \gamma(v(x, y)). \tag{13}$$

The problem satisfied by the reduced pressure is then to find $v \in H$ such that for any $\phi \in H$:

$$\int\int_{\Omega} (h(v)^3 \nabla v \cdot \nabla(\phi - v) + \lambda \frac{\partial h(v)}{\partial x}(\phi - v))\, dx\, dy \geqslant 0, \tag{14}$$

where

$$h(v)(X) = f(X) + \int\int_{\Omega} k(Z)\gamma(v(X - Z))\, dZ \tag{15}$$

and f and k are as before.

An important physical quantity in applications is known as the *reciprocal iso-viscous pressure* (see [2,3,9] and the references therein). It is given by

$$\alpha^* = \left[\int_0^\infty \frac{\mu(0)}{\mu(s)} ds\right]^{-1}. \tag{16}$$

We will find it more convenient to use a related quantity, namely,

$$A = \int_0^\infty \frac{1}{\mu(s)} ds = \frac{1}{\mu_0 \alpha^*}. \tag{17}$$

The existence of a solution to problem (14)–(15) was established in [23] under the assumption that $A = +\infty$ ($\alpha^* = 0$).

Although various formulas have been used for the viscosity pressure relation, all experimental measures (see [2,3,29] and the references therein) reflect two characteristics:

- The viscosity is an increasing function of the pressure.
- The reciprocal iso-viscous pressure α^* is non-zero. i.e., A is finite.

The most common forms of the function μ used in engineering applications are the Roelands formula (Viscosity–Pressure)

$$\mu(p) = \exp(C[-1 + (1 + cp)^z]), \tag{18}$$

where C, z, c are constants, and the simpler, more popular, Barus relation

$$\mu(p) = \mu_0 \exp(\alpha p), \tag{19}$$

where α and μ_0 are positive constants.

The Roelands formula is relatively recent (1966) and has been tested for pressures between zero and 0.5 Gpa . See [2,3,24].

For the case where A was assumed to be finite (non-zero α^*) existence results were established for problem (14)–(15) assuming that μ follows Barus's law (19) (see [18,23,30]). But in all of those cases, the existence was proved under the small data assumption.

We will prove existence of a solution to problem (14)–(15) without restriction on the size of the data. In Section 3 our results are proved for any function μ which satisfies conditions (5) and (6). In the last section, we will make further assumptions on the function μ. The additional restriction is asymptotic in nature. We will essentially assume that for values of p larger than an arbitrary value $p*$, we have $\mu(p) = (p+Q)^\beta$, where Q is a positive constant and $\beta \in (1, 3/2)$. Notice that such a function μ could agree with the Roelands formula over any range of p for which the Roelands formula has been tested. Also, since $\beta > 1$, such a function μ would yield a finite value for A (nonzero α^*). This kind of pressure–viscosity relations were used in [12].

3. The approximate problem and first related estimates

In this section we will first consider a cut-off approximation of γ by way of a small parameter ε, so defining an approximate problem of 14. Classical $W^{1,2}$ estimates of the reduced pressure (independent of ε) are established. We also prove an L^∞ estimate of the reduced pressure and an L^1 estimate of the pressure. These estimates are proved without any assumption on the size of the data.

For $\varepsilon > 0$ fixed, we define

$$\gamma_\varepsilon(s) = \begin{cases} \gamma(s); & 0 < s \leqslant A - \varepsilon, \\ \gamma(A-\varepsilon); & s \geqslant A - \varepsilon. \end{cases}$$

We start by stating the existence of a solution to the approximate problem.

Proposition 1. *For $\varepsilon > 0$ fixed, there exists a v_ε in H such that*

$$\iint_\Omega \left(h_\varepsilon^3 \nabla v_\varepsilon \cdot \nabla(\phi - v_\varepsilon) + \lambda \frac{\partial h_\varepsilon}{\partial x}(\phi - v_\varepsilon) \right) dx\, dy \geqslant 0, \tag{20}$$

$$\forall \phi \in H, \tag{21}$$

where

$$h_\varepsilon = f + d_\varepsilon \tag{22}$$

and

$$d_\varepsilon = \int\int \gamma_\varepsilon(v_\varepsilon(x-s, y-t)) \frac{ds\,dt}{\sqrt{s^2+t^2}}. \tag{23}$$

Furthermore, $v_\varepsilon \in C^{0,\alpha}(\Omega)$.

Proof. This proposition is proved in [18,23]. For the convenience of the reader we will give an outline of the proof. First, consider the following linear free boundary value problem: Given a positive function u we get $h_\varepsilon(u)$ by substituting u to v_ε in (23). For such an h_ε we solve the free boundary value problem (20) using classical results from [15,19], for example. To finish the proof it is enough to show that the operator maps u to v_ε has a fixed point. $\square$

Proposition 2. (1) *There exists a positive constant C such that*

$$\|v_\varepsilon\|_{W_0^{1,2}(\Omega)} \leqslant C. \tag{24}$$

(2) *There exists a* v *in* H *such that a subsequence* v_{ε_n} (*which will henceforth be denoted* v_ε) *weakly converges towards* v *for the* $W_0^{1,2}(\Omega)$ *norm.*

Proof. Part 2 of the proposition follows directly from part 1 and from compactness results in $W^{1,2}$. Next we prove part 1.

Setting $\phi = 0$ in the equality 20 we find

$$\int\int_\Omega h_\varepsilon^3 \,|\nabla v_\varepsilon|^2 \,dx\,dy \;\leqslant\; -\lambda \int\int_\Omega \frac{\partial h\varepsilon}{\partial x} v_\varepsilon \,dx\,dy \tag{25}$$

$$= \lambda \int\int_\Omega h_\varepsilon \frac{\partial v_\varepsilon}{\partial x}\,dx\,dy \tag{26}$$

$$= \lambda \int\int_\Omega \frac{1}{\sqrt{h_\varepsilon}} h_\varepsilon^{3/2} \frac{\partial v_\varepsilon}{\partial x}\,dx\,dy. \tag{27}$$

Using the Cauchy–Schwartz inequality and Young inequality we get that the last expression above satisfy

$$\lambda \left| \int\int_\Omega \left(\frac{1}{\sqrt{h_\varepsilon}}\right) \left(h_\varepsilon^{3/2} \frac{\partial v_\varepsilon}{\partial x}\right) dx\,dy \right|$$

$$\leqslant \lambda \left(\int\int_\Omega \frac{1}{h_\varepsilon}\,dx\,dy\right)^{1/2} \left(\int\int_\Omega h_\varepsilon^3 \left(\frac{\partial v_\varepsilon}{\partial x}\right)^2 dx\,dy\right)^{1/2}$$

$$\leqslant \frac{\lambda^2}{2} \int\int_\Omega \frac{1}{h_\varepsilon}\,dx\,dy + \frac{1}{2}\int\int_\Omega h_\varepsilon^3 \,(\nabla v_\varepsilon)^2 \,dx\,dy. \tag{28}$$

It then follows from (27) that

$$\iint_{\Omega} h_{\varepsilon}^{3} |\nabla v_{\varepsilon}|^{2} \, dx\, dy \leqslant \lambda^{2} \iint_{\Omega} \frac{dx\, dy}{h_{\varepsilon}} \leqslant \frac{\lambda^{2}}{h_{0}} |\Omega|.$$

Since $h_{\varepsilon}(x, y) \geqslant h_{0}$ $\forall (x, y) \in \Omega$ and $\forall \varepsilon > 0$ it follows from the above inequality that

$$\iint_{\Omega} |\nabla v_{\varepsilon}|^{2} \, dx\, dy \leqslant \frac{\lambda^{2}}{h_{0}^{4}} |\Omega|, \tag{29}$$

where $|\Omega|$ is the measure of Ω. From this estimate we deduce that there exists a $v \in H$ and a subsequence v_{ε_n} such that

$$v_{\varepsilon_n} \longrightarrow v \text{ weakly in } W_{0}^{1,2}(\Omega) \text{ as } \varepsilon_{n} \rightarrow 0. \qquad \square$$

Now, we will prove a somewhat technical proposition by using the same approach as in the Lemma of Stampacchia [26, p. 93], or [19] to get L^{∞} estimates for solutions of variational inequalities.

Proposition 3. *The following estimate is valid*:

$$\|v_{\varepsilon}\|_{L^{\infty}(\Omega)} \leqslant c_{0} |\Omega|^{1/2} 2^{9} \left[\frac{\lambda^{2}}{h_{m\varepsilon}^{4}} \right]^{3/2} \leqslant c_{0} |\Omega|^{1/2} 2^{9} \left[\frac{\lambda^{2}}{h_{0}^{4}} \right]^{3/2} \tag{30}$$

with $h_{m\varepsilon} = \min_{(x,y)\in\Omega} h_{\varepsilon}(x, y)$ *and* c_{0} *a positive constant independent of* ε.

Proof. We will temporarily drop the subscript ε. We will resume it at the end of this section. For $k \in \mathbb{R}$ we set

$$v^{k} = \begin{cases} v - k; & v \geqslant k, \\ \quad 0; & v \leqslant k \end{cases}$$

and

$$B_{k} = \{(x, y) \in \Omega;\ v(x, y) \geqslant k\}. \tag{31}$$

Notice that if $v \in H$ then $\forall k \geqslant 0$, $v^{k} \in H$. Setting $\phi = v^{k} + v_{\varepsilon}$ in (20) and using once again the Cauchy–Schwartz inequality and Young inequality and proceeding as we did in the proof of (24) we get that

$$\iint_{B_k} h^{3} \left| \nabla v^{(k)} \right|^{2} \, dx\, dy \leqslant \lambda^{2} \iint_{B_k} \frac{dx\, dy}{h}. \tag{32}$$

Using that $h \geqslant h_m \equiv \min_\Omega h(x, y)$, we get that

$$\int\int_{B_k} h^3 \left|\nabla v^{(k)}\right|^2 \, dx\, dy \leqslant \frac{\lambda^2}{h_m} |B_k|, \tag{33}$$

where $|B_k|$ is the measure of the set B_k. We get from (33) that

$$\int\int_{B_k} \left|\nabla v^{(k)}\right|^2 \, dx\, dy \leqslant \frac{\lambda^2}{h_m h_m^3} |B_k|. \tag{34}$$

Using embedding theorems we see that there exists a constant $C > 0$ which depends only on Ω such that:

$$\begin{aligned} \int\int_{B_k} \left|\nabla v^{(k)}\right|^2 \, dx\, dy &= \int\int_{\Omega} \left|\nabla v^{(k)}\right|^2 \, dx\, dy \\ &\geqslant C \left(\int\int_{\Omega} \left|v^{(k)}\right|^3 \, dx\, dy \right)^{2/3} \\ &= C \left(\int\int_{B_k} \left|v^{(k)}\right|^3 \, dx\, dy \right)^{2/3}. \end{aligned} \tag{35}$$

Hence

$$\left(\int\int_{B_k} \left|v^{(k)}\right|^3 \, dx\, dy \right)^{2/3} \leqslant \frac{\lambda^2}{C h_m h_m^3} |B_k| \leqslant \frac{\lambda^2}{C h_m^4} |B_k|. \tag{36}$$

Let $l > k$. Then we have that

$$\int\int_{B_k} \left|v^{(k)}\right|^3 \, dx\, dy \geqslant \int\int_{B_l} \left|v^{(k)}\right|^3 \, dx\, dy \geqslant \int\int_{B_l} (l-k)^3 \, dx\, dy \geqslant (l-k)^3 |B_l|, \tag{37}$$

and by (36), $\forall l \geqslant k > 0$ we have that

$$(l-k)^2 |B_l|^{2/3} \leqslant \frac{\lambda^2}{C h_m^4} |B_k|. \tag{38}$$

Setting $\sigma(s) = |B_s|$ we then have that

$$\sigma(l) \leqslant \bar{c}_0 \frac{1}{(l-k)^3} (\sigma(k))^{3/2} \quad \text{where } \bar{c}_0 = \left(\frac{\lambda^2}{C h_m^4} \right)^{3/2}. \tag{39}$$

By a Lemma of Stampacchia (see [26, p. 93]) we then have

$$\sigma(s) = 0 \ \ \forall s \geqslant D = \bar{c}_0 |\Omega|^{1/2} 2^9. \tag{40}$$

Restating in terms of our earlier notation, we see that

$$\sup_{\Omega} v_\varepsilon \leqslant D_\varepsilon, \quad \text{where } D_\varepsilon = \left(\frac{\lambda^2}{Ch_m^4}\right)^{3/2} |\Omega|^{1/2} 2^9 \leqslant \left(\frac{\lambda^2}{Ch_0^4}\right)^{3/2} |\Omega|^{1/2} 2^9. \qquad \square \tag{41}$$

It easily follows from what we have done that the limit v of the sequence v_ε is in L^∞; but, in fact, v satisfies a better estimate in $L^\infty(\Omega)$. We will show next that in fact v is bounded by A. For this purpose we will start by establishing that the sequence of “pressures" associated with the reduced pressures v_ε forms a sequence bounded in $L^1(\Omega)$. This L^1 estimate of the pressure is of independent interest and of physical significance. Indeed the L^1 norm of the pressure represents the load.

Proposition 4. *There exists a constant C independent of* ε *such that*

$$\int\int_\Omega \gamma_\varepsilon(v_\varepsilon) dx \leqslant C\ .$$

Proof. From the definitions of D_ε and $h_{m\varepsilon}$, we have

$$h_{m\varepsilon} \geqslant \int\int_\Omega \gamma_\varepsilon(v_\varepsilon(s,t))\, ds\, dt$$

$$h_{m\varepsilon} = \left(\frac{\lambda^2}{C}\right)^{1/4} |\Omega|^{1/(12)} 2^{3/2} D_\varepsilon^{-1/6} = c_1 D_\varepsilon^{-1/6},$$

where c_1 is a constant which does not depend on ε.

Now from (30), we get

$$\|v_\varepsilon\|_{L^\infty(\Omega)}^{-1/6} \geqslant D_\varepsilon^{-1/6} \geqslant \frac{1}{c_1} \int\int_\Omega \gamma_\varepsilon(v_\varepsilon(s,t))\, ds\, dt. \tag{42}$$

Let us assume now that $\int\int_\Omega \gamma_\varepsilon(v_\varepsilon)\, ds\, dt$ tends to infinity as ε tends to zero. Then by (42) $|v_\varepsilon|_{L^\infty(\Omega)}$ would tend to zero. Thus, as ε tends to zero, $|v_\varepsilon|_{L^\infty(\Omega)} \leqslant \frac{A}{2} < A - \varepsilon$, so the monoticity of γ and the definition of γ_ε imply that

$$\int\int_\Omega \gamma_\varepsilon(v_\varepsilon(s,t))\, ds\, dt = \int\int_\Omega \gamma(v_\varepsilon(s,t))\, ds\, dt \leqslant \int\int_\Omega \gamma\left(\frac{A}{2}\right) ds\, dt < +\infty,$$

which contradicts the assumption that $\iint_\Omega \gamma_\varepsilon(v_\varepsilon)\, ds\, dt$ tends to infinity and finishes the proof. $\square$

We will need the following technical result.

Lemma 5. *Let $E \subset R^2$ be the ball centered at the origin of finite radius $R > 0$. Assume that $u_n \geqslant 0$ is a sequence of radial functions converging strongly in $L^2(E)$ toward a radial function $u \geqslant 0$. We will assume that the sequence u_n converges almost everywhere in E to u. Assume also that for n fixed and for all r_1 and r_2 such that $0 \leqslant r_1 \leqslant r_2 \leqslant R$ we have that $u_n(r_1) \geqslant u_n(r_2)$. Let τ be a positive number.*

Set $\Omega_\tau = \{(x, y) \in E\colon\ u(x, y) \geqslant A + \tau\}$. Assume that Ω_τ has a non-zero measure. Set $\Omega_\tau^n = \{(x, y) \in E : u_k(x, y) \geqslant A\}$. Then there exists $n_0 > 0$ such that $\forall n \geqslant n_0,\ |\Omega_\tau^n| \geqslant \frac{1}{2}|\Omega_\tau|$.

Proof. Since the function u is also radially decreasing it follows that Ω_τ is a ball centered at the origin of radius $r_\tau > 0$. Further, there exists $r^* \in (r_\tau/\sqrt{2}, r_\tau)$ such that $u_n(r^*)$ converges to $u(r^*)$. Since u is decreasing and $r^* \leqslant r_\tau$ it follows that $u(r^*) \geqslant u(r_\tau) \geqslant A + \tau > A$. It then follows from the convergence of $u_n(r^*)$ that there exists n_0 such that for all $n \geqslant n_0$ $u_n(r^*) \geqslant A$. Since u_n is radially decreasing it then follows that $|\Omega_\tau^n|$ contains the ball centered at the origin of radius r^*. Using the fact that $r^* \geqslant r_\tau/\sqrt{2}$ it follows that $|\Omega_\tau^n| \geqslant \frac{1}{2} r_\tau^2 \pi \geqslant \frac{1}{2}|\Omega_\tau|$. This proves the Lemma. $\square$

Lemma 6. *Let $E = (-M, +M) \times (-M, +M) \subset R^2$ and let v_n be a sequence of functions of $L^2(E)$ which converges almost everywhere to v. If*

$$\Omega_\tau = \{(x, y) \in E,\ v(x, y) \geqslant A + \tau\},$$
$$\Omega_\tau^n = \{(x, y) \in E,\ v_n(x, y) \geqslant A\}$$

and $|\Omega_\tau| \neq 0$ then there exists $n_0 > 0$ such that $\forall n \geqslant n_0,\ |\Omega_\tau^n| \geqslant \dfrac{1}{2}|\Omega_\tau|$.

Proof. First we extend v_n and v by zero to all of the ball centered at the origin of radius M. Now we will take the Schwarz decreasing rearrangement of these functions and call them u_n and u respectively. It is well known (see Kawholl in [9, p. 23], for example) that u_n will converge strongly in $L^2(B(0, M))$ to the function u. $\square$

Now we can take a subsequence of u_n which converges to u almost everywhere in the ball. We then have all the conditions of the lemma above. By definition of the rearrangement we have that the measure of the set on which v is greater than or equal to $A + \tau$ is the same as the measure of the set on which u is greater than or equal to $A + \tau$. Similarly the measure of the set on which v_n is greater than or equal to A is the same as the measure of the set where u_n is greater than or equal to A. Hence, the results of Lemma 1 apply also to sets where v and v_n are greater than $A + \tau$ and A respectively.

Theorem 7. *The limit v of the sequence v_ε satisfies $\|v\|_{L^\infty(\Omega)} \leqslant A$.*

Proof. We will once again proceed by contradiction. Let us assume that there is a positive number τ such that $\Omega_\tau = \{(x, y) \in \Omega : \ v(x, y) \geqslant A + \tau\}$ has a non-zero measure.

Applying Lemma (6) to a sequence v_ε which converges almost everywhere to v, it follows that there exists ε_0 such that if $\varepsilon < \varepsilon_0$, then $v_\varepsilon(x, y) \geqslant A$ on a set Ω_τ^ε with $|\Omega_\tau^\varepsilon| \geqslant \frac{1}{2}|\Omega_\tau|$. This would induce from the definition of γ_ε that if $\varepsilon \leqslant \varepsilon_0$, then $\gamma_\varepsilon(v_\varepsilon) = \gamma(A - \varepsilon)$ on Ω_τ^ε. Hence,

$$\int\int_\Omega \gamma_\varepsilon(v_\varepsilon)\, dx\, dy \geqslant \frac{1}{2}|\Omega_\tau|\gamma(A - \varepsilon). \qquad \square$$

This last term goes to infinity as ε goes to zero and this contradicts our Proposition (4).

Proposition 8. *The limit v of the sequence v_ε satisfies $\gamma(v) \in L^1(\Omega)$ and $\int\int_\Omega \gamma(v)\, dx\, dy \leqslant \Gamma$.*

Proof. Let τ be a positive number and introduce $v_\varepsilon^\tau(xy) = \inf(v_\varepsilon(x, y), A - \tau)$. Since v_ε converges strongly in L^2 to v, it can easily be verified that v_ε^τ converges strongly in L^2 to the function v^τ given by $v^\tau(x, y) = \inf(v(x, y), A - \tau)$. Now, for τ fixed, $v_\varepsilon^\tau \leqslant v_\varepsilon \forall \varepsilon$. Therefore $\gamma_\varepsilon(v_\varepsilon^\tau) \leqslant \gamma_\varepsilon(v_\varepsilon) \forall \varepsilon$ and from Proposition 4,

$$\int\int \gamma_\varepsilon(v_\varepsilon^\tau)\, dx\, dy \leqslant \int\int \gamma_\varepsilon(v_\varepsilon)\, dx\, dy \leqslant c, \tag{43}$$

for some constant c. Since $v_\varepsilon^\tau \leqslant A - \tau$ it follows that for ε small enough $\gamma_\varepsilon(v_\varepsilon^\tau) = \gamma(v_\varepsilon^\tau) \leqslant \gamma(A - \tau) \leqslant c$. From this and the fact that v_ε^τ converges to v^τ in L^2 we deduce that $\gamma_\varepsilon(v_\varepsilon^\tau)$ converges to $\gamma(v^\tau)$ in $L^1(\Omega)$. $\square$

From this and the above inequality it follows that $\int\int \gamma(v^\tau)\, dx\, dy \leqslant c, \quad \forall \tau$. Next we will let τ go to zero. The sequence $\gamma(v^\tau) \to \gamma(v)$ as τ goes to zero. It then follows from the monotone convergence theorem that $\int\int_\Omega \gamma(v)\, dx\, dy \leqslant c$.

4. Existence result for the reduced problem

To prove that v is a solution of the reduced problem, we need supplementary convergence results for h_ε. To get them, we will consider additional restrictions on the behavior of the piezo-viscosity law for p near infinity. Specifically, we will assume that there exists a positive number $p*$ such that

$$\mu(p) = (p + Q)^\beta \quad \beta > 1, \quad Q > 0 \quad \text{for} \quad p \geqslant p*, \tag{44}$$

where Q and β are constants.

An easy calculation shows that in this case we have that

$$A = \int_0^\infty \frac{1}{\mu(s)} ds = \int_0^{p*} \frac{1}{\mu(s)} ds + \frac{(p^* + Q)^{(1-\beta)}}{\beta - 1}. \tag{45}$$

Also, for $s \geqslant a_1 \equiv \int_0^{p*} \frac{1}{\mu(s)} ds$,

$$\gamma(s) = ((\beta - 1)(A - s))^{\frac{1}{1-\beta}} - Q. \tag{46}$$

Remark 9. In fact, what follows remains valid if we simply require that $\mu(p) \simeq (p+Q)^\beta$ (in the sense that is $\mu(p) - (p+Q)^\beta = o(p)$), when $p \simeq +\infty$. In particular, we can allow μ to follow either Barus formula or Roelands formula for finite values of p and simply require asymptotic behavior of the precedent type.

The aim here is to show that there exists a constant c independent of ε such that there exists $s > 1$ with

$$\int\int_\Omega (\gamma_\varepsilon(v_\varepsilon))^s dx \leqslant c, \tag{47}$$

so that in turn we get

$$\|h_\varepsilon\|_{W^{1,s}(\Omega)} \leqslant c. \tag{48}$$

The basic idea is to find σ, with $\sigma \geqslant 0$, such that $\|\gamma_\varepsilon(v_\varepsilon)^\sigma\|_{W^{1,2}(\Omega)} \leqslant c$, which implies that $\|\gamma_\varepsilon(v_\varepsilon)^\sigma\|_{L^r(\Omega)} \leqslant c$ for any $r > 1$. The required convergence will be obtained for any $s = \sigma r$. The proof will be a two-step procedure. First, we introduce the function

$$\delta_\varepsilon(v_\varepsilon) = (Q + \gamma_\varepsilon(v_\varepsilon))^\alpha \psi(v_\varepsilon),$$

with ψ a cut-off function defined in Lemma (11) and α a parameter chosen such that $\|\nabla v_\varepsilon \cdot \sqrt{\delta'_\varepsilon(v_\varepsilon)}\|_{L^2(\Omega)}$ is bounded.

Secondly, σ is chosen such that $\nabla v_\varepsilon \cdot \sqrt{\delta'_\varepsilon(v_\varepsilon)}$ behaves like $\nabla(\gamma_\varepsilon(v_\varepsilon))^\sigma$ when v_ε is near A.

Remark 10. Recall that $\gamma_\varepsilon(v_\varepsilon)$ is uniformly bounded for v_ε away from A. The main purpose of what follows is to show that (47) holds for v_ε near A.

Lemma 11. *Let $a_2 = A - a_1$ and $\psi(s)$ be a function in $C^2(R)$ with $\psi(t) = 0$ if $t < a_1 + \frac{a_2}{3}$, $\psi(t) = 1$ if $t > a_1 + \frac{2a_2}{3}$, $\psi'(t) \geqslant 0$. Then for $\varepsilon > 0$, $\delta_\varepsilon = (Q + \gamma_\varepsilon(v_\varepsilon))^\alpha \psi(v_\varepsilon)$ lies in $W_0^{1,2}(\Omega)$, for all $\alpha \geqslant 0$.*

Proof. This is obvious, since $\gamma_\varepsilon(v_\varepsilon)$ is bounded and $\psi \geqslant 0$. The introduction of ψ is necessary to ensure that δ_ε vanishes on the boundary. $\square$

Lemma 12. *The following inequality is valid*:

$$\int\int_\Omega h_\varepsilon^3 \,|\nabla v_\varepsilon|^2\, \delta'_\varepsilon(v_\varepsilon)\, dx\, dy \leqslant c \int\int_\Omega \delta'_\varepsilon(v_\varepsilon)\, dx\, dy,$$

where c is a positive constant independent of ε.

Proof. Using $\phi = v_\varepsilon + \delta_\varepsilon$ as test function in (44), we obtain

$$\int\int_\Omega h_\varepsilon^3 \,|\nabla v_\varepsilon|^2\, \delta'_\varepsilon(v_\varepsilon)\, dx\, dy \leqslant \lambda \int\int_\Omega h_\varepsilon \left|\frac{\partial v_\varepsilon}{\partial x}\right| \delta'_\varepsilon(v_\varepsilon)\, dx\, dy.$$

As $\delta_\varepsilon \geqslant 0$, $\psi \geqslant 0$, $\psi' \geqslant 0$ and $\delta'_\varepsilon \geqslant 0$ the inequality above can be rewritten in the form

$$\int\int_\Omega h_\varepsilon^3 \,|\nabla v_\varepsilon|^2\, \delta'_\varepsilon(v_\varepsilon)\, dx\, dy \leqslant \lambda \int\int_\Omega \left[h_\varepsilon^{3/2}\sqrt{\delta'_\varepsilon(v_\varepsilon)}\left|\frac{\partial v_\varepsilon}{\partial x}\right|\right] \cdot \left[\frac{\sqrt{\delta'_\varepsilon(v_\varepsilon)}}{\sqrt{h_\varepsilon}}\right] dx\, dy$$

Applying the Cauchy–Schwartz inequality, we obtain the result. $\square$

Lemma 13. *Assume that ψ can be chosen such that*

$$\psi'(s) \leqslant c^*\psi(s) \quad \forall s > a_1 + \frac{A - a_1}{2} \tag{49}$$

and that $2 - \alpha - \beta \geqslant 0$. Then there exist positive numbers C and M, independent of ε, such that

$$\delta'_\varepsilon(s) \leqslant M + C(Q + \gamma_\varepsilon(s))\psi(s) \quad \forall s. \tag{50}$$

First, assumption (49) would easily be satisfied by any ψ which is C^1 and satisfies the conditions of Lemma (11).

Set

$$B_\varepsilon(v_\varepsilon) = Q + \gamma_\varepsilon(v_\varepsilon) = \begin{cases} ((\beta - 1)(A - v_\varepsilon))^{1/(1-\beta)} & \text{if} \quad a_1 \leqslant v_\varepsilon \leqslant A - \varepsilon, \\ \dfrac{1}{(\varepsilon(\beta - 1))^{1/(\beta-1)}} & \text{if} \quad v_\varepsilon \geqslant A - \varepsilon \end{cases}$$

so that for $v_\varepsilon \geqslant a_1$, $\delta'_\varepsilon(v_\varepsilon) = \alpha B_\varepsilon^{\alpha-1}(v_\varepsilon)\ B'_\varepsilon(v_\varepsilon)\psi(v_\varepsilon) + B_\varepsilon^{\alpha}(v_\varepsilon)\psi'(v_\varepsilon)$. We will consider the various possibilities (assuming ε small with respect to $a_2/3$).

- For $v_\varepsilon(s) \leqslant a_1 + \frac{a_2}{3}$, then $\delta'_\varepsilon(v_\varepsilon) \equiv 0$ the required inequality holds.
- For $a_1 + \frac{a_2}{3} \leqslant v_\varepsilon(s) \leqslant a_1 + 2\dfrac{a_2}{3}$, then due to the definition of $B_\varepsilon(v_\varepsilon)$, we have that $B'_\varepsilon(v_\varepsilon) = B_\varepsilon^{\beta}(v_\varepsilon)$ and therefore

$$\begin{aligned}\delta'_\varepsilon(v_\varepsilon) &= \alpha \cdot B_\varepsilon^{\alpha-1}(v_\varepsilon) \cdot B'_\varepsilon(v_\varepsilon) \cdot \psi(v_\varepsilon) + \psi'(v_\varepsilon) B_\varepsilon^{\alpha}(v_\varepsilon)\\ &\leqslant \left[\alpha B_\varepsilon^{\alpha-1+\beta}(v_\varepsilon) + c^* B_\varepsilon^{\alpha}(v_\varepsilon)\right]\psi(v_\varepsilon).\end{aligned}$$

  As $B_\varepsilon(v_\varepsilon)$ is bounded uniformly with respect to ε the proof is finished.
- For $a_1 + 2\dfrac{a_2}{3} \leqslant v_\varepsilon(s) \leqslant A$, then

$$\delta'_\varepsilon(v_\varepsilon) = \begin{cases} \alpha((\beta-1)(A-v_\varepsilon))^{\frac{\alpha-1+\beta}{1-\beta}} & \text{if} \quad v_\varepsilon \leqslant A-\varepsilon,\\ 0 & \text{if} \quad v_\varepsilon \geqslant A-\varepsilon \end{cases}$$

  and

$$(Q+\gamma_\varepsilon(v_\varepsilon))\psi(v_\varepsilon) = \begin{cases} ((\beta-1)(A-v_\varepsilon))^{1/(1-\beta)} & \text{if} \quad v_\varepsilon \leqslant A-\varepsilon,\\ \dfrac{1}{(\varepsilon(\beta-1))^{\frac{1}{(\beta-1)}}} & \text{if} \quad v_\varepsilon \geqslant A-\varepsilon. \end{cases}$$

Thus, as soon as $\alpha - 1 + \beta \leqslant 1$, the desired inequality is obtained.

Let us now prove

Proposition 14. *With the assumptions of Lemmas* (11) *and* (13),

$$\int\!\!\int_\Omega |\nabla v_\varepsilon|^2\, \delta'_\varepsilon\, dx\, dy \leqslant c \quad (\textit{independently of } \varepsilon).$$

Proof. From Lemma (12) and (13), we have

$$\begin{aligned}\int\!\!\int_\Omega |\nabla v_\varepsilon|^2\, \delta'_\varepsilon\, dx\, dy &\leqslant c \int\!\!\int_\Omega \delta'_\varepsilon\, dx\, dy\\ &\leqslant c \int\!\!\int_\Omega [M + c(Q+\gamma_\varepsilon(v_\varepsilon))\psi(v_\varepsilon)]\, dx\, dy.\end{aligned}$$

Applying Proposition (4) and the assumptions for ψ, we get the result. $\square$

Proposition 15. *With the assumptions of Lemma* (11) *and* (13) *for* ψ, *if* $1 < \beta < \frac{3}{2}$ *and* $r > 1$, *then* $\gamma_\varepsilon(v_\varepsilon)$ *is bounded in* $L^r(\Omega)$ *independently of* ε.

Proof. Set $g_\varepsilon(v_\varepsilon) = (\gamma_\varepsilon(v_\varepsilon))^\theta \psi(v_\varepsilon)$. Then

$$\begin{aligned}\iint_\Omega |\nabla g_\varepsilon|^2\, dx\, dy &= \theta^2 \iint_\Omega (\gamma_\varepsilon(v_\varepsilon))^{2(\theta-1)} \cdot (\gamma'_\varepsilon)^2\, |\nabla v_\varepsilon|^2 \cdot \psi^2\, dx\, dy \\ &\quad +2\theta \iint_\Omega (\gamma_\varepsilon(v_\varepsilon))^{2\theta-1} \cdot \gamma'_\varepsilon \cdot \psi \cdot |\nabla v_\varepsilon|^2 \cdot \psi'\, dx\, dy \\ &\quad + \iint_\Omega (\gamma_\varepsilon(v_\varepsilon))^{2\theta} \cdot (\psi')^2 \cdot |\nabla v_\varepsilon|^2\, dx\, dy.\end{aligned}$$

Let us split each of these integrals into three parts. In the first part, $v_\varepsilon(x,y) \leqslant a_1 + 2\dfrac{A-a_1}{3}$, so that all terms involved are bounded independently of ε. In the second part, $v_\varepsilon(x,y) \geqslant A-\varepsilon$. In this case, $\psi \equiv 1$ and $\gamma_\varepsilon(v_\varepsilon)$ is constant, so $\nabla g_\varepsilon = 0$. In the last one, we have $a_1 + 2\dfrac{A-a_1}{3} \leqslant v_\varepsilon(x,y) \leqslant A - \varepsilon$. Recalling that in this range $\psi \equiv 1$ and $\psi' \equiv 0$ we then get

$$\iint_\Omega |\nabla g_\varepsilon|^2\, dx\, dy \leqslant c + \theta^2 \iint_{\Omega^*} \gamma_\varepsilon(v_\varepsilon)^{2(\theta-1)} \cdot (\gamma'_\varepsilon)^2\, |\nabla v_\varepsilon|^2\, dx\, dy, \tag{51}$$

where Ω^* is given by

$$\Omega^* = \left\{ \Omega \cap \left\{ a_1 + 2\frac{A-a_1}{3} \leqslant v_\varepsilon(x,y) \leqslant A - \varepsilon \right\} \right\}. \tag{52}$$

Recalling now that for small ε, $a_1 + 2\dfrac{A-a_1}{3} < A - \varepsilon$, we then get for $v_\varepsilon \geqslant a_1 + 2\dfrac{A-a_1}{3}$,

$$\delta_\varepsilon(v_\varepsilon) = (Q + \gamma_\varepsilon(v_\varepsilon))^\alpha$$

and

$$\gamma_\varepsilon(v_\varepsilon) = \delta_\varepsilon(v_\varepsilon)^{1/\alpha} - Q.$$

Hence,

$$\gamma'_\varepsilon(v_\varepsilon) = \frac{1}{\alpha}\delta_\varepsilon^{\frac{1-\alpha}{\alpha}} \cdot \delta'_\varepsilon(v_\varepsilon) = \frac{1}{\alpha}(Q + \gamma_\varepsilon(v_\varepsilon))^{1-\alpha} \cdot \delta'_\varepsilon(v_\varepsilon).$$

Moreover, from the definition of $\gamma_\varepsilon = \gamma$ in (44), (45) and (46) we get

$$\gamma_\varepsilon(v_\varepsilon) = \gamma(v_\varepsilon) = ((\beta-1)(A-v_\varepsilon))^{\frac{1}{1-\beta}} - Q,$$
$$\gamma'_\varepsilon(v_\varepsilon) = ((\beta-1)(A-v_\varepsilon))^{\frac{\beta}{1-\beta}} = (\gamma_\varepsilon(v_\varepsilon)+Q)^\beta.$$

With a view toward applying Proposition (14), then (51) is rewritten as

$$\begin{aligned}
&\iint_\Omega |\nabla g_\varepsilon|^2\, dx\, dy \\
&\quad \leqslant c + \frac{\theta^2}{\alpha}\iint_{\Omega^*} \gamma_\varepsilon(v_\varepsilon)^{2(\theta-1)}(\gamma_\varepsilon(v_\varepsilon)+Q)^\beta(\gamma_\varepsilon(v_\varepsilon)+Q)^{1-\alpha}\delta'(v_\varepsilon)\,|\nabla v_\varepsilon|^2\, dx\, dy \\
&\quad \leqslant c + \frac{\theta^2}{\alpha}\iint_{\Omega^*} [\gamma_\varepsilon(v_\varepsilon)+Q]^{2(\theta-1)+\beta+1-\alpha}\cdot\delta'(v_\varepsilon)\,|\nabla v_\varepsilon|^2\, dx\, dy.
\end{aligned}$$

Choosing now $\theta = \dfrac{\alpha+1-\beta}{2} > 0$, we find that $2(\theta-1)+\beta+1-\alpha = 0$. This allows us to use Proposition (14) and then deduce that $\|g_\varepsilon\|_{W^{1,2}(\Omega)}$ is bounded uniformly with respect to ε. This in turn implies that $(\gamma_\varepsilon(v_\varepsilon))$ is bounded in $L^r(\Omega)$ for any $r > 1$. Let us remark that the choice $\beta < \dfrac{3}{2}$ is the best one to ensure each of $\alpha+1-\beta\geqslant 0$, $2-\alpha-\beta\geqslant 0$ and $\beta > 1$. $\square$

Theorem 16. *The limit v of the sequence v_ε is a solution of the reduced problem: Find $v \in H$ such that*

$$\iint (h^3\nabla v\cdot\nabla(\phi-v) + \lambda\frac{\partial h}{\partial x}(\phi-v))\, dx\, dy \geqslant 0 \quad \forall\phi\in H \text{ and where} \tag{53}$$

$$h(x,y) = f + d \tag{54}$$

where

$$f = h_0 + \frac{x^2+y^2}{R^2} \tag{55}$$

and

$$d = \iint \gamma(v(x-s, y-t))\frac{1}{\sqrt{s^2+t^2}}\, ds\, dt \tag{56}$$

with γ defined by (44) *and* $1 < \beta < 3/2$.

Proof. Since the Fourier transform of the kernel $K(s,t) = \frac{1}{\sqrt{s^2+t^2}}$ is given by the function $\frac{1}{\sqrt{\xi_1^2+\xi_2^2}}$ (see [25] or [27]) it follows from the properties of the product of convolution that if $\gamma_\varepsilon(v_\varepsilon)$ is in $L^r(\Omega)$ then

$$h_\varepsilon(v_\varepsilon) = \delta + \iint_\Omega \gamma_\varepsilon(v_\varepsilon(x-s, y-t)\frac{1}{\sqrt{s^2+t^2}}\,ds\,dt$$

is in $W^{1,r}(\Omega)$. Furthermore, since $\gamma_\varepsilon(v_\varepsilon)$ is uniformly bounded in L^r it follows that $h_\varepsilon(v_\varepsilon)$ is uniformly bounded in $W^{1,r}(\Omega)$.

It is easy to see then that $h_\varepsilon(v_\varepsilon)$ converges in the Holder space $C^{0,\alpha}(\Omega)$ (see Adams [1] or Nečas [22], for example) towards $h(v)$ for any $\alpha > 0$.

So we can pass to the limit in all terms of (44) with the exception of the quadratic one,

$$\iint_\Omega h_\varepsilon^3 \nabla v_\varepsilon \cdot \nabla v_\varepsilon\,dx\,dy.$$

This last term is rewritten as

$$\iint_\Omega (h_\varepsilon^3 - h^3(v))\nabla v_\varepsilon \cdot \nabla v_\varepsilon\,dx\,dy + \iint_\Omega h^3(v)\nabla v_\varepsilon \cdot \nabla v_\varepsilon\,dx\,dy,$$

in which the first integral tends to zero, since $|\nabla v_\varepsilon|$ is bounded in L^2, and the second one satisfies the inf-limit property

$$\iint_\Omega h^3(v)\nabla v \cdot \nabla v\,dx\,dy \leqslant \liminf \iint_\Omega h^3(v)\nabla v_\varepsilon \cdot \nabla v_\varepsilon\,dx\,dy,$$

which allows us to conclude the proof. □

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