Semigroup Varieties with the Amalgamation Property

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1. INTRODUCTION AND SUMMARY

Amalgamation questions for classes of semigroups which form varieties of universal algebras have been extensively studied in recent years by Hall [6], Imaoka [10] and the author [1, 2]. In [6], Hall determined which varieties of inverse semigroups have the weak or strong amalgamation property. His answer is necessarily dependent on the solution to the corresponding problem for varieties of groups; this solution is not yet complete (see [13, Problem 6]).

In [10], Imaoka determined which varieties of bands have the weak or strong amalgamation property, and in [1] the author obtained a partial answer to the corresponding question for varieties of completely regular semigroups. The incompleteness of the solution for varieties of groups also hampered the latter investigation, as did the difficulties presented by the lattice of varieties of completely simple semigroups. (In [11], Jones obtained a description of this lattice in terms of certain subgroups of a free group of countable rank; but a description in terms of identities seems difficult to obtain.)

For semigroup varieties, the additional assumption of commutativity leads to the following complete solution: a variety of commutative semigroups has the weak (strong) amalgamation property if and only if it is a variety of inflations of semilattices of abelian groups [2].

The main result of this paper is that a variety of semigroups has the weak (strong) amalgamation property if and only if it is a variety of inflations of completely regular semigroups and the subvariety consisting of all its completely regular semigroups has the same property. Crucial to the proof of this result is the characterization in [3] of those identities which determine varieties of inflations of completely regular semigroups.

The following is an open question. Does there exist a non-abelian variety of groups with all its finite members abelian, with all its members having...
periods dividing some fixed positive integer, and which has the weak amalgamation property? If this question can be answered in the negative, then we have determined precisely which varieties of semigroups have the weak or strong amalgamation property.

2. PRELIMINARIES

For background on semigroups, the reader is referred to [4], [9] or [14], on varieties of semigroups to [5], and on the amalgamation properties to [7] or [8].

In what follows, all varieties are subvarieties of the variety of all semigroups. Semigroups which are unions of groups will sometimes be referred to simply as unions of groups; such semigroups are also called completely regular semigroups.

A semigroup $S$ is said to be an inflation of a subsemigroup $T$ if $S^2 \subseteq T$ and there exists an idempotent homomorphism of $S$ upon $T$.

If $u$ is a word (in a free semigroup) involving finitely many variables, and $x$ is a variable which may or may not appear in $u$, then $|x|^u$ denotes the number of occurrences of $x$ in $u$. Let $|u|$ denote the length of the word $u$. If $u$ involves no variables other than $x$ and $y$, it will often be written as $f(x, y)$, and $|x|^u$ as $|x|^y$.

Where only part of a word is under consideration, the remainder of the word will sometimes be denoted $(\ldots)$; for example, $x(\ldots)y$ denotes a word which begins in $x$ and ends in $y$.

If $\mathcal{V}$ is a variety, then $\mathcal{I}(\mathcal{V})$ denotes the set of all identities satisfied by all semigroups in $\mathcal{V}$.

If $\{w_1 = w_1', \ldots, w_n = w_n'\}$ is a finite set of identities, then the variety determined by this set will be denoted $\{w_1 = w_1', \ldots, w_n = w_n'\}$.

Result 1 (from [3, Theorem 1]). Let $f(x, y)$ be a word of length at least 3 and which neither begins nor ends in $xy$. If a semigroup $S$ satisfies the identity $xy = f(x, y)$, then $S$ is an inflation of a completely regular semigroup.

Result 2 [2, Theorem 1]. A variety $\mathcal{V}$ has the weak (strong) amalgamation property if and only if the variety $\mathcal{V}'$ of all inflations of semigroups from $\mathcal{V}$ has the same property.

3. VARIETIES WITH AMALGAMATION

We study four examples of amalgams which are not weakly embeddable within certain varieties of semigroups. Using these examples we obtain infor-
mation about the structure of semigroups in a variety which has the weak amalgamation property.

From the first example we deduce that if \( S \) belongs to a variety which has the weak amalgamation property, then \( S \) satisfies an identity \( xy = f(x, y) \), with \( f(x, y) \) having a form described below.

**Example 1.** (This example is essentially due to Kimura [12].) Let \( U = \{u, v, 0\} \) be a three-element null semigroup. Extend the multiplication of \( U \) to one of \( A = U \cup \{a\} \) by defining \( au = ua = v \) and making all other products equal to 0.

Similarly, extend the multiplication of \( U \) to one of \( B = U \cup \{b\} \) by defining \( bv = vb = u \) and making all other products equal to 0. Clearly, \( A \) and \( B \) are isomorphic.

Now the amalgam \( (A, B; U) \) is not weakly embeddable in any semigroup; for, suppose that \( W \) is a semigroup containing \( A \) and \( B \). Then, in \( W \), \( u = vb = aub = a0 = 0 \).

**Lemma 1.** If a variety \( \mathcal{T} \) has the weak amalgamation property, then \( \mathcal{T}(\mathcal{T}') \) includes an identity of the form \( xy = f(x, y) \), where \( |x| \geq 1, |y| \geq 1 \) and \( |f(x, y)| \geq 3 \).

**Proof.** If \( \mathcal{T} \) has the weak amalgamation property, then \( \mathcal{T} \) must not include the semigroup \( A \) of Example 1. Hence \( \mathcal{T}(\mathcal{T}') \) includes an identity of the form \( xy = w \), where \( |w| \geq 3 \) (since any identity not satisfied by \( A \) implies an identity of this form; see [2, Theorem 2]). It is easy to show that if \( |w| \geq 3 \), then the identity \( xy = w \) implies an identity \( xy = f(x, y) \), where \( |x| \geq 1, |y| \geq 1 \) and \( |f(x, y)| \geq 3 \).

**Corollary.** If a variety \( \mathcal{T} \) has the weak amalgamation property, then there exists a positive integer \( n \) such that \( \mathcal{T} \subseteq [x^2 = x^{n+2}] \). Hence every group in \( \mathcal{T} \) has period dividing \( n \).

**Proof.** By Lemma 1, there exists a word \( f(x, y) \) of length at least 3 such that the identity \( xy = f(x, y) \) is satisfied by every semigroup in \( \mathcal{T} \). Put \( n = |f(x, y)| - 2 \). Then

\[
\mathcal{T} \subseteq [xy = f(x, y)] \\
\subseteq [x^2 = x^{n+2}].
\]

**Lemma 2.** If a variety \( \mathcal{T} \) has the weak amalgamation property, \( S \in \mathcal{T} \) and \( e \) is an idempotent of \( S \), then \( eSe \) is a union of groups.

**Proof.** Let \( C_2 = \{0, b\} \) be a two-element null semigroup. Adjoin an identity element 1 to form \( C_2^1 \). Put \( T = C_2^1 \times C_2^1 \), and \( W = T \setminus \{(1, 1)\} \). The
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semigroup \( C = \{(1, 0), (0, 1), (0, 0), (b, 0), (0, b)\} \) is an ideal of \( W \), and \( W/C \cong A \), the semigroup of Example 1. So if \( \mathcal{F}^- \) has the weak amalgamation property, then \( C \notin \mathcal{F}^- \).

Let \( a \in eSe \) and assume that \( a \notin \{a^2, a^3, \ldots\} = Y \). Put \( X = \{e, a, a^2, \ldots\} \). Then \( X/Y \in \mathcal{F}^- \), but \( X/Y \cong C \). So our assumption is false, and \( \langle a \rangle \) is a group.

The second example enables us to prove that all completely regular semigroups in a variety which has the weak amalgamation property are normal bands of groups.

**Example 2** (T. E. Hall; see [10; 7, Remark 7]). Let \( \mathcal{U} = \{f, g, h\} \) be a three-element right zero semigroup. Extend the multiplication of \( \mathcal{U} \) to one of \( S = \mathcal{U} \cup \{e\} \) so as to make \( S \) a right regular band with \( g < e, h < e \) and \( fe = g \).

Extend the multiplication of \( \mathcal{U} \) to one of \( T = \mathcal{U} \cup \{x, y\} \) so as to make \( T \) a right normal band whose order relations are precisely \( x < f, x < h \) and \( y < g \), with \( \{x, y\} \) being a right zero subsemigroup of \( T \).

Then the amalgam \( (S, T; U) \) is not weakly embeddable in any semigroup; for, suppose that \( W \) is a semigroup containing \( S \) and \( T \). Then, in \( W \), \( x = xh = xhe = xe = xe = xfe = xg = y \).

**Lemma 3** (T.E. Hall, verbal communication). If a variety \( \mathcal{F}^- \) has the weak amalgamation property, then every union of groups in \( \mathcal{F}^- \) is a normal band of groups.

**Proof.** If \( \mathcal{F}^- \) includes any completely regular semigroup which is not a normal band of groups, then \( \mathcal{F}^- \) contains either the variety of all right regular bands or the variety of all left regular bands (see the proof of [1, Theorem 1]). Hence \( \mathcal{F}^- \) includes either the semigroups \( S, T \) and \( U \) of Example 2 or their dual semigroups. So \( \mathcal{F}^- \) does not have the weak amalgamation property.

**Lemma 4.** Suppose that a variety \( \mathcal{F}^- \) has the weak amalgamation property, with \( n \) chosen so that \( \mathcal{F}^- \subseteq \{x^2 = x^{n+2}\} \), and that \( S \in \mathcal{F}^- \). If \( e \) is an idempotent of \( S \), then \( eSe \) is a normal band of groups, and hence satisfies the identity \( (axa)^n = (ayxa)^n \). So \( S \) satisfies the condition: for all \( a, x, y \in S \) and for any idempotent \( e \in S \),

\[ (eaeexayeae)^n = (eaeeyexeae)^n. \]  

**Proof.** If \( e \) is an idempotent of \( S \), then \( eSe \) is a union of groups (by Lemma 2), and is therefore (by Lemma 3) a normal band of groups. By the
corollary to Lemma 1, each of these groups has period dividing \( n \). Now any normal band of groups satisfies the completely regular semigroup identity

\[
(axya)(axya)^{-1} = (ayxa)(ayxa)^{-1}
\]

(see, for example, [15]). For a semigroup which is a union of groups having periods dividing \( n \), this identity is equivalent to \((axya)^n = (ayxa)^n\). Clearly, \( eSe \) satisfies the latter identity for every idempotent \( e \in S \) if and only if \( S \) satisfies condition (1) above.

**Remark 1.** If \( n \) is a positive integer, put

\[
m = \begin{cases} 
n & \text{if } n \geq 2 \\ 
2 & \text{if } n = 1.
\end{cases}
\]

Then condition (1) is equivalent to the identity

\[
(b^{"ab”}xb^{"y}xb^{"ab”})^n = (b^{"ab”}yb^{"y}xb^{"ab”})^n. 
\]

Together with the results already presented in this section, Example 3 enables us to prove that all semigroups in a variety which has the weak amalgamation property are ideal extensions of completely regular semigroups by null semigroups.

**Example 3.** Let \( Y_2 = \{0, e\} \) be a two-element semilattice. Extend the multiplication of \( Y_2 \) to one of \( D = Y_2 \cup \{a\} \) by defining \( ae = a \) and making all other products involving \( a \) equal to 0.

Let \( S = \{u, v, 0, f, g\} \) be the subsemigroup of \( D^2 \) given by \( u = \{a, 0\}, v = \{a, a\}, 0 = \{0, 0\}, f = \{e, e\} \) and \( g = \{e, 0\} \). Put \( U = \{u, v, 0, f\} \). Extend the multiplication of \( U \) to one of \( T = U \cup \{h\} \) by defining \( h^2 = h, hf = fh = h, uh = vh = v, \) and making all other products involving \( h \) equal to 0. It is easy to show that \( S \) and \( T \) are isomorphic.

Suppose that the amalgam \((S, T; U)\) can be weakly embedded in a semigroup \( W \) where, for some positive integer \( n \), \( W \) satisfies condition (1) of Lemma 4. Then, in \( W \), \( u = vg = uhg \), so that \( u = u(hg)^n \). Also, \( v = uh = vgh = ugh \), so that \( v = v(gh)^n = u(gh)^n \). Then

\[
(hg)^n = (fghf)^n = (fffghff)^n = (ffghff)^n = (fgh)^n,
\]

so that \( u = u(hg)^n = u(gh)^n = v \). This shows that \((S, T; U)\) is not weakly embeddable within any variety having the weak amalgamation property.
Lemma 5. *The semigroup* $D$ of Example 3 *generates the variety*
\[ xy = xy^2, zxy = zyx, x^2y - y^2x \].

*Proof.* It is clear that $D$ satisfies each of these identities. Now $D$ contains a non-trivial semilattice, and hence any identity satisfied by $D$ must be homotypical (that is, both sides involve the same variables). Every homotypical identity has exactly one of the following five forms:

(a) $xu = xv$ where $|x_u|_u = |x_v|_v = 0$;
(b) $xu = xv$ where $|x_u|_u = 0$, $|x_v|_v \geq 1$;
(c) $xu = xv$ where $|x_u|_u, |x_v|_v \geq 1$;
(d) $xu = yv$ where $|x_u|_u, |y_v|_v \geq 1$;
(e) $xu = yv$ where $|x_u|_u = 0$.

Any identity from (a) or (c) is implied by the conjunction of $xy = xy^2$ and $zxy = zyx$. Any identity from (d) is implied by the conjunction of $xy = xy^2$, $zxy = zyx$ and $x^2y = y^2x$.

If $xu = xv$ is an identity as in (b), then assigning the value $a$ from the semigroup $D$ to the variable $x$ and the value $e$ to all other variables results in $xu$ having value $a$ and $xv$ having value $0$. So $D$ does not satisfy this identity. Similarly, if $xyu = yv$ is an identity as in (e), then assigning value $a$ to $x$ and $e$ to all other variables results in $xu$ having value $a$ and $yv$ having value $0$. So $D$ does not satisfy this identity.

**Theorem 1.** *If a variety* $\mathcal{V}$ *has the weak amalgamation property and* $S$ *is an* $\mathcal{V}$ *semigroup in* $\mathcal{P}$, *then* $S^2$ *is regular.*

*Proof.* By Lemma 1, $\tilde{\mathcal{I}}(\mathcal{P})$ includes an identity $xy = f(x,y)$, where $|x|_x \geq 1$, $|y|_y \geq 1$ and $|f(x,y)| \geq 3$. Now $f(x,y)$ has exactly one of the following forms:

1. $xyg(x,y) \geq 0$;
2. $(a)$ $xy^{n+1}, n \geq 1$,
   (b) $xyg(x,y) y^2, |x|_x \geq 1$,
   (c) $xyg(x,y) x, |g(x,y)| \geq 0$;
3. (a) $x^{n+1}y, n \geq 1$,
   (b) $x^2g(x,y) xy, |y|_y \geq 1$,
   (c) $yg(x,y) xy, |g(x,y)| \geq 0$;
4. (a) $x^2g(x,y) y^2, |g(x,y)| \geq 0$,
   (b) $x^2g(x,y) x, |y|_y \geq 1$,
   (c) $yg(x,y) y^2, |x|_x \geq 1$,
   (d) $yg(x,y) x, |g(x,y)| \geq 1$. 
Suppose that $S$ satisfies the identity $xy = f(x, y)$. If $f(x, y)$ is as in (1), then clearly $S^2$ is regular. If $f(x, y)$ is as in (4), then by Result 1, $S$ is an inflation of a completely regular semigroup, and hence $S^2$ is (completely) regular.

To complete the proof of the theorem, we need three lemmas.

**Lemma 6.** Let $D$ be the semigroup of Example 3, and let $D'$ be its dual semigroup. Suppose that $r(Y)$ includes an identity as in (2)(a) [(3)(a)], but that $P$ does not include $D$ [D' 1. Then $f(Y)$ also includes an identity as in (2)(b) [(3)(b)] or (4).

**Proof.** By the proof of Lemma 5, any identity not satisfied by $D$ must be

(i) a heterotypical identity, or

(ii) $xu = xv$, where $|x_u| = 0$, $|x_v| \geq 1$, or

(iii) $xu = yv$, where $|x_u| = 0$.

Now suppose that $\bar{f}(Y)$ includes an identity as in (2)(a) and also an identity as in (i) above. For a variable occurring on one side of the identity from (i) and not on the other side, replace all occurrences of that variable by $x$. Replace all other variables by $y$. This gives us an identity $y^r = h(x, y)$, where $r \geq 1$ and $|x_y| \geq 1$. So if $S$ is in $\bar{f}$ and $x, y \in S$, we have

$$xy = xy^{n+1} \text{ (from (2)(a))} = xy^m \quad \text{(for some } m \geq r + 2)$$

$$= xy^r y^{m-r} = xh(x, y) y^{m-r}.$$

Thus, $\bar{f}(Y)$ includes an identity as in (2)(b) or (4)(a).

Suppose we have (2)(a) and (ii) above. Since we can assume that $xu = xv$ is homotypical, this identity implies an identity $x = x^{r+1}$ where $r \geq 1$ (if $|u| = 0$) or an identity $xy^r = xh(x, y)$ where $r \geq 1$, $|x_y| \geq 1$ and $|y_y| \geq 1$. Now $x = x^{r+1}$ implies $xy = (xy)^{r+1}$, which is an identity as in (1). Otherwise, we have

$$xy = xy^{n+1} = xy^m \quad \text{(for some } m \geq r + 2)$$

$$= xy^r y^{m-r} = xh(x, y) y^{m-r},$$

as before.

Suppose we have (2)(a) and (iii) above. Assuming that $xu = yv$ is homotypical, this identity implies an identity $xv = yh(x, y)$ where $r \geq 1$ and $|x_y| \geq 1$. Then

$$xy = xy^{n+1} = xy^m \quad \text{(for some } m \geq r + 2)$$

$$= xy^r y^{m-r} = yh(x, y) y^{m-r}.$$

Thus, $\bar{f}(Y)$ includes an identity as in (4)(c).
Dually, (3)(a) together with an identity to exclude $D'$ gives an identity as in (3)(b), (4)(a) or (4)(b).

**Lemma 7.** Any identity from (2)(c) [(3)(c)] implies an identity as in (2)(b) [(3)(b)] or as in (1).

**Proof.** Suppose we have an identity as in (2)(c). Then

$$xy = f(x, y) = xg(x, y)x = xf(y, g(x, y)x)$$

$$= xyg(x, y)yg(y, g(x, y)x)y.$$

Thus, we have an identity as in (1) or (2)(b).

**Lemma 8.** Any identity from (2)(b) or (3)(b) implies an identity as in (1).

**Proof.** Suppose we have an identity as in (3)(b). Then

$$xy = f(x, y)$$

$$= x^2g(x, y)xy$$

$$= x^{m_1}y^{n_1}x^{m_2}y^{n_2} \ldots x^{m_k}y$$

$$= x^{m_1}f(y^{n_1}, x^{m_2}y^{n_2} \ldots x^{m_k}y)$$

$$= x^{m_1}(y^{n_1})^{m_1}(x^{m_2}y^{n_2} \ldots x^{m_k}y)^{n_1}(\ldots)xy$$

$$= x^{m_1}y^{m_1}(x^{m_2}y^{n_2} \ldots x^{m_k}y)^{n_2}(\ldots)xy$$

$$= x^{m_1}y^{m_1}(x^{m_2}y^{n_2} \ldots x^{m_k}y)^{n_3}(\ldots)xy$$

$$\vdots$$

$$= x^{m_1}y^{m_1}x^{m_2}y^{m_2} \ldots x^{m_k-1}y^{m_k-1}x^{m_k}y(\ldots)xy$$

$$= x^{m_1}y^{m_1}x^{m_2}y^{m_2} \ldots x^{m_k-1}y^{m_k-1}x^{m_k}f(y, (\ldots))xy$$

$$= x^{m_1}y^{m_1}x^{m_2}y^{m_2} \ldots x^{m_k-1}y^{m_k-1}x^{m_k}y^{m_1}(\ldots)xy$$

$$= f(x, y^{m_1})(\ldots)xy$$

$$= xy^{m_1}(\ldots)xy.$$

Thus, we have an identity as in (1). Dually, an identity from (2)(b) implies an identity as in (1).
We now return to the proof of the theorem. Suppose \( f(x, y) \) is as in (2)(a) or (3)(a). Since \( \mathcal{F} \) has the weak amalgamation property, the semigroups \( D \) and \( D' \) (of Example 3 and its dual) do not belong to \( \mathcal{F} \). Hence, by Lemma 6, \( \mathcal{I}(\mathcal{F}) \) includes an identity as in (2)(b), (3)(b) or (4).

Suppose \( f(x, y) \) is as in (2)(c) or (3)(c). Then, by Lemma 7, \( \mathcal{I}(\mathcal{F}) \) includes an identity as in (1), (2)(b) or (3)(b).

Thus, \( \mathcal{I}(\mathcal{F}) \) includes an identity from (1), (2)(b), (3)(b) or (4). Since (by Lemma 8) any identity from (2)(b) or (3)(b) implies an identity as in (1), we have that \( \mathcal{I}(\mathcal{F}) \) includes an identity from (1) or (4), as required.

**Theorem 2.** If a variety \( \mathcal{F} \) has the weak amalgamation property, and \( S \) is a regular semigroup in \( \mathcal{F} \), then \( S \) is a union of groups.

**Proof.** Assume that \( S \) includes an element \( a \) which is not completely regular. If \( S \) has a kernel \( K \), then \( K \) is simple and hence completely simple (since \( S \) is periodic, by the corollary to Lemma 1); therefore, \( a \not\in K \). So the principal factor \( J(a)/I(a) \) is 0-simple, and hence completely 0-simple.

Construct the semigroup \( T \) from the partial semigroup \( J_a \cup \{0\} \) by defining \( xy = 0 \) if \( xy \not\in J_a \). Then \( T \in \mathcal{F} \), since \( T \cong J(a)/I(a) \). Let \( a' \) be an inverse of \( a \) and put \( e = a'a \). Then the three-element subsemigroup \( \{a, e, 0\} \) of \( T \) is isomorphic to the semigroup \( D \) of Example 3. Since \( \mathcal{F} \) has the weak amalgamation property, \( D \in \mathcal{F} \). Thus, our initial assumption was false, and \( a \) is completely regular.

**Corollary.** If a variety \( \mathcal{F} \) has the weak amalgamation property, and \( S \in \mathcal{F} \), then \( S^2 \) is a union of groups. Hence \( S \) satisfies the identity \( xy = (xy)^{n+1} \), for some positive integer \( n \).

**Lemma 9.** Let \( S \) be a semigroup satisfying the identity \( xy = (xy)^{n+1} \). Let \( \{m_1, n_1, m_2, n_2, \ldots, m_k, n_k\} \) be a list of positive integers. Then \( S \) satisfies an identity

\[
xy = x y^m x^{n_m} y^{m_2} x^{n_2} \cdots x^{m_k} y^{n_k} v(x, y),
\]

where \( v(x, y) \) is some word involving \( x \) and \( y \).

**Proof.** The method of proof is as follows. Begin with the third variable (that is, the second \( x \)) appearing in \( (xy)^{n+1} \). We increase the index of this power of \( x \) by 1, and then continue to increase it until it reaches \( m_1 \). We then increase the index of the second \( y \) until it reaches \( n_1 \). This process is continued until the result is achieved.

The following equations hold in \( S^1 \), where \( x \) and \( y \) are arbitrary elements of \( S \) and \( x^0 \) is understood to mean the identity element 1 of \( S^1 \):
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\[ xy' = (xy)^{n+1} = xyxy(xy)^{n-1} = (xyx)^{n+1} y(xy)^{n-1} \]
\[ = (xyx)(xyx)(\ldots) = xyx^2(\ldots) = (xyx^2)^{n+1}(\ldots) \]
\[ = (xyx^2)(xyx^2)(\ldots) = xyx^3(\ldots) = \ldots \]
\[ = xyx^{m1}(\ldots) \quad \text{(end of stage 1)} \]
\[ = x(yx^{m1})^{n+1}(\ldots) = xyx^{m1}y(\ldots) = x(yx^{m1}y)^{n+1}(\ldots) \]
\[ = x(yx^{m1}y)(yx^{m1}y)(\ldots) = xyx^{m1}y^2(\ldots) = \ldots \]
\[ = xyx^{m1}y^{n1}(\ldots) \quad \text{(end of stage 2)} \]
\[ = xyx^{m1-1}y^{n1}(\ldots) = xyx^{m1-1}(xy^{n1})^{n+1}(\ldots) \]
\[ = xyx^{m1-1}y^{n1}x(\ldots) = xyx^{m1-1}(xy^{n1}x)^{n+1}(\ldots) \]
\[ = xyx^{m1-1}(xy^{n1}x)(yx^{n1}x)(\ldots) = xyx^{m1}y^{n1}x^2(\ldots) \]
\[ = \ldots = xyx^{m1}y^{n1}x^{m2}(\ldots) \quad \text{(end of stage 3)} \]
\[ = \ldots = xyx^{m1}y^{n1}x^{m2}y^{n2} \ldots x^{mkn}y(\ldots), \]

as required.

Together with the information we have already obtained on the structure of those semigroups belonging to a variety which has the weak amalgamation property, the fourth example enables us to show that such a variety consists entirely of inflations of completely regular semigroups.

**Example 4.** Let \( E = \{ e, f, g \} \) be a three-element right zero semigroup. Extend the multiplication of \( E \) to one of \( F = E \cup \{ a \} \) by defining \( a^2 = e \), \( fa = g \), \( ea = e \), \( ga = e \) and \( ax = ex \) for all \( x \) in \( E \).

Similarly, extend the multiplication of \( E \) to one of \( G = E \cup \{ b \} \) by defining \( b^2 = e \), \( gb = f \), \( eb = e \), \( fb = e \) and \( by = ey \) for all \( y \) in \( E \). Clearly, \( F \) and \( G \) are isomorphic.

Suppose that the amalgam \( (F, G; E) \) can be weakly embedded in a semigroup \( W \). Then, in \( W \), \( g = fa = gba \). Suppose also that \( W \) satisfies an identity \( xy = (xy)^{n+1} \). Then, by Lemma 9, \( W \) satisfies an identity \( xy = xy^{n}v(x, y) \), whence

\[ ba = bab^2v(b, a) = baev(b, a) = bev(b, a) = ev(b, a). \]

Now \( ea = e \) and \( eb = e \), so that by successively premultiplying each power of \( b \) and each power of a appearing in \( v(b, a) \) by \( e \), we eventually get \( ev(b, a) = e \). So \( g = gba = gev(b, a) = ge = e \).

**Remark 2.** The amalgam \( (F, G; E) \) is not weakly embeddable in any semigroup satisfying a non-trivial identity \( xy = f(x, y) \); the proof of this is slightly longer than that given above.
LEMMA 10 [3, Result 3]. The semigroup $F$ of Example 4 generates the variety $[xy = zxy]$.

LEMMA 11. Let $\mathcal{V}$ be a variety of ideal extensions of completely regular semigroups by null semigroups. Let $F$ be the semigroup of Example 4, and let $F'$ be its dual semigroup. Then $\mathcal{V}$ is a variety of inflations of completely regular semigroups if and only if $\tilde{I}(\mathcal{V})$ includes neither $F$ nor $F'$.

Proof. Since $F$ is not an inflation of a completely regular semigroup, one-half of the lemma is obvious.

Clearly, $\mathcal{V}$ consists entirely of ideal extensions of completely regular semigroups by null semigroups if and only if $\tilde{I}(\mathcal{V})$ includes the identity $xy = (xy)^{n+1}$, for some positive integer $n$. Suppose that $F' \notin \mathcal{V}$. Using the dual of Lemma 10, it is routine to show that any identity not satisfied by $F'$ implies an identity having one of the following forms:

(a) $x^{m_1}y^{n_1} \cdots x^{m_k}y^{n_k} = y^k(x, y)$,
(b) $x^{m_1}y^{n_1} \cdots x^{m_k}y^{n_k} = x^2h(x, y)$,

where in either case $k \geq 1$, $m_i \geq 1$ for $i = 1, \ldots, k$, $n_i \geq 1$ for $i = 1, \ldots, k - 1$, $n_k \geq 0$ and $|h(x, y)| \geq 0$. If $\tilde{I}(\mathcal{V})$ includes an identity from (a), then the following equations hold in any semigroup belonging to $\mathcal{V}$:

\begin{align*}
xy &= (xy)^{n+1} = xyx^{m_1}y^{n_1} \cdots x^{m_k}y^{n_k} (\cdots) \quad \text{(by Lemma 9)} \\
 &= xyxyh(x, y)(\cdots) \quad \text{(by the identity (a))} = xy^2(\cdots) \\
 &= xy^m(\cdots) \quad \text{(for some $m > m_1$)} = xy^m y^{m-m_1}(\cdots) \\
 &= (xy^m)^{n+1} (\cdots) = xy^m(xy^m)^n (\cdots) = xy^m xy(\cdots) \\
 &= xy^m(xy)^{n+1} (\cdots) = xy^m x^{n_1}y^{n_2} x^{n_3} \cdots y^{n_k} x^{n_k}(\cdots) \\
\end{align*}

by an obvious modification of the proof of Lemma 9. Thus,

$xy = xxh(y, x)(\cdots) = x^2(\cdots)$.

If $\tilde{I}(\mathcal{V})$ includes an identity from (b), then

\begin{align*}
xy &= (xy)^{n+1} = xyx^{n_1}y^{n_2} x^{n_3} \cdots y^{m_k} x^{n_k}(\cdots) \\
 &= xy^2h(y, x)(\cdots) = xy^m(\cdots) \quad \text{(for some $m > n_1$)} \\
 &= xy^{n_1}y^{m-n_1}(\cdots) = (xy^{n_1})^{n+1} (\cdots) = xy^{n_1} xy(\cdots) \\
 &= xy^{n_1}(xy)^{n+1} (\cdots) = xy^{n_1} x^{m_2}y^{n_2} \cdots x^{m_k}y^{n_k}(\cdots) \\
 &= x^2h(x, y)(\cdots) .
\end{align*}
THE AMALGAMATION PROPERTY

If \( F \) is also excluded from \( \mathcal{Y} \), the dual procedure can be applied. We obtain an identity \( xy = (\ldots) y^2 \). Then

\[
xy = (xy)^{n+1} = xy(xy)^{n-1} xy = x^2(\ldots) y^2.
\]

By Result 1, \( \mathcal{Y} \) is a variety of inflations of completely regular semigroups.

**Theorem 3.** If a variety \( \mathcal{Y} \) has the weak amalgamation property, then \( \mathcal{Y} \) is a variety of inflations of completely regular semigroups.

**Proof:** By the corollary after Theorem 2, \( \mathcal{Y} \) consists entirely of ideal extensions of completely regular semigroups by null semigroups. Also, \( \mathcal{Y} \) must include neither the semigroup \( F \) of Example 4 nor its dual semigroup. Therefore, by Lemma 11, every semigroup in \( \mathcal{Y} \) is an inflation of a completely regular semigroup.

**Theorem 4.** The following varieties of semigroups have the strong amalgamation property:

(a) every variety of inflations of orthodox normal bands of abelian groups, and

(b) every variety of inflations of completely simple semigroups with only abelian subgroups.

Conversely, any variety of semigroups having the weak (strong) amalgamation property is one of the above or is

(c) a non-abelian variety \( \mathcal{H} \) of groups with all its finite members abelian, the variety \( CS(\mathcal{H}) \) of all completely simple semigroups whose subgroups lie in \( \mathcal{H} \), or a subvariety of \( CS(\mathcal{H}) \) containing \( \mathcal{H} \), or

(d) the variety of all inflations of semigroups from a variety as in (c).

**Proof:** From Theorems 3 and 6 of \([1]\), any variety of completely regular semigroups as in (a) or (b) has the strong amalgamation property, and any other variety of completely regular semigroups having the weak or strong amalgamation property must be a variety as in (c). The theorem then follows from Result 2 and Theorem 3.

**Remark 3.** If a variety \( \mathcal{H} \) of groups as in (c) has the weak (strong) amalgamation property then so do the corresponding varieties of left groups, right groups, rectangular groups and completely simple semigroups. To each of the above there corresponds a variety as in (d) having the same property. We cannot answer whether or not any other variety as in (c) has the weak or strong amalgamation property.
COROLLARY (from [2, Theorem 3]). A variety \( \mathcal{V} \) of commutative semigroups has the weak (strong) amalgamation property if and only if \( \mathcal{V} \) is a variety of inflations of semilattices of abelian groups.

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