Multiple Nevanlinna–Pick interpolation with both interior and boundary data and its connection with the power moment problem

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Abstract

The so-called Hankel vector approach is used to handle three boundary versions of the multiple Nevanlinna–Pick interpolation problem in the Nevanlinna class \( \mathcal{N} \) involving both interior and boundary data. It turns out that each of these boundary interpolation problems can be reduced to what amounts to a certain truncated (standard or nonstandard) Hamburger moment problem, associated with the Hankel vector of the former, with some possible constraint on distribution functions that assign no mass to any of the real nodes. In particular, this leads to solvability criteria for each of these interpolation problems and the description of solutions by using results from theory of moments. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction and formulation of boundary NP interpolation problem

In [7] we considered the complete solution of the multiple Nevanlinna–Pick interpolation problem in the Nevanlinna class \( \mathcal{N} \) (the NP(\( \mathcal{N} \)) problem) along the
so-called Hankel vector approach, which allows us to reduce the NP(avenport) problem to what amounts to a certain nonstandard truncated Hamburger moment problem associated with the Hankel vector of that NP(avenport) problem (see also [8] for a matrix version thereof).

In the present paper, we continue our investigations of the NP(avenport) problem and are primarily interested in three boundary versions of the NP(avenport) problem, involving both interior and boundary data. These boundary problems do arise, in particular, in the theory of molecular interactions [25] and in connection with the synthesis of circuits with losses by the Darlington method [12]. Digging a little deeper, we can show that the Hankel vector approach to the solving of these boundary NP interpolation problems is also appropriate and allows us to connect these problems with a (standard or nonstandard) truncated Hamburger moment problem with or without some additional constraint on the distribution functions. It is natural to view the latter problem can be viewed as a boundary NP problem for which all interpolation nodes coincide with a single point, infinity. Thanks to such a connection, the theory of moments enables one not only to find existence and uniqueness conditions for the solutions of each of the boundary NP problems under consideration, but also to describe the full solutions, and so on.

The boundary NP problems have a long history. Nevanlinna [22] did at first his work in the unit disk with the help of a modified Schur algorithm. In the 1930s, Krein [19] studied a boundary NP problem in the class \( \mathcal{N} \) with real and simple nodes only (which is one particular situation of the BNP(avenport) problem below), by using a method of Riesz [26]. In a paper [14] published in 1937, Kotelyanskii considered specifically a more general boundary NP problem in the class \( \mathcal{N} \) involving both interior and boundary data (which is the same as the BNP(avenport) problem below).

However, the main theorems there on solvability criteria are incorrect. After that a number of results related to the boundary NP problems (in essence to the BNP(avenport) problem and its analogue in the class \( \mathcal{C} \) (Caratheodory) or \( \mathcal{S} \) (Schur)) for the scalar functions and certain intricate matrix analogues have been addressed by many different methods. We refer to [2–6,11,15–17,21,23–25] for more information; among others, the papers [23,24] do contain the solution of a matrix version of the BNP(avenport) problem by the method of the Fundamental Matrix Inequality in the Potapov’s theory.

It should be noted that a corresponding result of Sarason [27] on boundary versions of the NP problems in the classes \( \mathcal{C} \) and \( \mathcal{S} \) stimulated the establishment of our theorems here. In Sarason’s paper [27], the boundary NP problems in the class \( \mathcal{C} \) with boundary data only are reduced to a certain truncated trigonometric moment problem, and both the boundary NP problems and the trigonometric moment problems can be obtained as particular cases of a general scheme. Besides, Sarason’s scheme gives only criteria for existence of solutions, whereas today descriptions of the set of interpolants are known as well. As stated in [9], there exists a subtle difference between the reduction of the NP(avenport) problem to the power moment problem associated with the Hankel vector and the reduction of the NP problem in the class \( \mathcal{C} \) (the NP(avenport) problem) to the trigonometric moment problem associated with the
For the former reduction, the associated power moment problem is nonstandard (i.e., only with the inequality in place of the equality in the last moment condition), whereas for the latter reduction, the moment problem is always standard (which, in essence, is nothing other than a Carathéodory coefficient problem). The corresponding phenomenon is proved, in this paper, to be valid in broad outline for the boundary versions of the NP problems under consideration. One intrinsic reason for this is that the Hermitian nonnegativity of a Toeplitz matrix automatically guarantees that it admits an extension of a larger nonnegative Toeplitz matrix, whereas for Hankel matrices the corresponding phenomenon fails—nonnegativity of the original Hankel matrix in general does not supply an extension of a larger Hankel matrix which is still nonnegative. It is the case if and only if the Hankel matrix is proper, i.e., its rank is concentrated in the upper left-hand corner. Beyond these, in the boundary case some special features in contrast with the original NP problem appear. Thus, the boundary NP problems in the class \( N \) and their connections with the power moment problems warrant detailed study.

Let us begin with the formulation of the naive boundary version of the NP problem in the class \( N \) of functions \( f(z) \) holomorphic in the open upper half plane \( \pi^+ \) with \( \text{Im}(f(z)) > 0 \). The boundary NP problem. Assume that there are given \( \theta \) distinct nodes \( z_1, \ldots, z_\theta \) in \( \pi^+ \) with respective multiplicities \( n_1, \ldots, n_\theta \) (each \( n_i \geq 1 \)) and the interpolated values \( c_{ik} \in \mathbb{C}, 1 \leq i \leq \theta, 0 \leq k \leq n_i - 1; s \) distinct real nodes \( \alpha_1, \ldots, \alpha_s \) with respective multiplicities \( 2m_1, \ldots, 2m_s \) (each \( m_j \geq 1 \)) and the interpolated values \( d_{jl} \in \mathbb{R}, 1 \leq j \leq s, 0 \leq l \leq 2m_j - 1 \). It is required to determine whether there is a function \( f(z) \in \mathcal{N} \) subject to

\[
\frac{1}{k!} f^{(k)}(z_i) = c_{ik}, \quad i = 1, \ldots, \theta, \quad k = 0, 1, \ldots, n_i - 1, \quad (1.1a)
\]

\[
\frac{1}{l!} f^{(l)}(\alpha_j) = d_{jl}, \quad j = 1, \ldots, s, \quad l = 0, 1, \ldots, 2m_j - 1 \quad (1.1b)
\]

and, if there is, to describe the family of all such \( f(z) \). Here and in the sequel, the values \( f(\alpha_j) \) and \( f^{(l)}(\alpha_j) \) \((l \geq 1)\) are real and to be interpreted as nontangential limits and angular derivatives of order \( l \), respectively.

Analogous to the case of the boundary NP problems in the class \( C \) [27], the BNP(\(N\)) problem is not the closest boundary analogues of the NP(\(N\)) problem.

The BNP(\(N\)) problem. A variant of the BNP(\(N\)) problem, is the same as the BNP(\(N\)) problem except with the equalities

\[
\frac{1}{(2m_j - 1)!} f^{(2m_j - 1)}(\alpha_j) = d_{j,2m_j - 1}, \quad j = 1, \ldots, s
\]

therein replaced by the following inequalities:

\[
\frac{1}{(2m_j - 1)!} f^{(2m_j - 1)}(\alpha_j) \leq d_{j,2m_j - 1}, \quad j = 1, \ldots, s. \quad (1.2)
\]
There is another variant of the BNP(N) problem in which in addition to Eqs. (1.1a) and (1.1b) the interpolants \( f(z) \in \mathcal{M} \) are required to satisfy

\[
\lim_{z \to \infty} \frac{f(z)}{z} = 0
\]

(1.3)

uniformly in each sector \( \pi \varepsilon(0) = \{ z \mid \varepsilon \leq \arg z \leq \pi - \varepsilon \}, \varepsilon \in (0, \pi/2) \). We shall refer to the last problem as the “BNP(\mathcal{M})” problem for short.

Let now

\[
r = \sum_{i=1}^{\theta} n_i, \quad m = \sum_{j=1}^{\tau} m_j,
\]

and \( N = r + m > 1 \). It is well known that there is a unique polynomial \( \omega(z) \), of degree at most \( 2N - 1 \), satisfying both Eqs. (1.1a) and (1.1b) and the equations

\[
\frac{1}{k!} f^{(k)}(z_i) = z_i^k, \quad i = 1, \ldots, \theta, \quad k = 0, 1, \ldots, n_i - 1.
\]

(1.4)

As usual, for any \( f(z) \in \mathcal{M} \) we have by reflecting that \( f(z) = \overline{f(\bar{z})} \) for any \( z \) with \( \text{Im} \ z < 0 \). Let \( a(z) \) be the annihilator polynomial, of degree \( 2N \), of interpolation nodes:

\[
a(z) = \prod_{i=1}^{\theta} (z - z_i)^{n_i} (\bar{z} - \bar{z}_i)^{\bar{n}_i} \prod_{j=1}^{\tau} (z - \alpha_j)^{2m_j}
\]

\[
= \sum_{i=0}^{2N} a_i z^i \quad (a_{2N} = 1),
\]

(1.5)

which together with \( \omega(z) \) contains all the data of these boundary problems. It is not difficult to check that both \( \omega(z) \) and \( a(z) \) have real coefficients. Let the Laurent series expansion of the rational function \( \omega(z)/a(z) \) at infinity be of the form

\[
\frac{w(z)}{a(z)} = \frac{h_0}{z} + \frac{h_1}{z^2} + \cdots + \frac{h_{2N-3}}{z^{2N-2}} + \frac{h_{2N-2}}{z^{2N-1}} + \frac{h_{2N-1}}{z^{2N}} + \cdots.
\]

(1.6)

All the \( h_p \) are obviously real and permit the representation

\[
h_p = 2 \text{Re} \left\{ \sum_{i=1}^{\theta} \frac{1}{(n_i - 1)!} \left( \frac{d}{d\xi} \right)^{n_i-1} \sum_{k=0}^{n_i} c_{ik}(\xi - z_i)^{n_i+k(p+1)} \right\} \left( \frac{1}{a(\xi)} \right)_{\xi = z_i}
\]

\[
+ \sum_{j=1}^{\tau} \frac{1}{(2m_j - 1)!} \left( \frac{d}{d\xi} \right)^{2m_j-1} \sum_{l=0}^{2m_j-1} d_{jl}(\xi - \alpha_j)^{2m_j+k(p+1)} \right\} \left( \frac{1}{a(\xi)} \right)_{\xi = \alpha_j}
\]

(1.7)

for \( p = 0, 1, \ldots \). We refer to the vector \( \mathbf{h} = (h_0, h_1, \ldots, h_{2N-2}) \) being sought via Eqs. (1.7) as the Hankel vector of the BNP(\mathcal{M}) or the BNP(\mathcal{M})' problem. With that Hankel vector \( \mathbf{h} \) we will associate the following two moment problems.
The truncated, nonstandard Hamburger moment (HM) problem associated with the Hankel vector \( h \) of the BNP(\( N \)) problem, consisting of finding the distribution functions \( \tau(u) \) on \( \mathbb{R} \), i.e., \( \tau(u) \) are nondecreasing and right-continuous functions on \( \mathbb{R} \) such that \( \tau(-\infty) = 0 \), that satisfy

\[
\begin{aligned}
    h_i &= \int_{-\infty}^{+\infty} u^i \, d\tau(u), \quad i = 0, 1, \ldots, 2N - 3, \\
    h_{2N-2} &\geq \int_{-\infty}^{+\infty} u^{2N-2} \, d\tau(u).
\end{aligned}
\]

The truncated, standard Hamburger moment (SHM) problem associated with the Hankel vector \( h \), which is the same as the HM problem (1.8) except with the equality in place of the inequality in the last moment condition, i.e.,

\[
\begin{aligned}
    h_i &= \int_{-\infty}^{+\infty} u^i \, d\tau(u), \quad i = 0, 1, \ldots, 2N - 2.
\end{aligned}
\]

Perhaps somewhat surprisingly, in Theorem 3.1 below it is shown that the BNP(\( N \)) problem is solvable if and only if the HM problem (1.8) is solvable, and in this case, there exists an intrinsic and explicit one-to-one correspondence between solutions \( f(z) \) to the BNP(\( N \)) problem and solutions \( \tau(u) \) to the HM problem (1.8), which is determined via the formula

\[
f(z) = \alpha(z) + a(z) \int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z}, \quad z \in \mathbb{R}^+.
\]

Also, we can amplify these results to some extent (see Theorem 5.2).

Furthermore, as distinguished from the assertions of Theorem 3.1 we can conclude that the BNP(\( N \)) problem is solvable if and only if the HM problem (1.8) with an additional constraint that some distribution function \( \tau(u) (-\infty < u < +\infty) \) assigns no mass to any of the real nodes \( \alpha_1, \ldots, \alpha_s \), i.e.,

\[
\tau(\{\alpha_j\}) = 0, \quad j = 1, \ldots, s,
\]

is solvable. Moreover, formula (1.10) also realizes a one-to-one correspondence between solutions \( f(z) \) to the BNP(\( N \)) problem and solutions \( \tau(u) \) to the HM problem with constraint (1.11). These assertions are given in Theorem 3.2 and further in Section 5.3.

Finally, the analogous implications concerning both the BNP(\( N \)) problem and the SHM problem with constraint (1.11) hold also and are made in Theorems 3.3, 5.7 and 5.8 below.

This mainly expository paper emphasizes that the idea of using the Hankel vector in question as a tool for transforming the interpolation problems into moment problems underlies the unifying approach of our analysis here. It turns out that the
treatments for the BNP($\mathcal{N}$ problem can be expanded to some extent so as to clarify the solution of the BNP($\mathcal{N}$ problem, and vice versa. Consequently, we are able to answer, at least in some cases, a remarkable question stated by Sarason [27, p. 233] (see Section 3.3 for details).

In addition to Section 1, the paper is divided into five sections. Sections 2 and 4 are of preliminary character; we bring the necessary background for the solution of the boundary NP problems under consideration. A portion of the results on angular derivatives and higher derivatives of a function in the class $\mathcal{N}$ is contained in Section 2. In Section 4, we present the material from the theory of moments, the first part of which are well known, and the last part seems to be new, relative to the description of solutions to the HM and the SHM problems both with constraint (1.11). In Section 3, we address ourselves to the exact connections of boundary NP problems in the class $\mathcal{N}$ with the associated moment problems in a new light of the Hankel vector approach. Thanks to these connections, the solvability conditions for the boundary NP problems and their full solutions are derived in a transparent way in Section 5 by using the results of the theory of moments cited in Section 4.

2. Angular derivatives and higher derivatives of a function in the class $\mathcal{N}$

Let $\pi_\varepsilon(t)$ denote the sector in $\pi^+$ with the vertex $t \in \mathbb{R}$ defined by

$$\pi_\varepsilon(t) = \{z \mid \varepsilon \leq \arg(z - t) \leq \pi - \varepsilon\}, \quad 0 < \varepsilon < \frac{\pi}{2}. \tag{2.1}$$

For a function $f(z)$ holomorphic in $\pi^+$, it is said to have an angular derivative at a boundary point $t \in \mathbb{R}$, if $f(z)$ has a nontangential limit at $t$, denoted by $f(t)$, and if the divided difference $[z, t]_f = [f(z) - f(t)]/(z - t)$ has a nontangential limit at $t$,

$$\lim_{z \to t} [z, t]_f = \lim_{z \to t} \frac{f(z) - f(t)}{z - t}. \tag{2.2}$$

where and in the sequel $z$ tends to $t \in \mathbb{R}$ along any nontangential path, i.e., $z \in \pi_\varepsilon(t)$ and $z \to t$ for any $\varepsilon \in (0, \pi/2)$. As a rule, we refer to the value of the limit (2.2) as the angular derivative of $f(z)$ at $t \in \mathbb{R}$, and denote it by $f'(t)$.

Consider next the extension of angular derivatives to higher derivatives. Let $t$ be real and let $p$ be a positive integer. We say successively that $f(z)$ has the $p$th angular derivative at $t \in \mathbb{R}$, denoted by $f^{(p)}(t)$, if the divided difference $[z, t]_f$ has the $(p - 1)$th angular derivative at $t$ ($f^{(0)}(t) = f(t)$). Thus, with the notation of divided differences of a function, by definition, the existence of $f^{(p)}(t)$ is equivalent to the existence of the nontangential limits of

$$\left[ z, \underbrace{t, \ldots, t}_k \right]_f$$

at $t$ for $k = 0, 1, \ldots, p - 1$, i.e., the limits
exist successively for \( k = 0, 1, \ldots, p - 1 \). In this case, \( \frac{1}{p!} f^{(p)}(t) \) coincides with the value of Eq. (2.3) with \( k = p - 1 \), so that

\[
\frac{1}{p!} f^{(p)}(t) = \left[ \underbrace{\ldots, \underbrace{t}, \ldots, t}_p \right]_f,
\]

and \( f(z) \) must have a priori all of its angular derivatives at \( t \) up to order \( p - 1 \). Also, we then have that the \( f^{(k)}(t) \) coincide with the nontangential limits of \( f^{(k)}(z) \) at \( t \) for \( k = 0, 1, \ldots, p \).

More specifically, let \( f(z) \) be a function of the class \( \mathscr{N} \). Then and only then it admits an integral representation (see, e.g., [1,20])

\[
f(z) = az + \beta + \int_{-\infty}^{+\infty} \left( \frac{1}{u - z} - \frac{u}{1 + u^2} \right) \, d\sigma_f(u) \quad \forall z \in \mathbb{P}^+,
\]

in which \( \alpha \geq 0 \), \( \beta \) is real, and \( \sigma_f(u) (-\infty < u < +\infty) \) is a distribution function such that

\[
\int_{-\infty}^{+\infty} (1 + u^2)^{-1} \, d\sigma_f(u) < +\infty,
\]

moreover, in this case

\[
\lim_{z \to \infty} \frac{f(z)}{z} = \alpha
\]

holds uniformly in each sector \( \pi_v(0) \).

It is well known that each \( f(z) \in \mathscr{N} \) has nontangential limits almost everywhere on the real axis. The following lemma characterizes when \( f(z) \in \mathscr{V} \) has all of real-valued angular derivatives at \( t \in \mathbb{R} \) up to order \( 2m - 1 \) \( (m \geq 1) \) in terms of \( \sigma_f(u) \) \( (-\infty < u < +\infty) \), and provides the representations for \( f^{(k)}(t) \), \( k = 0, 1, \ldots, 2m - 1 \). A number of results have been presented in [10,11,13,26], [27, Lemma 1], [28, pp. 46–59] and [29] for angular derivatives of order 1 or more of a function. These sources deal mostly with the class \( \mathscr{C} \) on the disc and/or the upper half plane. But the analysis, especially for the latter class, is quite similar to the class \( \mathscr{N} \) considered in this paper.
**Lemma 2.1.** Let \( t \in \mathbb{R} \) be given and let \( f(z) \in \mathcal{N} \) admit representation (2.4). Then the derivatives \( f^{(k)}(t) \) exist and are real for \( k = 0, 1, \ldots, 2m - 1 \) (\( m \geq 1 \)) if and only if

\[
\int_{-\infty}^{+\infty} \frac{1}{(u-t)^{2m}} \, d\sigma_f(u) < +\infty.
\]  

(2.6)

In that case,

\[
f^{(k)}(t) = \begin{cases} 
\alpha t + \beta + \int_{-\infty}^{+\infty} \frac{1}{u-t} - \frac{u}{1+u} \, d\sigma_f(u) & \text{if } k = 0, \\
\alpha \delta_{ik} + \int_{-\infty}^{+\infty} \frac{k^i}{(u-t)^i} \, d\sigma_f(u) & \text{if } 1 \leq k \leq 2m - 1,
\end{cases}
\]

(2.7)

where the symbol \( \delta_{ik} \) stands for Kronecker \( \delta \), and \( f^{(k)}(t) \geq 0 \) whenever \( k \) is odd.

**Proof.** We suppose first that Eq. (2.6) holds. Then \( \sigma_f([t]) = 0 \) obviously. Let

\[
\tau(u) = \int_{-\infty}^{u} \frac{1}{(s-t)^{2m}} \, d\sigma_f(s), \quad -\infty < u < +\infty,
\]

(2.8)

a distribution function on \( \mathbb{R} \) with bounded variation there. For the case of \( k = 0, 1 \), it can be directly checked that the \( f^{(k)}(t) \) make sense and have representation (2.7) with \( k = 0, 1 \). To prove the general case, we assume that Eq. (2.7) is true for \( k = n \) \((n < 2m - 1)\) as an induction hypothesis, and consider the case of \( k = n + 1 \). Then, by induction and by Eq. (2.8),

\[
\begin{bmatrix} 
\text{n+1 times} \\
\text{z, t, \ldots, t} \\
f
\end{bmatrix} = \int_{-\infty}^{+\infty} \frac{d\sigma_f(u)}{(u-z)(u-t)^{n+1}} \\
= \int_{|u-t| > 1} \frac{d\sigma_f(u)}{(u-z)(u-t)^{n+1}} + \int_{|u-t| \leq 1} \frac{u-t)^{2m-n-1}}{(u-z)} \, d\tau(u).
\]

(2.9)

Since in the case when \(|u-t| \leq 1\) and \( z \in \pi\varepsilon(t) \)

\[
\left| \frac{u-t}{u-z} \right|^{2m-n-2} \leq \left| \frac{u-t}{u-z} \right|^{n+1} + \left| \frac{z-t}{u-z} \right|^{1+ \frac{1}{\sin \varepsilon}}.
\]

Lebesgue’s convergence theorem implies that

\[
\lim_{z \to t} \int_{|u-t| \leq 1} \frac{(u-t)^{2m-n-2}}{(u-z)} \, d\tau(u) = \int_{|u-t| \leq 1} (u-t)^{2m-n-2} \, d\tau(u).
\]

Thus, from Eq. (2.9) we deduce that

\[
\lim_{z \to t} \begin{bmatrix} 
\text{n+1 times} \\
\text{z, t, \ldots, t} \\
f
\end{bmatrix} = ... \]
exists and has the representation
\[
\frac{1}{(n+1)!} f^{(n+1)}(t) = \frac{d}{dt} \left[ \frac{d}{dt} \left[ \cdots \frac{d}{dt} \left[ f(t) \right] \cdots \right] \right] = \int_{|u-t|>1} \frac{d\sigma_f(u)}{(u-t)^{n+2}} + \int_{|u-t|<1} \frac{d\sigma_f(u)}{(u-t)^{n+2}}
\]
as needed.

Conversely, we suppose that the \( f^{(k)}(t) \) exist and are real for \( k = 0, 1, \ldots, 2m - 1 \) \( (m \geq 1) \). Then for \( z = t + iy \) \( (y > 0) \),
\[
\frac{1}{(2m-1)!} f^{(2m-1)}(t) = \lim_{z \to t} \Re \left[ \frac{d^{2m-1}}{dz^{2m-1}} \right]
\]
\[
= \lim_{z \to t} \Re \int_{-\infty}^{+\infty} \frac{d\sigma_f(u)}{(u-z)(u-t)^{2m-1}}
\]
\[
= \lim_{y \to 0} \int_{-\infty}^{+\infty} \frac{d\sigma_f(u)}{(u-t)^2 + y^2)(u-t)^{2m-2}} < +\infty.
\]

From Fatou’s theorem, we deduce condition (2.6). This completes the proof of the Lemma 2.1.

The following lemma is needed in the sequel (see also [11, Lemma 8.1] for a similar result).

Lemma 2.2. Let \( t \in \mathbb{R} \) be given and let \( f(z) \in A \) admit representation (2.4). Then
\[
\lim_{z \to t} (z-t) f(z) = -(\sigma_f(t) - \sigma_f(t^{-1})) = -\sigma_f(|t|).
\]  

(2.10)

Proof. Observe that if \( z \in \pi_{r}(t) \) with some \( \varepsilon \in (0, \pi/2) \) is such that \( |z-t| < 1 \), then
\[
|u-z| \geq |z-t| \sin \varepsilon, \quad -\infty < u < +\infty
\]
and thus
\[
\left| (z-t) \left( \frac{1}{u-z} - \frac{u}{1+u^2} \right) \right| = \left| \frac{1+uz}{u-z} \right| \frac{|z-t|}{1+u^2} \leq \left( |t| + 1 + \frac{(|t| + 1)^2}{\sin \varepsilon} \right) \frac{1}{1+u^2}.
\]

From the last inequality and the fact that
\[
\lim_{z \to t} (z-t) \left( \frac{1}{u-z} - \frac{u}{1+u^2} \right) = \begin{cases} 0 & \text{if } u \neq t, \\ -1 & \text{if } u = t, \end{cases}
\]


we deduce by Lebesgue’s convergence theorem that
\[
\lim_{z \to t} (z - t) f(z) = \lim_{z \to t} (z - t) \int_{-\infty}^{+\infty} \left( \frac{1}{u - z} - \frac{u}{1 + u^2} \right) \, d\sigma_f(u)
\]
\[
= -(\sigma_f(t) - \sigma(t - 0)) = -\sigma_f([t]),
\]
as needed. □

We remark that the value of \(\sigma_f([t])\) given in Eq. (2.10) can be viewed as the mass at the point \(t \in \mathbb{R}\) of the distribution function \(\sigma_f(u)\) on \(\mathbb{R}\). Thus, if \(t \in \mathbb{R}\) is a mass point of \(\sigma_f(u)\) \((-\infty < u < +\infty)\), then \(\lim_{z \to t} f(z)\) diverges to \(\infty\).

**Lemma 2.3.** Let \(a(z)\) be as in Eq. (1.5), and \(\tau(u)\) \((-\infty < u < +\infty)\) be a distribution function such that
\[
\int_{-\infty}^{+\infty} \, d\tau(u) < +\infty.
\]
Then the function \(\psi(z)\) defined by
\[
\psi(z) = a(z) \int_{-\infty}^{+\infty} \frac{1}{u - z} \, d\tau(u) \quad \text{(2.11)}
\]
is subject to
\[
\frac{1}{k!} \psi^{(k)}(z_i) = 0, \quad i = 1, \ldots, \theta, \quad k = 0, 1, \ldots, n_i - 1, \quad \text{(2.12)}
\]
\[
\prod_{j=1}^{\theta} (\alpha_j) = 0, \quad j = 1, \ldots, s, \quad k = 0, 1, \ldots, 2m_j - 2, \quad \text{(2.13)}
\]
\[
\frac{1}{(2m_j - 1)!} \psi^{(2m_j - 1)}(\alpha_j) = -\frac{\theta}{2m_j} \prod_{i=1}^{\theta} |\alpha_j - z_i|^{2n_i} \prod_{k \neq j} (\alpha_j - \alpha_k)^{2m_k} \tau([\alpha_j]), \quad j = 1, \ldots, s. \quad \text{(2.14)}
\]

**Proof.** Rewrite \(\psi(z)\) in the form
\[
\psi(z) = \frac{a(z)}{z - \alpha_j} \int_{-\infty}^{+\infty} \frac{1}{u - z} \, d\tau(u) \quad \text{(2.15)}
\]
for \(j = 1, \ldots, s\). From Lemma 2.2 we deduce
\[
\lim_{z \to \alpha_j} (z - \alpha_j) \int_{-\infty}^{+\infty} \frac{1}{u - z} \, d\tau(u) = -\tau([\alpha_j]), \quad j = 1, \ldots, s,
\]
so that the $\psi^{(l)}(\alpha_j)$, $j = 1, \ldots, s$, $l = 0, 1, \ldots, 2m_j - 2$ are well defined, and

$$\frac{1}{l!} \psi^{(l)}(\alpha_j) = 0, \quad j = 1, \ldots, s, \quad l = 0, 1, \ldots, 2m_j - 2.$$ 

In the case of $l = 2m_j - 1$, we have by Eq. (2.15)

$$\lim_{z \to \alpha_j} \left[ \frac{2m_j - 1}{z, \alpha_j, \ldots, \alpha_j} \right] \psi = \prod_{i=1}^\theta (\alpha_j - z_i) \prod_{k \neq j} (\alpha_j - \alpha_k)^2 \lim_{z \to \alpha_j} \int_{-\infty}^{+\infty} \left( \frac{u - \alpha_j}{u - z} - 1 \right) d\tau(u) = - \prod_{i=1}^\theta (\alpha_j - z_i) \prod_{k \neq j} (\alpha_j - \alpha_k)^2 \tau([\alpha_j]).$$

Thus, $\psi^{(2m_j-1)}(\alpha_j)$ makes sense and Eq. (2.14) holds. The proof of assertion (2.12) is plain.  

3. On the boundary NP problems in the class $\mathcal{M}$ and the associated power moment problems

3.1. Connection of the BNP($\mathcal{M}$) problem to the HM problem

One can deduce from representation (2.4) the equivalency of the BNP($\mathcal{M}$) problem to the problem of finding $\alpha \geq 0$, real $\beta$ and a distribution function $\sigma_f(u)$ ($-\infty < u < +\infty$) such that the corresponding function $f(z)$ of form (2.4) is a solution to the BNP($\mathcal{M}$) problem.

**Theorem 3.1.** The BNP($\mathcal{M}$) problem is solvable if and only if the HM problem is solvable. In that case, there exists a one-to-one correspondence between solutions $f(z)$ to the BNP($\mathcal{M}$) problem and solutions $\tau(u)$ to the HM problem, which is determined by the formula

$$f(z) = \sigma(z) + a(z) \int_{-\infty}^{+\infty} \frac{1}{u - z} d\tau(u), \quad z \in \pi^+. \quad (3.1)$$

More precisely, if $f(z) \in \mathcal{M}$ is a solution to the BNP($\mathcal{M}$) problem, and admits representation (2.4), then the distribution function $\tau(u)$ ($-\infty < u < +\infty$) defined via the formula

$$d\tau(u) = \begin{cases} \frac{1}{a(u)} d\sigma_f(u) & \text{if } u \not\in \{\alpha_1, \ldots, \alpha_s\}, \\ \frac{d_{j,2m_j-1}}{\prod_{i=1}^\theta (\alpha_j - z_i)^2 \prod_{k \neq j} (\alpha_j - \alpha_k)^2} & \text{if } u = \alpha_j, \\ j = 1, \ldots, s \end{cases} \quad (3.2)$$
makes sense, where
\[ \tilde{d}_{j,2m_j-1} = \frac{1}{(2m_j - 1)!!} f^{(2m_j-1)}(a_j) \]
is real, and is a solution to the HM problem such that
\[ h_j = \int_{-\infty}^{+\infty} u^j \, d\tau(u) + \delta_{j,2N-2}a, \quad j = 0, 1, \ldots, 2N - 2. \] (3.3)

Conversely, if \( \tau(u) \ (\pm \infty < u < +\infty) \) is a solution to the HM problem, then \( f(z) \) defined via formula (3.1) is a solution to the BNP(\( \mathcal{A} \)) problem, and further admits representation (2.4) in which
\[ \alpha = h_{2N-2} - \int_{-\infty}^{+\infty} u^{2N-2} \, d\tau(u), \]
\[ \beta = h_{2N-1} + \alpha a_{2N-1} - \int_{-\infty}^{+\infty} \left( u^{2N-1} - \frac{u a(u)}{1 + u^2} \right) \, d\tau(u) \] (3.4)
and
\[ d\sigma_f(u) = a(u) \, d\tau(u). \] (3.5)

Proof. We assume first that the BNP(\( \mathcal{V} \)) problem is solvable, and \( f(z) \in \mathcal{V} \) is a solution thereof with representation (2.4). Let \( \tilde{\omega}(z) \) be a real coefficient polynomial of degree at most \( 2N - 1 \), which is an interpolant of the same conditions as Eqs. (1.1a), (1.1b) and (1.4) except with \( \tilde{d}_{j,2m_j-1} \) in place of \( d_{j,2m_j-1}, j = 1, \ldots, s \), where
\[ \tilde{d}_{j,2m_j-1} = \frac{1}{(2m_j - 1)!!} f^{(2m_j-1)}(a_j), \quad j = 1, \ldots, s. \]

By Lemma 2.1 we then have
\[ \int_{-\infty}^{+\infty} \frac{1}{(u - a_j)^{2m_j}} \, d\sigma_f(u) < +\infty, \quad j = 1, 2, \ldots, s \] (3.6)
or equivalently
\[ \int_{-\infty}^{+\infty} \prod_{j=1}^{s} (u - a_j)^{-2m_j} \, d\sigma_f(u) < +\infty. \]

Then \( \sigma_f([a_j]) = 0 \) for \( j = 1, \ldots, s \) and
\[ \int_{-\infty}^{+\infty} \frac{1}{a(u)} \, d\sigma_f(u) < +\infty. \] (3.7)

Owing to the fact that \( a(u) > 0 \) for all \( u \in \mathbb{R} \) except for \( u = a_j, j = 1, \ldots, s \), the function \( \tilde{\tau}(u) \ (\pm \infty < u < +\infty) \) which is the same as the function \( \tau(u) \ (\pm \infty < u < +\infty) \) that is defined by Eq. (3.2) but with \( \tilde{\tau}([a_j]) = 0 \) in place of the values
of $\tau(\{\alpha_j\})$ determined by Eq. (3.2) makes sense and is a distribution function with bounded variation on $\mathbb{R}$. Thus, $f(z)$ can be rewritten as

$$f(z) = \hat{\omega}(z) + a(z) \int_{-\infty}^{+\infty} \frac{d\tilde{\tau}(u)}{u - z},$$

(3.8)
in which

$$\hat{\omega}(z) = az + \beta + \int_{-\infty}^{+\infty} \left( \frac{a(u) - a(z)}{u - z} - \frac{u a(u)}{1 + u^2} \right) d\tilde{\tau}(u)$$
is a polynomial of degree at most $2N - 1$. By Eq. (3.8) and Lemma 2.3, we deduce that

$$\hat{\omega}(z) = \hat{\omega}(z), \quad \phi(z) = \int_{-\infty}^{+\infty} \frac{d\tilde{\tau}(u)}{u - z} \in \mathcal{N} \quad \forall \zeta \in \pi^+.$$  

A further computation by Eqs. (1.6) and (2.4) shows that for sufficiently large value of $|z|$ and $z \in \pi_\epsilon(0)$ ($\epsilon > 0$),

$$\phi(z) = \frac{f(z) - \hat{\omega}(z)}{a(z)}$$

$$= \frac{f(z)}{a(z)} - \frac{\omega(z)}{a(z)} + \frac{\omega(z) - \hat{\omega}(z)}{a(z)}$$

$$= -\sum_{j=1}^{s} \tau(\{\alpha_j\}) - \frac{h_0}{z} - \frac{h_1}{z^2} - \cdots - \frac{h_{2N-2} - \alpha}{z^{2N-1}} - \cdots$$

$$= -\frac{h_0}{z} - \frac{\sum_{j=1}^{s} \tau(\{\alpha_j\})}{z} - \frac{h_1}{z^2} - \frac{\sum_{j=1}^{s} \alpha_j \tau(\{\alpha_j\})}{z^2}$$

$$\cdots = \frac{h_{2N-2} - \alpha - \sum_{j=1}^{s} \alpha_j^{2N-2} \tau(\{\alpha_j\})}{z^{2N-1}} - \cdots.$$  

(3.9)

Then the well-known Nevanlinna–Hamburger theorem (see, e.g., [1,18]) implies that

$$\int_{-\infty}^{+\infty} u^k d\tilde{\tau}(u) = h_k - \alpha \delta_{k,2N-2} - \sum_{j=1}^{s} \alpha_j^k \tau(\{\alpha_j\}), \quad k = 0, 1, \ldots, 2N - 2,$$

and thus $\tau(u)$ is a solution to the HM problem such that Eqs. (3.3) hold.

Conversely, we assume that $\tau(u)$ ($-\infty < u < +\infty$) is a solution of the HM problem. Put

$$\alpha = h_{2N-2} - \int_{-\infty}^{+\infty} u^{2N-2} d\tau(u) \quad \text{and} \quad d\sigma(u) = a(u) d\tau(u).$$

Then $\alpha \geq 0$, $d\sigma(u) \geq 0$ for $u \in \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} (1 + u^2)^{-1} d\sigma(u) < +\infty.$$
obviously. Define the function \( f(z) \) via Eq. (3.1). It is easy to see that \( f(z) \) is holomorphic in \( \pi^+ \) and subject to Eqs. (1.1a).

In order to prove that \( f(z) \) satisfies Eqs. (1.1b) for \( j = 1, \ldots, s \), \( l = 0, 1, \ldots, 2m_j - 2 \), and inequalities (1.2) for \( j = 1, \ldots, s \) Lemma 2.3 is applicable.

To show that \( f(z) \) defined via Eq. (3.1) is a solution to the BNP, we need only show that \( f(z) \) belongs to the class \( \mathcal{N} \).

Let \( h_0, h_1, \ldots, h_{2N-1} \) be defined via Eq. (1.6), and let \( \mathcal{H} \) be the linear space of all rational functions. Define a linear functional \( \mathcal{L}_u \) on \( \mathcal{H} \) by

\[
\mathcal{L}_u [r(u)] = \sum_{j=0}^{2N-1} r_j h_j \quad \forall r(u) \in \mathcal{H},
\]

whenever the Laurent series expansion of \( r(u) \in \mathcal{H} \) at infinity is of the form

\[
r(u) = \sum_{j=-\infty}^{+\infty} r_j u^j,
\]

in which the number of nonzero \( r_j \) (\( j \geq 0 \)) in Eq. (3.11) is finite. Then, in particular,

\[
\mathcal{L}_u [u^j] = h_j, \quad j = 0, 1, \ldots, 2N - 1.
\]

A direct calculation leads to

\[
\mathcal{L}_u \left\{ \frac{a(u)}{u-z} \right\} = \mathcal{L}_u \left\{ \frac{a(u) - a(z)}{u-z} \right\} = \omega(z).
\]

Rewrite \( f(z) \) defined by Eq. (3.1) in the form

\[
f(z) = p(z) + \int_{-\infty}^{+\infty} \left( \frac{1}{u-z} - \frac{u}{1+u^2} \right) a(u) \, d\tau(u),
\]

where

\[
p(z) = \omega(z) - \int_{-\infty}^{+\infty} \left( \frac{a(u) - a(z)}{u-z} - \frac{u a(u)}{1+u^2} \right) \, d\tau(u),
\]

a polynomial of \( z \), of degree at most \( 2N - 1 \), and the integral on the right-hand side of Eq. (3.14) stands for a Nevanlinna function, since

\[
\int_{-\infty}^{+\infty} (1+u^2)^{-1} a(u) \, d\tau(u) < +\infty.
\]

We will show that \( p(z) = \alpha z + \beta \) (\( \beta \) real), and thus \( f(z) \in \mathcal{N} \).

Notice that

\[
\frac{a(u) - a(z)}{u-z} = \sum_{j+k=0}^{2N-1} a_{j+k+1} u^j z^k.
\]

Therefore, by Eqs. (3.13) and (3.15), we have

\[
p(z) = \sum_{j+k \leq 2N-1} a_{j+k+1} h_j z^k
\]
\[- \sum_{j+k \leq 2N-1 \atop j \neq 2N-1} a_{j+k+1}(h_j - \alpha \delta_{j,2N-2})z^k \]
\[- \int_{-\infty}^{+\infty} \left( u^{2N-1} - \frac{ua(u)}{1+u^2} \right) d\tau(u) \]
\[= h_{2N-1} + \alpha a_{2N-1} + \alpha z \]
\[- \int_{-\infty}^{+\infty} \left( u^{2N-1} - \frac{ua(u)}{1+u^2} \right) d\tau(u) \]
\[= \alpha z + \beta, \quad \text{(3.16)} \]

where \( \beta \) is real, and equal to
\[\beta = h_{2N-1} + \alpha a_{2N-1} - \int_{-\infty}^{+\infty} \left( u^{2N-1} - \frac{ua(u)}{1+u^2} \right) d\tau(u), \]

noting that there exists some polynomial \( G(z) \) of degree at most \( 2N \) such that
\[u^{2N-1} - \frac{ua(u)}{1+u^2} = \frac{G(u)}{1+u^2}, \]
whence the last integral makes sense and is real. Thus, from Eqs. (3.14) and (3.16) it follows that \( f(z) \) has an integral representation of the form
\[f(z) = \alpha z + \beta + \int_{-\infty}^{+\infty} \left( \frac{1}{u-z} - \frac{u}{1+u^2} \right) a(u) \, d\tau(u) \]
or what is the same, \( f(z) \in \mathcal{V}' \), and \( d\sigma_f(u) = a(u) \, d\tau(u) \). This completes the proof of the theorem. \( \square \)

3.2. Connection of the BNP(\( \mathcal{V} \)) (or the BNP(\( \mathcal{V} \))\(^n \)) problem to the HM (or the SHM) problem with constraint (3.11)

The chief objective of this subsection is to examine both the connection between the BNP(\( \mathcal{V} \)) problem and the HM problem with constraint (1.11), and the connection between the BNP(\( \mathcal{V} \))\(^n \) problem and the SHM problem with the same constraint (1.11). These two questions are answered by Theorems 3.2 and 3.3, respectively, and can be considered as particular situations of the connection given in Theorem 3.1.

**Theorem 3.2.** The BNP(\( \mathcal{V} \)) problem is solvable if, and only if, the HM problem with constraint (1.11) is solvable. In that case, the formula
\[f(z) = \omega(z) + a(z) \int_{-\infty}^{+\infty} \frac{1}{u-z} \, d\tau(u) \quad \text{(3.17)} \]
realizes a one-to-one correspondence between solutions \( f(z) \) to the BNP(\( \mathcal{V} \)) problem and solutions \( \tau(u) \) \((-\infty < u < +\infty)\) to the HM problem with the constraint (1.11). More precisely, if \( f(z) \in \mathcal{V} \) is a solution to the BNP(\( \mathcal{V} \)) problem and admits representation (2.4), then the distribution function \( \tau(u) \) \((-\infty < u < +\infty)\) defined via the formula

\[
\begin{aligned}
d\tau(u) &= \begin{cases} 
\frac{1}{a(u)} \, d\sigma_f(u), & \text{if } u \notin \{\alpha_1, \ldots, \alpha_s\}, \\
0, & \text{if } u \in \{\alpha_1, \ldots, \alpha_s\}
\end{cases} 
\end{aligned}
\]  

(3.18)

makes sense and is a solution to the HM problem with constraint (1.11) such that

\[
\begin{aligned}
h_j &= \int_{-\infty}^{+\infty} u^j \, d\tau(u) + \delta_{j,2N-2} \alpha, \quad j = 0, 1, \ldots, 2N-2.
\end{aligned}
\]  

(3.19)

Conversely, if \( \tau(u) \) \((-\infty < u < +\infty)\) is a solution to the HM problem such that \( \tau([\alpha_j]) = 0, j = 1, \ldots, s, \) then \( f(z) \) defined via formula (3.17) is a solution to the BNP(\( \mathcal{V} \)) problem, and admits representation (2.4) in which

\[
\begin{aligned}
\alpha &= h_{2N-2} - \int_{-\infty}^{+\infty} u^{2N-2} \, d\tau(u), \\
\beta &= h_{2N-1} + a \alpha_{2N-1} - \int_{-\infty}^{+\infty} \left( u^{2N-1} - \frac{u a(u)}{1 + u^2} \right) \, d\tau(u),
\end{aligned}
\]

and

\[
d\sigma_f(u) = a(u) \, d\tau(u).
\]

Let us turn to the second connection just mentioned. As stated above, an arbitrary \( f(z) \in \mathcal{V} \), which admits representation (2.4), has the property that the limit relation (2.5) holds uniformly in each sector \( \pi_t(0) \). Thus, if \( f(z) \) is a solution to the BNP(\( \mathcal{V} \)) problem, then and only then the distribution function \( \tau(u) \) \((-\infty < u < +\infty)\) defined by \( \tau(u) = \int_{-\infty}^{u} a(s)^{-1} \, d\sigma_f(s) \) makes sense and is a solution to the SHM problem with constraint (1.11), in accordance with Theorem 3.2 with \( \alpha = 0 \) therein. Hence, as a simple consequence of Theorem 3.2 we have:

**Theorem 3.3.** The BNP(\( \mathcal{V} \)) problem is solvable if, and only if, the SHM problem with constraint (1.11) is solvable. In that case, there exists a one-to-one correspondence between solutions \( f(z) \) to the BNP(\( \mathcal{V} \)) problem and solutions \( \tau(u) \) \((-\infty < u < +\infty)\) to the SHM problem with constraint (1.11), which is determined by formula (3.17). Moreover, a precise description for it, parallel to that made in Theorem 3.2, is also valid.

Two remarks are in order. First of all, the statements of Theorems 3.1–3.3 say actually that the BNP(\( \mathcal{V} \)) problem rather than the BNP(\( \mathcal{V} \)) problem is the closest boundary analogous of the classical NP(\( \mathcal{V} \)) problem with multiple nodes in the sense
that each of the BNP(, \cal V)' problem and the NP(, \cal V) problem supports a solvability criterion, which rests on the Hermitian nonnegativity of the Hankel matrix built on the Hankel vector of the former problem.

Secondly, as for BNP(, \cal V)'' problem, it is actually the closest analogues of problem (\cal ONP) in [27]. Furthermore, Theorem 3.3 might provide a necessary background for the study of the boundary NP problems in the class \cal V with infinitely many nodes. This and related problems will be dealt with in separate investigations by adopting the Hankel vector approach to both of the infinite boundary NP problem and the associated full Hamburger moment problem.

3.3. On a question of Sarason

It is noteworthy to emphasize that the established connections of the boundary NP problems with the associated moment problems are on the basis of the so-called divisor–remainder formulation of form (3.1):

$$f(z) = \omega(z) + a(z) \int_{-\infty}^{+\infty} \frac{1}{u-z} \, d\tau(u), \quad z \in \pi^+,$$

in which \omega(z) is the unique polynomial interpolant, of degree at most 2N − 1, of the interpolation conditions for the BNP(, \cal V) problem (rather than that for the BNP(, \cal V)' problem) together with Eq. (1.4) (rather than only the indicated interpolation conditions) and therefore it has all real coefficients. As to the BNP(, \cal V)' problem, this amounts to a transfer from its solutions \{f(z)\} via the Hankel vector b to the solutions \{\tau(u)\} to the HM problem, and vice versa. In general, thanks to the proof of Theorem 3.1 we conclude that for a solution \{f(z)\} to the BNP(, \cal V)' problem, the conditions of \{d_{j,2m_j-1} = d_{j,2m_j-1}\} hold true for each j if and only if the associated distribution function \{\tau(u)\} assigns no mass to any of the real nodes, what is more important, if and only if \{f(z)\} is a solution to the BNP(, \cal V) problem [to the HM problem with constraint (1.11)]. In this sense, the connection given in Theorem 3.2 can be viewed as a particular case of the connection given in Theorem 3.1.

Thus, we become accidentally aware of an answer to a question stated by Sarason in a transparent way. He said in [27, p. 233], “It is unclear, at least to this author, to what extent the preceding treatments can be expanded so as to clarify Problems (\cal ONP) and (\cal ONP), for example to find a criterion for the solution set to be nonempty”. In this statement the problems (\cal ONP) and (\cal ONP)', are variants of the BNP(, \cal V)' problem in the Schur and Caratheodory classes, respectively, whereas problems (\cal ONP) and (\cal ONP)' are the corresponding variants of the BNP(, \cal V) problem in the present paper.

Furthermore, the last implication can be reversed at all, i.e., starting from Theorem 3.2, one can directly deduce Theorem 3.1 by using the conditions

$$\frac{1}{(2m_j - 1)!} f^{(2m_j - 1)}(\alpha_j) = \hat{d}_{j,2m_j-1}, \quad j = 1, \ldots, s$$
in place of inequalities (1.2) (and therefore the BNP(N′) problem can be considered as a certain new BNP(N) problem, and Theorem 3.2 then is applicable for it). As a result, the statement of Theorem 3.1 is nothing other than a simple consequence of Theorem 3.2.

The preceding reasoning suggests in substance an equivalency of both Theorems 3.1 and 3.2, and a similarity of both the BNP(N′) and the BNP(N) problems.

4. Material from the theory of power moments

In view of the exposition of the simple connections between solutions to the boundary NP problems in the class N and solutions to the truncated (standard or nonstandard) Hamburger moment problem, it makes sense to apply the rather thoroughly developed machinery of theory of power moments to the boundary NP problems.

In Section 4.1, we will recall the material from the theory of moments needed for what follows, referring to [1,7,20] for details. In Section 4.2, we will consider the general solutions to the HM problem with constraint (1.11) and to the SHM problem with constraint (1.11) in the indeterminate case, which seem to be new.

Let \( h = (h_0, h_1, \ldots, h_{2N-2}) \) be the (real) Hankel vector of the BNP(N′) (or the BNP(N/0)) problem, which is the moment sequence of the HM (or the SHM) problem. Denote by 

\[
H_{NN}[h] = \begin{pmatrix} h_0 & h_1 & \cdots & h_{2N-2} \\ h_1 & h_2 & \cdots & h_{2N-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{2N-2} & h_{2N-3} & \cdots & h_{3N-4} \end{pmatrix}
\]

the real Hankel matrix of order \( N \), built on \( h \). Let \( n_1 = \text{rank} \, H_{NN}[h] \). If, further, \( H_{NN}[h] \) is Hermitian nonnegative and singular, then there exists a unique real \( \xi \) with \( 0 \leq \xi < h_{2N-2} \) such that \( h = h_p + h_d \), \( h_p = (h_0, h_1, \ldots, h_{2N-3}, h_{2N-2} - \xi) \), \( h_d = (0, \ldots, 0, \xi) \) and

\[
H_{NN}[h] = H_{NN}[h_p] + H_{NN}[h_d],
\]

where \( H_{NN}[h_p] \) and \( H_{NN}[h_d] \) are the proper and degenerate parts of \( H_{NN}[h] \), respectively, \( H_{NN}[h_p] \geq 0 \), \( H_{NN}[h_d] \geq 0 \), and \( \text{rank} \, H_{NN}[h] = \text{rank} \, H_{NN}[h_p] + \text{rank} \, H_{NN}[h_d] \) (see Lemma 2.7 of [7]).

4.1. The solution of the HM problem and of the SHM problem

Consider first the HM problem. It is well known that the HM problem is solvable if and only if

\[
H_{NN}[h] \geq 0.
\]

In the case when \( H_{NN}[h] \geq 0 \) is singular, then and only then the solution \( \tau(u) \) is unique. A distinction should be made between the case when \( H_{NN}[h] \geq 0 \) is degenerate and the opposite. If \( H_{NN}[h] \geq 0 \) is degenerate, i.e., \( H_{NN}[h] = H_{NN}[h_d] \), the unique solution to the HM problem is \( \tau(u) \equiv 0 (\infty < u < +\infty) \) obviously. In the
opposite case, i.e., \( H_{HN}[h] \neq 0 \), it is easy to see that \( R = \text{rank} \; H_{HN}[h] \) is equal to \( n_1 \) if \( \xi = 0 \) or \( n_1 - 1 \) if \( \xi > 0 \), and that there exists a pair of coprime polynomials \( p_{n_1}(z) \) and \( \gamma_{n_1}(z) \), where \( p_{n_1}(z) \) is the \( R \)th orthogonal polynomial with respect to the positive sequence \( \{h_k\}_{0}^{2R-2} \), and \( \gamma_{n_1}(z) \) is a polynomial of the second kind defined by

\[
\gamma_{n_1}(z) = L_u \left\{ \frac{p_{n_1}(u) - p_{n_1}(z)}{u - z} \right\}
\]

(see Eq. (3.10) for definition of \( L_u \)). Then, the unique solution \( \tau(u) \) \((-\infty < u < +\infty)\) to the HM problem is determined by the formula

\[
\int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma_{n_1}(z)}{p_{n_1}(z)},
\]

and by the Stieltjes inversion formula. Thus, \( \tau(u) \) defined by Eq. (4.4) assigns no mass to any of the real nodes \( \alpha_1, \ldots, \alpha_s \) if and only if \( p_{n_1}(\alpha_j) \neq 0 \) for \( j = 1, \ldots, s \). (Noting that \( \gcd(\gamma_{n_1}(z), p_{n_1}(z)) = 1 \) and that \( p_{n_1}(z) \) has only real zeros.)

Furthermore, if \( H_{HN}[h] > 0 \), the HM problem is indeterminate, i.e., it has infinitely many solutions. Let \( p(z) \) of degree \( N \) and \( q(z) \) of degree \( N - 1 \) denote, respectively, the \( N \)th and the \( (N - 1) \)th orthogonal polynomials with respect to the positive sequence \( \{h_k\}_{0}^{2N-2} \), and \( \gamma(z) \) and \( \delta(z) \) be polynomials of the second kind, defined by

\[
\gamma(z) = L_u \left\{ \frac{p(u) - p(z)}{u - z} \right\} \quad \text{and} \quad \delta(z) = L_u \left\{ \frac{q(u) - q(z)}{u - z} \right\}.
\]

Then the general solution \( \tau(u) \) \((-\infty < u < +\infty)\) of the HM problem is representable as a linear fractional transformation

\[
\int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)},
\]

(4.6)

where \( v(z) \) is a certain member of the class \( \mathcal{N} \) augmented by the constant \( \infty \).

Let us turn to the SHM problem. Obviously, the solution set of the SHM problem, if a solution exists, is contained in the solution set of the HM problem. It is known that the SHM problem is solvable if and only if \( H_{NN}[h] \) is Hermitian nonnegative and proper, i.e., either \( H_{NN}[h] > 0 \) or \( H_{NN}[h] \geq 0 \) is singular with \( H_{NN}[h] = H_{NN}[h] \). Moreover, in the case when \( H_{NN}[h] > 0 \) is singular and proper, the SHM problem is determinate, and the unique solution coincides with the function \( \tau(u) \) \((-\infty < u < +\infty)\) that is defined by Eq. (4.4). In the case when \( H_{NN}[h] > 0 \), the SHM problem is indeterminate, and its general solution \( \tau(u) \) \((-\infty < u < +\infty)\) is representable as a linear fractional transformation of form (4.6) but with the free parameter \( v(z) = \infty \) or \( v(z) \in \mathcal{N} \) subject to \( \lim_{z \to -\infty} v(z)/z = 0 \) uniformly in each sector \( \pi_{\epsilon}(0) \).
4.2. On constraint (1.11) for both the HM and the SHM problems in the indeterminate case

In the case of \( H_{NN}^T U > 0 \), both the HM and the SHM problems with constraint (1.11) are obviously indeterminate, since for any real \( c \neq 0 \) such that \( p(\alpha_j)c - q(\alpha_j) \neq 0, \) \( j = 1, \ldots, s \) (the number of such real \( c \) is infinite),

\[
\int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma(z)c - \delta(z)}{p(z)c - q(z)}
\]

is a certain solution to the SHM problem (and thus to the HM problem) that assigns no mass to all the real nodes. We shall dwell in a little more detail on these two moment problems, because they are needed in Section 5. For the HM problem with constraint (1.11) we have:

**Theorem 4.1.** Let \( H_{NN}^T U > 0 \). The general solution \( \tau(u) \) to the HM problem with constraint (1.11) is representable as the linear fractional transformation (4.6), in which the free parameter \( v(z) \in \mathcal{N} \cup \{\infty\} \) is determined in the following manner: either \( v(z) = \infty \) if \( p(\alpha_j) \neq 0 \) for \( j = 1, \ldots, s \) or \( v(z) \) runs over the class \( \mathcal{N} \) such that \( \sigma_v(\alpha_j) = 0 \) if \( p(\alpha_j) = 0 \), and one of the following conditions holds if \( p(\alpha_j) \neq 0 \):

(a) \( v(z) \) has a nontangential limit \( v(\alpha_j) \) at \( \alpha_j \) satisfying \( p(\alpha_j)v(\alpha_j) - q(\alpha_j) \neq 0 \);
(b) \( v(z) \) has no a real angular derivative of order 1 at \( \alpha_j \), i.e.,

\[
\int_{-\infty}^{+\infty} \frac{1}{(u - \alpha_j)^2} \, d\tau(u) = +\infty.
\]

**Proof.** We first prove that given a function \( v(z) \in \mathcal{N} \) in the way made in the statement of the theorem, which is of a representation of form (2.4) but with \( v \) in place of \( f \), the \( \tau(u) \) defined by

\[
\phi(z) = \int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)}
\]

is subject to Eq. (1.11), and thus is some solution to our moment problem.

As for the case when \( p(\alpha_j) \neq 0 \) for \( j = 1, \ldots, s \) and \( v(z) = \infty \), the corresponding result is plain.

In the case when \( v(z) \) runs over the class \( \mathcal{N} \) such that \( \sigma_v(\alpha_j) = 0 \) if \( p(\alpha_j) = 0 \) and one of the conditions (a) and (b) holds if \( p(\alpha_j) \neq 0 \), we are only to prove that \( \tau((\alpha_j)) = 0 \) for that \( j \).

If \( p(\alpha_j) \neq 0 \) and (a) holds for some \( j \), then \(-\tau((\alpha_j)) = \lim_{z \to \alpha_j} (z - \alpha_j)\phi(z) = 0 \) by Lemma 2.2.

Now let \( p(\alpha_j) \neq 0 \) and (b) holds for that \( j \). If \( \sigma_v((\alpha_j)) \neq 0 \) for the same \( j \), then by Lemma 2.2 again, \( \lim_{z \to \alpha_j} v(z) = \infty \), so that \( \phi(\alpha_j) = -\gamma(\alpha_j)/p(\alpha_j) \)
and \( \tau((\alpha_j)) = 0 \) for that \( j \) obviously. If \( \sigma_v((\alpha_j)) = 0 \) for that \( j \), i.e., \( \lim_{z \to \alpha_j} (z - \alpha_j)\phi(z) = 0 \)
\( \alpha_j \nu_j = 0 \), we have also that \( \tau(\{\alpha_j\}) = 0 \) for that \( j \). We suppose in the opposite
that \( \tau(\{\alpha_j\}) \neq 0 \) for that \( j \). Then, from
\[
\lim_{z \to \alpha_j} (z - \alpha_j)(\gamma(z)v(z) - \delta(z)) = \gamma(\alpha_j) \lim_{z \to \alpha_j} (z - \alpha_j)v(z) = 0,
\]
we deduce that \( \lim_{z \to \alpha_j} (p(z)v(z) - q(z)) = 0 \) and thus \( v(z) \) has a non-tangential
limit at \( \alpha_j \) such that
\[
v(\alpha_j) = \lim_{z \to \alpha_j} v(z) = \frac{q(\alpha_j)}{p(\alpha_j)} \in \mathbb{R}.
\]
Furthermore, since \( \gamma(z)q(z) - \delta(z)p(z) = \sigma \) (see, e.g., [7]), and
\[
-\tau(\{\alpha_j\}) = \lim_{z \to \alpha_j} (z - \alpha_j)\phi(z)
\]
\[
= -\frac{\sigma}{(p(\alpha_j))^2} \lim_{z \to \alpha_j} \left( \frac{v(z) - q(z)/p(z)}{z - \alpha_j} \right)^{-1}
\]
\[
\neq 0,
\]
\( (v(z) - q(z)/p(z))/(z - \alpha_j) \) has a nonzero and real non-tangential limit at \( \alpha_j \), and therefore
\[
\frac{v(z) - q(\alpha_j)/p(\alpha_j)}{z - \alpha_j} = \frac{v(z) - v(\alpha_j)}{z - \alpha_j}
\]
has also a real non-tangential limit at \( \alpha_j \). Thus, \( v(z) \) has a real non-tangential limit at
\( \alpha_j \) and the real angular derivative at \( \alpha_j \), i.e., by Lemma 2.1, for that \( j \),
\[
\int_{-\infty}^{+\infty} \frac{d\sigma_v(u)}{(u - \alpha_j)^2} < +\infty, \quad (4.8)
\]
a contradiction with condition (b).

Let us turn to the case of \( p(\alpha_j) = 0 \) and \( \nu_j(\{\alpha_j\}) = 0 \) for some \( j \). Then we will show that \( \tau(\{\alpha_j\}) = 0 \) for the same \( j \). But the condition that \( p(\alpha_j) = 0 \) implies that
\( q(\alpha_j) \neq 0 \), so that, by Lemma 2.2, we have
\[
-\tau(\{\alpha_j\}) = \lim_{z \to \alpha_j} (z - \alpha_j)\phi(z)
\]
\[
= -\lim_{z \to \alpha_j} \frac{\gamma(z)(z - \alpha_j)v(z) - (z - \alpha_j)\delta(z)}{p(z) - p(\alpha_j)} \frac{z - \alpha_j}{z - \alpha_j}v(z) - q(z)
\]
\[
= -\frac{\gamma(\alpha_j)\sigma_v(\{\alpha_j\})}{p'(\alpha_j)\sigma_v(\{\alpha_j\}) + q(\alpha_j)}
\]
\[
= 0.
\]
Conversely, we suppose that \( \tau(u) (-\infty < u < +\infty) \) is some solution to the HM
problem with constraint (1.11) \( \tau(\{\alpha_j\}) = 0 \) for \( j = 1, \ldots, s \). Then it follows from
Eq. (4.6) that
\[ \phi(z) = \int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)} \]  
(4.9)

for some \( v(z) \). We show that this \( v(z) \) should be subject to one of the conditions made in the statement of the theorem. Let \( v(z) \) be not the constant \( 1 \), and \( p(j) = D_0 \) for some \( j \). We will verify that if \( v(z) \in \mathcal{V} \) does not meet condition (a) for that \( j \), then (b) must hold for that \( j \). For the opposite, we have Eq. (4.8) for that \( j \), so that \( \tau([\alpha_j]) = -\lim_{z\to\alpha_j}(z - \alpha_j)v(z) = 0 \) for \( j \). Since \( p(\alpha_j)v(\alpha_j) - q(\alpha_j) = 0 \) for \( j \), \( p(\alpha_j) \neq 0 \) and \( \gamma(\alpha_j)v(\alpha_j) - \delta(\alpha_j) \neq 0 \) for the same \( j \) obviously. From

\[ \tau([\alpha_j]) = -\lim_{z\to\alpha_j}(z - \alpha_j)\frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)} = 0 \]

for that \( j \), we deduce

\[ \lim_{z\to\alpha_j} \frac{v(z) - q(z)/p(z)}{z - \alpha_j} = \infty \]

for the same \( j \), so that

\[ \lim_{z\to\alpha_j} \frac{v(z) - q(\alpha_j)/p(\alpha_j)}{z - \alpha_j} = \lim_{z\to\alpha_j} \frac{v(z) - v(\alpha_j)}{z - \alpha_j} = \infty \]

for that \( j \), which is impossible, since \( v(z) \) has the angular derivative at \( \alpha_j \).

It remains to consider the case of \( p(\alpha_j) = 0 \) for some \( j \). We will show that \( \sigma_e(\{\alpha_j\}) = 0 \), where \( v(z) \) is defined via Eq. (4.9). For the opposite, we have that \( \sigma_e(\{\alpha_j\}) = -\lim_{z\to\alpha_j}(z - \alpha_j)v(z) \neq 0 \) for that \( j \), so that, since \( \gamma(\alpha_j) \neq 0 \) and \( p'(\alpha_j)\sigma_e(\{\alpha_j\}) + q(\alpha_j) \) is finite,

\[ \tau([\alpha_j]) = -\lim_{z\to\alpha_j}(z - \alpha_j)\phi(z) = \frac{\gamma(\alpha_j)v(\alpha_j)}{p'(\alpha_j)\sigma_e(\{\alpha_j\}) + q(\alpha_j)} \neq 0, \]
a contradiction with Eq. (1.11). The proof of Theorem 4.1 is complete. \( \square \)

Similarly, for the SHM problem with constraint (1.11) we have:

**Theorem 4.2.** Let \( H_{NN}[h] > 0 \). Then the general solution \( \tau(u) \) \((-\infty < u < +\infty)\) to the SHM problem with constraint (1.11) is representable as the linear fractional transformation (4.6), in which the free parameter \( v(z) \in \mathcal{V} \cup \{\infty\} \) is determined in the following manner: either \( v(z) = \infty \) if \( p(\alpha_j) \neq 0 \) for \( j = 1, \ldots, s \) or \( v(z) \) runs over the class \( \mathcal{V} \) satisfying in addition to one of the conditions made in the statement of Theorem 4.1,

\[ \lim_{z\to\infty} \frac{v(z)}{z} = 0 \]

(4.10)

uniformly in each sector \( \pi_e(0) \).
5. The solution of the boundary NP problems in the class $\mathcal{A}$

This section is devoted to find the existence and uniqueness conditions for the solutions to the BNP(\(\mathcal{A}\)), the BNP(\(\mathcal{A}\)), and the BNP(\(\mathcal{A}\)) problems, and to describe their solutions by using the results given in the previous sections.

5.1. The generalized Pick matrix for the BNP(\(\mathcal{A}\)) problem

Let $\omega(z)$ and $h = (h_0, h_1, \ldots, h_{2N-2})$ be given as before. Denote by
\[
x = \left(\begin{array}{c}
\delta_1 	ext{ times} \\
\vdots \\
\delta_{\theta+s} 	ext{ times}
\end{array}\right) x_1, \ldots, x_{\theta+s}, \ldots, x_1
\]
(5.1)
the collection of interpolation nodes, in which $x_i = z_i$, $\delta_i = n_i$ if $1 \leq i \leq \theta$ and $x_i = \alpha_i$, $\delta_i = m_i$ if $\theta + 1 \leq i \leq \theta + s$. By definition, the generalized Pick matrix of the BNP(\(\mathcal{A}\)) problem is in fact a generalized Hermitian Loewner matrix of order $N$ defined via the formula
\[
L_x = (L_{ij})_{i,j=1}^{N+s},
\]
in which
\[
L_{ij} = (l_{ij}^{kl})_{k,l=0}^{N+s} \in \mathbb{C}^{n \times s}
\]
with entries $l_{ij}^{kl}$ determined by the divided differences of $\omega(z)$:
\[
l_{ij}^{kl} = \left[\frac{\omega^{k+l}(z) - \omega^{k+l}(x_i)}{k!l!} \frac{\partial^k \omega \partial^l \omega}{\partial \lambda^k \partial \mu^l} \right]_{\lambda = x_i, \mu = x_j}
\]
if $x_i \neq x_j$,
\[
= \frac{1}{(k + l + 1)!} \omega^{k+l+1}(x_i)
\]
if $x_i = x_j$.

Let
\[
b(z) = \prod_{i=1}^{\theta+s} (z - x_i)^{\delta_i},
\]
\[
b_{ik}(z) = \frac{b(z)}{(z - x_i)^{\delta_i + 1}}, \quad i = 1, \ldots, \theta + s, \quad k = 0, 1, \ldots, \delta_i - 1.
\]
and let $W_x$ of order $N$ be the transition matrix from the standard bases $\{1, z, \ldots, z^{N-1}\}$ of the space of polynomials of degree at most $N - 1$ to the interpolation bases $\{b_{ik}(z)\}, i = 1, \ldots, \theta + s, k = 0, 1, \ldots, \delta_i - 1$, that is to say,
Obviously, $W_X$ is invertible.

**Lemma 5.1** [7,30]. Let $L_X$, $W_X$ and $H_{NN}[h]$ be defined as before. Then

$$L_X = W_X H_{NN}[h] W_X^*.$$  \hspace{1cm} (5.4)

As a consequence of Lemma 5.1, we have a useful assertion:

$$L_X \succ 0 \quad \text{if and only if} \quad H_{NN}[h] \succ 0 \quad (>0).$$  \hspace{1cm} (5.5)

In the case when $L_X \succ 0$ is singular (or what is the same, $H_{NN}[h] \succ 0$ is singular), from Eq. (4.1) we deduce that $L_X$ has a unique quasidirect composition of the form

$$L_X = L_p + L_d \overset{def}{=} W_X H_{NN}[h_p] W_X^* + W_X H_{NN}[h_d] W_X^*,$$  \hspace{1cm} (5.6)

where $L_p \succ 0$ and $L_d \succ 0$, referred to as the proper and degenerate parts of $L_X$, respectively, such that rank $L_X = \text{rank} L_p + \text{rank} L_d$. In what follows, we shall call $L_X \succeq 0$ proper if $L_X > 0$ or if $L_X \succeq 0$ is singular with $L_X = L_p$. Furthermore, we shall call $L_X \succeq 0$ of rank $R$ is saturated if all its Loewner submatrices of order $R$ are nonsingular. Here by an $R \times R$ Loewner submatrix of $L_X$ we mean a certain generalized Loewner matrix of order $R$, which is the same as $L_X$ in Eqs. (5.1)–(5.3) but with $t_i$ therein replaced by $t_i$, $0 \leq t_i \leq t_i$, $i = 1, \ldots, \theta + s$ such that $\sum_{i=1}^{\theta+s} t_i = R$. It is easy to check that $L_X \succeq 0$ of order $R$ is proper and saturated if and only if the $BNP(A')$ problem has a proper rational solution of degree $R$, also if and only if $H_{NN}[h] \succ 0$ is proper, and there exists a solution $\tau(u)$ to the HM problem such that Eq. (1.11) holds.

### 5.2. The solution of the $BNP(A')$ problem

In view of Theorem 3.1 and Lemma 5.1 together with the results made in Section 4.1, we have:

**Theorem 5.2.** The $BNP(A')$ problem is solvable if and only if $L_X \succeq 0$, also, if and only if, $H_{NN}[h] \succeq 0$, or what is the same, the HM problem is solvable. Moreover:

(a) In the case when $H_{NN}[h] \succeq 0$ is singular, the $BNP(A')$ problem has only one solution $f(z)$, determined by $\omega(z)$ of degree at most one if $H_{NN}[h] = H_{NN}[h_1]$ and by

$$f(z) = \omega(z) - a(z)\gamma_{n_1}(z)/p_{n_1}(z)$$  \hspace{1cm} (5.7)

if $H_{NN}[h] \neq H_{NN}[h_1]$, which is a rational function of degree exact $n_1$, where $\gamma_{n_1}(z)$ and $p_{n_1}(z)$ are as in Eq. (4.4).

(b) In the case of $H_{NN}[h] > 0$, the $BNP(A')$ problem has infinitely many solutions, which can be parametrized by the linear fractional transformation

\[ f(z) = \frac{\omega(z) - a(z)\gamma_{n_1}(z)/p_{n_1}(z)}{1 - \gamma_{n_1}(z)/p_{n_1}(z)} \]
\[ f(z) = \alpha(z) - \alpha(z) \frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)}, \]  
(5.8)

where \( v(z) \) runs over the class \( \mathcal{V} \cup \{ \infty \} \), and \( p(z), q(z), \gamma(z), \delta(z) \) are as in Eqs. (4.5) and (4.6).

We remark that in Eq. (5.7) even if \( p_{n_1}(\alpha_j) = 0 \) for some \( j \), then, by Lemma 2.3, \( \alpha_j \gamma_{n_1}(z)/p_{n_1}(z) \) has all of the angular derivatives at \( \alpha_j \in \mathbb{R} \) up to order \( 2m_j - 1 \), similarly for Eq. (5.8).

5.3. The solution of the BNP(\( \mathcal{V} \)) problem

From Theorem 3.2 together with certain results made in Section 4, we deduce:

**Theorem 5.3.** The BNP(\( \mathcal{V} \)) problem is solvable if and only if

\[ H_{NN}[h] > 0 \text{ or } H_{NN}[h] \geq 0 \text{ is singular and there exists some solution } \tau(u) \ (-\infty < u < +\infty) \text{ to the HM problem that assigns no mass to any of real nodes } \alpha_1, \ldots, \alpha_s. \]

To present an alternative solvability criterion for the BNP(\( \mathcal{V} \)), a definition is needed, which is a slight extension of the notion of minimally positive matrices introduced in [27].

Let \( \mathcal{D}_x \) denote the set of all \( N \times N \) positive semidefinite diagonal matrices of the form \( D(a) = \text{diag}[0, \ldots, D_{n_1}] \), where \( a = (a_1, \ldots, a_s)^T \geq 0 \), and \( D_{n_1} \) is a diagonal matrix of order \( m \) defined by

\[ D_{n_1} = \text{diag} \left[ \text{diag} \left( \underbrace{0, \ldots, 0}_{m_1-1 \text{ times}}, a_i \right) \right]_{i=1}^s. \]

(5.9)

**Definition 5.4.** An \( N \times N \) positive semidefinite matrix \( L \) is said to be \( \mathcal{D}_x \)-minimally positive if there is no nonzero matrix \( D(a) \in \mathcal{D}_x \) such that \( L \geq D(a) \). (Each \( \mathcal{D}_x \)-minimally positive matrix is obviously singular.)

**Theorem 5.5.** The BNP(\( \mathcal{V} \)) problem is solvable if and only if \( L_x > 0 \) or \( L_x \) is \( \mathcal{D}_x \)-minimally positive.

**Proof.** Suppose that the BNP(\( \mathcal{V} \)) problem is solvable, so is the BNP(\( \mathcal{V} \)) problem. Then, by Theorem 5.2, we have \( L_x > 0 \). If \( L_x > 0 \) is singular but not \( \mathcal{D}_x \)-minimally positive, then the BNP(\( \mathcal{V} \)) problem has only one solution \( f(z) \) and there exists \( 0 \neq D(a) \in \mathcal{D}_x \) such that \( L_x = L_x - D(a) \geq 0 \). But \( \bar{L}_x \) is obviously a singular generalized Loewner matrix with the same off-diagonal entries as that of \( L_x \), so that the BNP(\( \mathcal{V} \)) problem with \( \bar{L}_x \) (rather than \( L_x \)) as its generalized Loewner matrix has only one solution \( \bar{f}(z) \). It is clear that \( \bar{f}(z) \) is also a solution to the BNP(\( \mathcal{V} \)) problem, and therefore \( f(z) = \bar{f}(z) \). This is impossible,
since \( L_x \neq \bar{L}_x \). Thus, \( L_x > 0 \) or \( L_x \) is \( \mathcal{D}_x \)-minimally positive if the BNP(\( \mathcal{N} \)) problem is solvable.

Conversely, we first suppose that \( L_x \) is \( \mathcal{D}_x \)-minimally positive. Then the HM problem has only one solution \( \tau(u) \) \((-\infty < u < +\infty)\). Let \( f(z) \) be the corresponding function defined by Eq. (3.1), then, by Theorem 5.2, \( f(z) \) is the unique solution to the BNP(\( \mathcal{N} \)'\( \mathcal{N} \)) problem but with \( f(z) \) in place of \( \omega(z) \) therein. We have that \( L_x - \bar{L}_x \in \mathcal{D}_x \) obviously, and therefore \( L_x = \bar{L}_x \), since \( L_x \) is \( \mathcal{D}_x \)-minimally positive. This implies that \( f(z) \) is also a solution to the BNP(\( \mathcal{N} \)) problem.

Secondly, we suppose that \( L_x > 0 \). Then the HM problem has infinitely many solutions of form (4.6). Since \( \gcd(p(z), q(z)) = 1 \), there exists a real \( c \) such that \( p(\alpha_j)c - q(\alpha_j) \neq 0 \) for \( j = 1, \ldots, s \). It is easy to see that the solution \( \tau(u) \) to the HM problem defined by
\[
\int_{-\infty}^{+\infty} \frac{d\tau(u)}{u - z} = -\frac{\gamma(z)c - \delta(z)}{p(z)c - q(z)}
\]
is subject to constraint (1.11). Let \( f(z) \) be defined via Eq. (3.1), in which \( \tau(u) \) is as in Eq. (5.10), then \( f(z) \) is a solution to the BNP(\( \mathcal{N} \)) problem by Theorem 3.2. \( \square \)

From Theorem 5.3 and Lemma 4.1 together with some results made in Section 4.1, we have:

**Theorem 5.6.** Let the BNP(\( \mathcal{N} \)) problem be given. Then:

(a) If \( H_{NN}[h] \geq 0 \) is singular and degenerate, then the BNP(\( \mathcal{N} \)) problem has only one solution, \( \omega(z) \), a polynomial of degree at most one.

(b) If \( H_{NN}[h] \geq 0 \) is singular and not degenerate, and if \( p_{n_1}(\alpha_j) \neq 0 \), \( j = 1, \ldots, s \), then the BNP(\( \mathcal{N} \)) problem has only one solution of the form
\[
f(z) = \frac{\gamma_{n_1}(z)}{p_{n_1}(z)}
\]
in which \( \gamma_{n_1}(z) \) and \( p_{n_1}(z) \) are as in Eq. (4.4). Here \( f(z) \) is a rational function of degree exact \( n_1 \).

(c) If \( H_{NN}[h] > 0 \), then the BNP(\( \mathcal{N} \)) problem has infinitely many solutions, which can be parametrized by the linear fractional transformation
\[
f(z) = \frac{\gamma(z)v(z) - \delta(z)}{p(z)v(z) - q(z)}
\]
where \( p(z), q(z), \gamma(z), \delta(z) \) are as in Eq. (4.6), and \( v(z) \in \mathcal{N} \cup \{\infty\} \) is determined in the same way as that made in the statement of Theorem 4.1.

We remark that the main theorems stating solvability criteria for the BNP(\( \mathcal{N} \)) problem in [14] are not true, as was pointed out in [27]. (But the solvability criterion stated in [14] supplemented by the properness of \( L_x \) is in fact applicable for the BNP(\( \mathcal{N} \)'\( \mathcal{N} \)) problem rather than the BNP(\( \mathcal{N} \)) problem, see Theorem 5.7 below.)
5.4. The solution of the BNP$(\mathcal{V})^\nu$ problem

As a consequence of Theorem 3.3 together with the related results given in Sections 4 and 5.1, we have:

**Theorem 5.7.** The BNP$(\mathcal{V})^\nu$ problem is solvable if and only if $H_{NN}[\mathbf{h}] > 0$ or $H_{NN}[\mathbf{h}] \geq 0$ is singular, proper and $p_{n_1}(\alpha_j) \neq 0$ for $j = 1, \ldots, s$, where $p_{n_1}(z)$ is as in Eq. (4.4), also if and only if $L_\mathbf{x}$ is positive semidefinite, proper and saturated.

Note that from Theorems 5.5 and 5.7 we deduce that for the singular $L_\mathbf{x} \geq 0$, the properness and saturatedness guarantee the condition of $S_\mathbf{x}$-minimal positivity; the inverse is not true. This observation leads to a subtlety of the BNP$(\mathcal{V})^\nu$ problem compared to a seemingly analogous problem ($\mathcal{O}$NP$\mathcal{P}$) in [27]. For the latter, if the Pick matrix, which always is proper in some sense stated in Section 1, is singular and positive semidefinite, then it is saturated if and only if it is minimally positive.

**Theorem 5.8.** Let the BNP$(\mathcal{V})^\nu$ problem be given. Then

(a) If $H_{NN}[\mathbf{h}] \geq 0$ is singular and proper, and $p_{n_1}(\alpha_j) \neq 0$, $j = 1, \ldots, s$, then the BNP$(\mathcal{V})^\nu$ problem has only one solution $f(z)$ of form (5.11). Here $f(z)$ is a proper rational function of degree exact $n_1$.

(b) If $H_{NN}[\mathbf{h}] > 0$, then the BNP$(\mathcal{V})^\nu$ problem has infinitely many solutions, which can be parametrized by the linear fractional transformation (5.12), where the parameter $v(z) \in \mathcal{V} \cup \{\infty\}$ is determined in the same way as that made in the statement of Theorem 4.2.

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References


