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Theoretical Computer Science 327 (2004) 175–195

Theoretical Computer Science

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A coalgebraic view on positive modal logic

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Abstract

Positive modal logic is the restriction of the modal local consequence relation defined by the class of all Kripke models to the propositional negation-free modal language. The class of *positive modal algebras* is the one canonically associated with PML according to the theory of the algebrization of logics (Lecture Notes in Logic, Springer, Berlin, 1996). A Priestley-style duality is established between the category of positive modal algebras and the category of $\mathbf{K}^+$*-spaces* in (J. IGPL 7 (6) (1999) 683). In this paper, we establish a categorical equivalence between the category $\mathbf{K}^+$ of $\mathbf{K}^+$-spaces and the category $\mathbf{Coalg}(\mathbf{V})$ of coalgebras of a suitable endofunctor $\mathbf{V}$ on the category of Priestley spaces.

Keywords: Positive modal logic; Positive modal algebra; Priestley space; Vietoris functor

1. Introduction

Overview on algebras, coalgebras and topological spaces. In recent years, researchers in logic and theoretical computer science have developed a growing interest in coalgebras as semantic structures for logical languages. The perspective taken by Moss [22], Rossiger [26], Kurz [19,20], Jacobs and Pattinson among others (see [20] also for a complete list of references) is to view coalgebras as abstract versions of state-based dynamical systems. Generalizing the view on modal logic as the logic of transition systems, formulas of logical languages arising as initial algebras of given (classes of) endofunctors on **Set** are interpreted in the corresponding final coalgebras, which play a similar role to canonical models.

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1 Partially supported by the Catalonian Grant 2001SGR-00017, by the Spanish Grant BFM2001-3329 and by the Grant 2001FI 00281 of the Generalitat de Catalunya.

doi:10.1016/j.tcs.2004.07.026

But there are also reasons of interest in coalgebras as semantic structures for logical languages that stem from algebraic logic, and are independent from the dynamical systems perspective on coalgebras. The fact that every logic is canonically associated with a class of algebras, in addition to the natural algebra/coalgebra duality, is what intuitively makes coalgebras a good candidate for the role of semantic structure for logics, from the general perspective of algebraic logic. An attempt in this direction is [7]. See also [23] for a different, but related perspective.

Following the algebraic logic perspective, topological spaces are easily brought into the picture in connection with coalgebras, via the *duality theory*. The theory of dualities is a well-established field of research in universal algebra, and consists in establishing categorical dualities between given classes of algebras and nice categories of topological spaces, possibly endowed with additional structure (see [6] for a general account). Some well-known dualities of this kind are the *Stone duality*, between Boolean algebras and Stone spaces, the *Jónsson–Tarski duality* between Boolean algebras with operators (BAOs) and descriptive general frames for the normal modal logic K, and the *Priestley duality*, between bounded distributive lattices and Priestley spaces. Since coalgebras are dual to algebras in a natural way, it seems reasonable to hope that topological spaces that are dual to interesting categories of algebras could be nicely represented as coalgebras. This is the case of the topological spaces that are dual to Heyting algebras (see [7], and the discussion at the beginning of Section 5). Independently from the algebraic logic and universal algebra perspective on the connection between coalgebras and topological spaces, the coalgebraic nature of topological spaces has been noticed by Gumm in [14], and Kurz and Pattinson [21] used topology to capture the notion of *finitary observational equivalence*, and find an adequate semantics to finitary modal logic in a coalgebraic setting. Intuitively, topologizing a set is a handy way of selecting all its relevant subsets (like for example, the ones that correspond to propositions of a logical language) and keeping at the same time cardinalities small. Topological spaces have been successfully applied to this purpose not only in logic and universal algebra, but also in theoretical computer science, *domain theory* being an outstanding example.

Dualities as well proved to be a useful tool of investigation in theoretical computer science: the Stone duality is the key tool Abramsky used in [1] to connect the denotational and the logical interpretations of the metalanguage of types and terms there introduced. The methodology he follows has many points in common with the one used by Jacobs in a later paper [15] to define the *Kripke polynomial functors* on **Set** and in his proof of the ‘soundness and completeness’ result on the associated Many-Sorted Coalgebraic Modal Logic. Jacob’s framework on **Set** was then extended to coalgebras over Stone spaces in [18]. A key ingredient in moving from **Set** to Stone spaces is to replace the powerset endofunctor $\mathcal{P}$ with the *Vietoris endofunctor* **K** *on Stone spaces*. Another relevant result in [18] is the categorical equivalence between the category **DGF** of descriptive general frames for the modal logic K and $\mathbf{Coalg}(\mathbf{K})$. The main result of the present paper extends this equivalence to the case of positive modal logic.

1.1. Positive modal logic

Intuitively, positive modal logic (PML) is what one gets when one drops the negation symbol in the language of the normal modal logic K. PML was introduced by Dunn [9], and it is the restriction of the modal local consequence relation defined by the class of all Kripke models to the propositional modal language whose connectives are $\wedge$, $\vee$, $\Box$, $\Diamond$, $\top$, $\bot$. Readers familiar with domain theory may think of it as a variation of the logic introduced by Abramsky [1], in case of the Plotkin powerdomain. PML and K have the same Kripke semantics, and the theorems of PML are exactly the theorems of K in which the negation does not occur. Differences show on the algebraic side, because dropping the negation corresponds to a move from BAOs to a class of distributive-lattice based algebras called *positive modal algebras* (see Definition 1 below) introduced by Dunn [9]. In [16], Jansana shows that the class of positive modal algebras is the one canonically associated with PML according to the theory of the algebraization of logics developed in [12], and this means that positive modal algebras are to PML what BAOs are to the modal logic K (and its associated local consequence relation). In [4], a Priestley-style duality is established between the category of positive modal algebras and the category of $\mathbf{K}^+$-*spaces* (see Definition 15 below), which are relational Priestley spaces and can be thought of as the 'descriptive general frames' of PML.

In this paper, we establish an equivalence between the category $\mathbf{K}^+$ of $\mathbf{K}^+$-spaces and the category $\mathbf{Coalg}(\mathbf{V})$ of coalgebras of a suitable endofunctor $\mathbf{V}$ on the category **Pri** of Priestley spaces. Like in the case of K [18], the definition of $\mathbf{V}$ is based on the Vietoris powerspace construction.

The category $\mathbf{Coalg}(\mathbf{V})$ obtained in this way provides a new coalgebraic semantics for PML, the standard one being the well-known representation of Kripke frames as coalgebras of the covariant powerset endofunctor $\mathcal{P}$ on the category **Set** of sets and set maps. We have already remarked that PML and K have the same Kripke semantics (hence, they have the same standard coalgebraic semantics), but different algebraic semantics (positive modal algebras and Boolean algebras with operators, respectively). The new semantics for PML presented here and the one for K given in [18] are capable to reflect this difference in the context of coalgebras. More in general, the categorical equivalences and dualities involved in the process of associating the new coalgebraic semantics with the two logics imply that the total amount of information about PML (and K, respectively) carried by the class of positive modal algebras (Boolean algebras with operators) is imported into the new coalgebraic semantics.

Outline of the paper. In Section 2 the basic notions are presented, together with some useful facts about them. Section 3 is about the definition of the *Vietoris endofunctor* $\mathbf{V}$ on Priestley spaces. The equivalence between $\mathbf{K}^+$ and $\mathbf{Coalg}(\mathbf{V})$ is established in Section 4. Sections 5 and 6 are about questions on connections between Intuitionistic Propositional Logic, its associated class of algebras (Heyting algebras) and the framework introduced here. Finally, some open problems are listed in Section 7.

2. Preliminaries

2.1. The algebraic semantics of PML

Positive modal algebras form the class of algebras canonically associated with PML, and so they are to PML what BAOs are for the normal modal logic K. Essentially, positive modal algebras are bounded distributive lattices with operators:

Definition 1 (*Positive modal algebra*). $\mathcal{A} = \langle A, \wedge, \vee, \Box, \Diamond, 0, 1\rangle$ is a *positive modal algebra* (PMA) iff $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, and $\Box$ and $\Diamond$ are unary operations satisfying the following axioms:

1. $\Box 1 = 1$,	2. $\Diamond 0 = 0$,
3. $\Box(a \wedge b) = \Box a \wedge \Box b$,	4. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
5. $\Box a \wedge \Diamond b \leqslant \Diamond(a \wedge b)$,	6. $\Box(a \vee b) \leqslant \Box a \vee \Diamond b$.

Analogously to the case of K, axioms 1–4 of the definition above say that the modal operators $\Box$ and $\Diamond$ are *normal*. In the case of K, $\Box$ and $\Diamond$ are interdefinable: $\Diamond := \neg\Box\neg$ and $\Box : \neg\Diamond\neg$, and so the same relation in Kripke frames is used to interpret both operators. In the case of PML, due to the lack of negation, $\Box$ and $\Diamond$ are not interdefinable any more, but since $\Box$ and $\Diamond$ are still interpreted using the same relation (recall that PML and K have the same Kripke semantics), the bond between them still exists and needs to be accounted for. This task is accomplished by the *connecting axioms* 5 and 6. The reader familiar with domain theory might have recognized them from the definition of the *Plotkin powerdomain* (see for example Definition 3.4.7 in [1]) where they also occur.

For every preorder $\langle X, \leqslant\rangle$, let $\mathcal{P}_{\leqslant}(X)$ be the collection of the $\leqslant$-*increasing* subsets of X, i.e. those subsets $Y \subseteq X$ such that if $x \leqslant y$ and $x \in Y$ then $y \in Y$. The $\geqslant$-increasing subsets of X are the $\leqslant$-*decreasing* ones. When there can be no confusion about the preorder $\leqslant$, we will refer to $\leqslant$-increasing and $\leqslant$-decreasing subsets as *increasing* and *decreasing subsets*, respectively. It holds that $\langle \mathcal{P}_{\leqslant}(X), \cap, \cup, \emptyset, X\rangle$ is a bounded distributive lattice.

A PML-*frame* [4] is a structure $\langle X, \leqslant, R\rangle$ such that X is a set, $\leqslant$ is a preorder on X (i.e. it is reflexive and transitive) and $R \subseteq X \times X$ such that

$$(\leqslant \circ R) \subseteq (R \circ \leqslant) \quad \text{and} \quad (\geqslant \circ R) \subseteq (R \circ \geqslant). \tag{1}$$

(Recall that if $S, T \subseteq X \times X$, then $x(S \circ T)y$ iff xSz and zTy for some $z \in X$.) Let $R_\Box = (R \circ \leqslant)$ and $R_\Diamond = (R \circ \geqslant)$. For every relation $S \subseteq X \times X$ and every $Y \subseteq X$, let

$$\Box_S(Y) = \{x \in X \mid S[x] \subseteq Y\} \text{ and } \Diamond_S(Y) = \{x \in X \mid S[x] \cap Y \neq \emptyset\}.$$

The properties in (1) are necessary and sufficient conditions for $\Box_{R_\Box}$ and $\Diamond_{R_\Diamond}$ (respectively) to be operations on $\mathcal{P}_{\leqslant}(X)$. In particular we have:

Example 2. For every PML-frame $\langle X, \leqslant, R\rangle$, $\langle \mathcal{P}_{\leqslant}(X), \cap, \cup, \Box_{R_\Box}, \Diamond_{R_\Diamond}, \emptyset, X\rangle$ is a positive modal algebra.

Moreover, it is well known that if the properties in (1) hold, then $R_\Box$ and $R_\Diamond$ are, respectively, the greatest elements of the sets $\{S \subseteq X \times X \mid \Box_S = \Box_R \text{ on } \mathcal{P}_{\leqslant}(X)\}$ and $\{S \subseteq X \times X \mid \Diamond_S \Diamond_R \text{ on } \mathcal{P}_{\leqslant}(X)\}$.

2.2. *The category* **Pri** *of Priestley spaces*

The category **Pri**, of ordered topological spaces and continuous and order-preserving maps between them, is dually equivalent to the category **BDL** of bounded distributive lattices and their homomorphisms according to the well-known Priestley duality [8]. As it was mentioned earlier, positive modal algebras are essentially bounded distributive lattices with operators, and the duality involving positive modal algebras will be based on Priestley duality in the same way as the duality between BAOs and descriptive general frames for K is based on the Stone duality. So Priestley spaces are to PML what Stone spaces are for the normal modal logic K.

Definition 3 (*Priestley space, cf. Davey and Priestley [8]*). A *Priestley space* is a structure $\mathbf{X}\langle X, \leqslant, \tau\rangle$ such that $\langle X, \leqslant\rangle$ is a partial order, $\langle X, \tau\rangle$ is a compact topological space which is *totally order-disconnected*, i.e. for every $x, y \in X$, if $x \nleqslant y$ then $x \in U$ and $y \notin U$ for some clopen increasing subset U of X.

Example 4. If $\mathcal{A} = \langle A, \wedge, \vee\rangle$ is a finite lattice and $\leqslant$ is the lattice order on A, then $\langle A, \leqslant, \mathcal{P}(A)\rangle$ is a Priestley space.

Example 5. If $\mathbf{X} = \langle X, \tau\rangle$ is a Stone space, then $\mathbf{I}(\mathbf{X}) = \langle X, =, \tau\rangle$ is a Priestley space.

Example 6. The Cantor space $\mathcal{C}$ with the order inherited by the real numbers is a Priestley space, for it is compact, and if $x, y \in \mathcal{C}$ such that $x \nleqslant y$, then any subset $U = \mathcal{C} \cap (a, +\infty)$ such that $y < a < x$ and $a \notin \mathcal{C}$ is a witness for the total order-disconnectedness.

A topological space is 0-*dimensional* iff it has a base of clopens (cf. [10]).

Lemma 7. *Let* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$ *be a compact ordered topological space, and let* $\mathcal{B}$ *be a collection of clopen subsets such that for every* $x, y \in X$, *if* $x \nleqslant y$ *then* $x \in B$ *and* $y \notin B$ *for some* $B \in \mathcal{B}$. *Then*
(1) $\mathbf{X}$ *is Hausdorff.*
(2) $\mathcal{B} \cup \{(X \backslash B) \mid B \in \mathcal{B}\}$ *is a subbase of* τ.
(3) $\mathbf{X}$ *is* 0-*dimensional, hence* $\langle X, \tau\rangle$ *is a Stone space.*

Corollary 8. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, $\mathbf{X}$ *is Hausdorff,* 0-*dimensional and*

$$\{U \mid U \textit{ clopen and increasing}\} \cup \{(X \backslash U) \mid U \textit{ clopen and increasing}\}$$

is a subbase of τ.

An immediate consequence of Corollary 8 is that for every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, the space $\mathbf{U}(\mathbf{X}) = \langle X, \tau\rangle$ is a Stone space.

For every preorder $\langle X, \leqslant\rangle$, every $Y \subseteq X$ and every $x \in X$, let $x\uparrow = \{y \in X \mid x \leqslant y\}$ and $x\downarrow = \{y \in X \mid y \leqslant x\}$, let $Y\uparrow \bigcup_{y \in y} y\uparrow$ and $Y\downarrow = \bigcup_{y \in Y} y\downarrow$. For every topological space $\mathbf{X}$, let $K(\mathbf{X})$ be the set of the closed subsets of $\mathbf{X}$.

Proposition 9 (*Palmigiano [24]*). *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$,
(1) $\leqslant$ *is a closed subset of the product space* $\mathbf{X} \times \mathbf{X}$.
(2) *For every* $F \in K(\mathbf{X})$, $F\uparrow$ *and* $F\downarrow$ *are closed subsets of* $\mathbf{X}$.
(3) *For every* $x \in X$, $x\uparrow$ *and* $x\downarrow$ *are closed subsets of* $\mathbf{X}$.

2.3. The closed and convex subsets

The collection of the closed and convex subsets of a Priestley space will play an important role in the definition of the equivalence.

Lemma 10. *Let* $\langle X, \leqslant\rangle$ *be a partial order, then the following are equivalent for every* $F \subseteq X$:
(1) $F = U\uparrow \cap V\downarrow$ *for some* $U, V \subseteq X$.
(2) $F = \bigcup_{x,y\in F}(x\uparrow \cap y\downarrow)$.
(3) *If* $x, y \in F$ *and* $x \leqslant y$, *then* $z \in F$ *for every* $z \in X$ *such that* $x \leqslant z \leqslant y$.

Definition 11 (*Convex subset*). A subset F of a partial order $\langle X, \leqslant\rangle$ is *convex* iff F satisfies any of the conditions of Lemma 10.

For every ordered topological space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$ let us denote $\mathbf{Kcv}(\mathbf{X})$ the collection of the closed and convex subsets of $\mathbf{X}$.

2.4. The Vietoris endofunctor K on Stone spaces

Here, we review the construction of the Vietoris space of a given topological space. This construction is very much related with the definition of the *Plotkin powerdomain* (see for example [1]), and it is functorial over the category of Stone spaces and continuous functions. The resulting endofunctor can be thought of as the topological counterpart of the covariant powerset endofunctor on **Set**, and it is used with this purpose in [18].

Definition 12 (*The Vietoris space, cf. Johnstone [17]*). Let $\mathbf{X} = \langle X, \tau\rangle$ be a topological space. The *Vietoris space associated with* $\mathbf{X}$ is the topological space $\mathbf{K}(\mathbf{X}) = \langle K(\mathbf{X}), \tau_V\rangle$, where $K(\mathbf{X})$ is the collection of the closed subsets of $\mathbf{X}$, and the topology τ_V is the one generated by taking

$$\{t(A) \mid A \in \tau\} \cup \{m(A) \mid A \in \tau\}$$

as a subbase, where for every $A \in \tau$, $t(A) = \{F \in K(\mathbf{X}) \mid F \subseteq A\}$ and $m(A) = \{F \in K(\mathbf{X}) \mid F \cap A \neq \emptyset\}$.

Lemma 13. *For every topological space* $\mathbf{X} = \langle X, \tau\rangle$, *every collection* $\{A_i \mid i \in I\} \subseteq \tau$ *and every clopen subset* U *of* X,
(1) $m(\bigcup_{i\in I} A_i) = \bigcup_{i\in I} m(A_i)$.
(2) $t(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} t(A_i)$.
(3) $m(X\backslash U) = K(\mathbf{X})\backslash t(U)$, *hence* $t(U)$ *is a clopen subset of* $\mathbf{K}(\mathbf{X})$.
(4) $t(X\backslash U) = K(\mathbf{X})\backslash m(U)$ *hence* $m(U)$ *is a clopen subset of* $\mathbf{K}(\mathbf{X})$.

Proposition 14 (*cf. Engelking [10]*). *For every topological space* $\mathbf{X} = \langle X, \tau \rangle$,
(1) *if* $\mathbf{X}$ *is compact and Hausdorff, then so is* $\mathbf{K}(\mathbf{X})$.
(2) *If* $\mathbf{X}$ *is* 0-*dimensional, then so is* $\mathbf{K}(\mathbf{X})$.
(3) *If* $\mathbf{X}$ *is a Stone space, then so is* $\mathbf{K}(\mathbf{X})$.

The assignment $\mathbf{X} \mapsto \mathbf{K}(\mathbf{X})$ can be extended to an endofunctor on the category **St** of Stone spaces and their continuous maps as follows [17]: For every $f \in Hom_{\mathrm{St}}(\mathbf{X}, \mathbf{Y})$ and every $F \in K(\mathbf{X})$, $\mathbf{K}(f)(F) := f[F]$. $\mathbf{K}$ is the *Vietoris endofunctor on Stone spaces*.

2.5. *The category* $\mathbf{K}^+$ *of* $\mathbf{K}^+$*-spaces*

The category of $\mathbf{K}^+$-spaces and their bounded morphisms is dually equivalent to the category of positive modal algebras and homomorphisms according to the duality established in [4]. This duality is based on the Priestley duality, in the same way as the Jónsson–Tarski duality for BAOs and descriptive general frames is based on the Stone duality. In the case of Jónsson–Tarski duality, any descriptive general frame is obtained by endowing a Stone space $\mathbf{X} = \langle X, \tau \rangle$ with a relation $R \subseteq X \times X$ such that $R[x] \in K(\mathbf{X})$ for every $x \in X$ (R is said to be *point closed*). This relation accounts for the modal operators, and all the other connectives are accounted for in the underlying Stone duality. In this case, $\mathbf{K}^+$-spaces, which are 'the descriptive general frames of PML', are essentially Priestley spaces endowed with a relation R that is *point closed-and-convex*. As in the Jónsson–Tarski case, R accounts for the modal operators, and all the other connectives are accounted for in the underlying Priestley duality.

Definition 15 ($\mathbf{K}^+$*-space, cf. Celani et al. [4, Definition 3.5]*). A $\mathbf{K}^+$*-space* is a structure $\mathcal{G}\langle X, \leqslant, R, \mathcal{A} \rangle$ such that $\leqslant$ is a partial order on X, $\mathcal{A}$ is a sublattice of $\langle \mathcal{P}_{\leqslant}(X), \cap, \cup, \emptyset, X \rangle$ and R is a binary relation on X such that the following conditions are satisfied:

D1. The space $\mathbf{X}_{\mathcal{G}} = \langle X, \leqslant, \tau_{\mathcal{A}} \rangle$, where $\tau_{\mathcal{A}}$ is the topology defined by taking $\{U \mid U \in \mathcal{A}\} \cup \{(X \backslash U) \mid U \in \mathcal{A}\}$ as a subbase, is a Priestley space such that $\mathcal{A}$ is the collection of the clopen increasing subsets of $\tau_{\mathcal{A}}$.
D2. $\mathcal{A}$ is closed under the operations $\Box_R$ and $\Diamond_R$.
D3. For every $x \in X$, $R[x]$ is a closed subset of $\mathbf{X}_{\mathcal{G}}$.
D4. For every $x \in X$, $R[x] = (R \circ \leqslant)[x] \cap (R \circ \geqslant)[x]$.

Condition D1 says that the algebra $\mathcal{A}$ and the topology $\tau_{\mathcal{A}}$ are easily recoverable from one another, so $\mathbf{K}^+$-spaces would be equivalently defined as Priestley spaces endowed with a relation satisfying conditions D3 and D4, and such that the algebra of the clopen increasing subsets is closed under $\Box_R$ and $\Diamond_R$.

Let us recall that for every $\mathbf{K}^+$-space $\mathcal{G}$, the collection of the closed and convex subsets of $\mathbf{X}_{\mathcal{G}}$ is

$$\begin{aligned}\mathbf{Kcv}(\mathbf{X}_{\mathcal{G}}) &= \{F \in K(\mathbf{X}_{\mathcal{G}}) \mid F U{\uparrow} \cap V{\downarrow} \text{ for some } U, V \in \mathcal{P}(X)\}\\ &= \{F \in K(\mathbf{X}_{\mathcal{G}}) \mid F = \bigcup_{x,y \in F} (x{\uparrow} \cap y{\downarrow})\}.\end{aligned}$$

Remark 16. Conditions D3 and D4 hold iff for every $x \in X$, $R[x] \in \mathbf{Kcv}(\mathbf{X}_{\mathcal{G}})$ (R is point closed-and-convex).

Lemma 17 (*cf. Celani et al. [4, Proposition 3.6]*). *For every* $\mathbf{K}^+$*-space* $\mathcal{G} = \langle X, \leqslant, R, \mathcal{A}\rangle$, *the frame* $\langle X, \leqslant, R\rangle$ *is a frame for Positive Modal Logic, i.e.*

$$(\leqslant \circ R) \subseteq (R \circ \leqslant) \text{ and } (\geqslant \circ R) \subseteq (R \circ \geqslant).$$

As a consequence of the lemma above, in every $\mathbf{K}^+$-space $\Box_R = \Box_{R_\Box}$ and $\Diamond_R = \Diamond_{R_\Diamond}$ (see discussion at the end of Section 2.1).

Definition 18 (*Morphism in* $\mathbf{K}^+$, *cf. Celani et al. [4, Definition 3.8]*). For all $\mathbf{K}^+$-spaces $\mathcal{G}_i = \langle X_i, \leqslant_i, R_i, \mathcal{A}_i\rangle$ $i = 1, 2$, a map $f : X_1 \longrightarrow X_2$ is a *bounded morphism* between $\mathcal{G}_1$ and $\mathcal{G}_2$ iff the following conditions are satisfied:

B1. f is order-preserving.
B2. For every $x, y \in X_1$, if $\langle x, y\rangle \in R_1$ then $\langle f(x), f(y)\rangle \in R_2$.
B3. If $\langle f(x), y'\rangle \in R_2$, then $f(z_1) \leqslant y' \leqslant f(z_2)$ for some $z_1, z_2 \in R_1[x]$.
B4. For every $U' \in \mathcal{A}_2$, $f^{-1}[U'] \in \mathcal{A}_1$.

Conditions B2 and B3 are the back-and-forth axioms of bounded morphisms in the case of PML.

Lemma 19. *Let* $\mathcal{G}_i = \langle X_i, \leqslant_i, R_i, \mathcal{A}_i\rangle$ $i = 1, 2$ *be* $\mathbf{K}^+$*-spaces. The following are equivalent for every map* $f : X_1 \longrightarrow X_2$:

(1) *f satisfies conditions* B1 *and* B4 *of Definition* 18.
(2) *f is a continuous and order preserving map between* $\mathbf{X}_{\mathcal{G}_1}$ *and* $\mathbf{X}_{\mathcal{G}_2}$.

Theorem 20 (*Celani et al. [4]*). *The category* **PMA** *of Positive Modal Algebras and their homomorphisms is dually equivalent to the category* $\mathbf{K}^+$ *of* $\mathbf{K}^+$*-spaces and their morphisms*.

3. The Vietoris endofunctor V on Pri

In this section, we are going to define an endofunctor **V** on the category of Priestley spaces, in such a way that the categories $\mathbf{K}^+$ and **Coalg**(**V**) will turn out to be isomorphic. Our starting points are the following facts: (a) For every Priestley space **X**, **U**(**X**) (see Corollary 8) is a Stone space, (b) For every Stone space **X**, **I**(**X**) (see Example 5) is a Priestley space, and (c) the Vietoris construction gives rise to the endofunctor **K** on Stone spaces.

For every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, $\mathbf{KU}(\mathbf{X}) = \langle K(X), \tau_V\rangle$ is a Stone space. So the question is whether we can endow $\mathbf{KU}(\mathbf{X})$ with a partial order $\leqslant^*$, in such a way that the resulting ordered space $\mathbf{K}^*(\mathbf{X}) = \langle K(X), \leqslant^*, \tau_V\rangle$ is a Priestley space, and for every $\mathbf{X} \in \mathbf{Pri}$, $\mathbf{Y} \in \mathbf{St}$,

$$\mathbf{UK}^*(\mathbf{X}) = \mathbf{KU}(\mathbf{X}) \quad \text{and} \quad \mathbf{K}^*\mathbf{I}(\mathbf{Y})\mathbf{IK}(\mathbf{Y}).$$

Our candidate for $\leqslant^*$ is the *Egli–Milner power order* $\leqslant^{\mathrm{EM}}$ [3,27]. We will see that this order does not meet all the requirements, i.e. for every Priestley space $\langle X, \leqslant, \tau\rangle$, the space $\langle K(X), \leqslant^{\mathrm{EM}}, \tau_V\rangle$ is not in general a Priestley space. The condition that fails is the antisymmetry of $\leqslant^{\mathrm{EM}}$ (see Example 28 below). However, this is the first step of the construction we are going to present. The Vietoris space endowed with $\leqslant^{\mathrm{EM}}$ is an instance of a more general construction called the *Vietoris power space* (cf. [3, Definition 2.36]).

3.1. The Egli–Milner power order

Definition 21 (*The Egli–Milner power order) (cf. Brink and Rewitzky [3, Definition 2.30]*). For every set X and every preorder $\leqslant$ on X, the *Egli–Milner power order of* $\leqslant$ is the relation $\leqslant^{\mathrm{EM}} \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: For every $Y, Z \subseteq X$,

$$Y \leqslant^{\mathrm{EM}} Z \quad \text{iff} \quad Y \subseteq Z{\downarrow} \text{ and } Z \subseteq Y{\uparrow}.$$

Clearly, if $\leqslant$ is the identity relation on X, then $\leqslant^{\mathrm{EM}}$ is the identity relation on $\mathcal{P}(X)$. The next two lemmas show that the Egli–Milner power order behaves well w.r.t. the order-preserving maps and w.r.t. the binary relations that satisfy the defining conditions of PML-frames (see Section 2.1):

Lemma 22. *For every order-preserving map $f : \langle X_1, \leqslant_1\rangle \longrightarrow \langle X_2, \leqslant_2\rangle$ between partial orders and every $Z, W \subseteq X$, if $Z \leqslant_1^{\mathrm{EM}} W$ then $f[Z] \leqslant_2^{\mathrm{EM}} f[W]$.*

Lemma 23. *For every partial order $\langle X, \leqslant\rangle$ and every binary relation R on X, the following are equivalent*:
(1) *For every $x, y \in X$, if $x \leqslant y$ then $R[x] \leqslant^{\mathrm{EM}} R[y]$.*
(2) $(\leqslant \circ R) \subseteq (R \circ \leqslant)$ *and* $(\geqslant \circ R) \subseteq (R \circ \geqslant)$.

3.2. The Vietoris power space

Definition 24 ($\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$). For every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, the *Vietoris power space of* $\mathbf{X}$ is the ordered space $\mathbf{K}^{\mathrm{EM}}(\mathbf{X}) = \langle K(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau_V\rangle$, where $\leqslant^{\mathrm{EM}}$ is the restriction of the Egli–Milner power order to $K(\mathbf{X}) \times K(\mathbf{X})$.

As $\leqslant^{\mathrm{EM}}$ is the identity relation on $K(\mathbf{X})$ whenever $\leqslant$ is the identity relation on $\mathbf{X}$, then $\mathbf{K}^{\mathrm{EM}}\mathbf{I}(\mathbf{Y})\mathbf{IK}(\mathbf{Y})$ for every $\mathbf{Y} \in \mathbf{St}$, which is one of the conditions we mentioned in the discussion at the beginning of Section 3.

Lemma 25. *For every ordered topological space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$ and every $A \in \tau$, if A is $\leqslant$-increasing, then $m(A)$ and $t(A)$ are $\leqslant^{\mathrm{EM}}$-increasing.*

The most important property of the Egli–Milner power order $\leqslant^{\mathrm{EM}}$ is stated in the item 2 of the next Lemma:

Lemma 26. *For every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$,*
(1) *for every $F, G \in K(\mathbf{X})$, if $F \not\leqslant^{\mathrm{EM}} G$, then there exists a clopen increasing $U \subseteq X$ such that either $F \in m(U)$ and $G \notin m(U)$, or $F \in t(U)$ and $G \notin t(U)$.*
(2) $\leqslant^{\mathrm{EM}}$ *is a closed subset of $K(\mathbf{X}) \times K(\mathbf{X})$ with the product topology.*

Proof. 1. If $F \not\leqslant^{\mathrm{EM}} G$, then either (a) there exists $z \in F$ such that for every $w \in G$ $z \not\leqslant w$, or (b) there exists $w \in G$ such that for every $z \in F$ $z \not\leqslant w$.

If (a), then, as **X** is totally order-disconnected, for every $w \in G$ there exists a clopen increasing $U_w \subseteq X$ such that $z \in U_w$ and $w \notin U_w$. Therefore $G \subseteq \bigcup_{w \in G}(X \setminus U_w)$, i.e. the subsets $(X \setminus U_w)$ form an open covering of G, and as G is compact (for it is a closed subset of the compact space **X**), then $G \subseteq \bigcup_{i=1}^{n}(X \setminus U_{w_i})$ for some $w_1, \ldots, w_n \in G$. Let $U = \bigcap_{i=1}^{n} U_{w_i}$. U is clopen increasing, $z \in F \cap U$ and $G \cap U = \emptyset$, hence $F \in m(U)$ and $G \notin m(U)$.

If (b), then, as **X** is totally order-disconnected, for every $z \in F$ there exists a clopen increasing $U_z \subseteq X$ such that $z \in U_z$ and $w \notin U_z$. Therefore $F \subseteq \bigcup_{z \in F} U_z$, i.e. the subsets U_z form an open covering of F, and as F is compact, then $F \subseteq \bigcup_{i=1}^{n} U_{z_i}$ for some $z_1, \ldots, z_n \in F$. Let $U = \bigcup_{i=1}^{n} U_{z_i}$. U is clopen increasing, $F \subseteq U$ and $w \in (G \setminus U)$, hence $F \in t(U)$ and $G \notin t(U)$.

2. Let $\langle F, G \rangle \notin \ \leqslant^{\mathrm{EM}}$. We have to show that $\langle F, G \rangle \in \mathcal{U}$ and $\mathcal{U} \cap \leqslant^{\mathrm{EM}} = \emptyset$ for some open subset $\mathcal{U} \in K(\mathbf{X}) \times K(\mathbf{X})$. As $F \not\leqslant^{\mathrm{EM}} G$, then by item (1) of this lemma, there exists a clopen increasing $U \subseteq X$ such that either (a) $F \in t(U)$ and $G \notin t(U)$, or (b) $F \in m(U)$ and $G \notin m(U)$.

If (a), then take $\mathcal{U} = t(U) \times (K(\mathbf{X}) \setminus t(U))$. $\langle F, G \rangle \in \mathcal{U}$. Let us show that if $\langle F', G' \rangle \in \mathcal{U}$, then $F' \not\leqslant^{\mathrm{EM}} G'$. As $\langle F', G' \rangle \in \mathcal{U}$, then $F' \in t(U)$, i.e. $F' \subseteq U$, and $G' \notin t(U)$, i.e. $G' \nsubseteq U$, hence there exists $w \in (G' \setminus U)$. Let us show that $z \not\leqslant w$ for every $z \in F'$: if $z \in F' \subseteq U$ and $z \leqslant w$, then, as U is increasing, $w \in U$, contradiction. Therefore $F' \not\leqslant^{\mathrm{EM}} G'$.

If (b), then take $\mathcal{U} = m(U) \times (K(\mathbf{X}) \setminus m(U))$. □

Corollary 27. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau \rangle$, $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$ *is totally order-disconnected, and the collection* $\{m(U), t(U) \mid U \subseteq X \textit{ clopen}, \ U \textit{ increasing or decreasing}\}$ *is a subbase of* τ_V.

Proof. The total order-disconnectedness immediately follows from item 1 of the Lemma 26, and from the fact that if $U \subseteq X$ is clopen increasing, then $m(U)$ and $t(U)$ are clopen increasing subsets of $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$ (see Lemmas 13 and 25). The second part of the statement immediately follows from item 1 of Lemma 26 and from Lemma 7. □

If $\leqslant$ is a preorder on a set X, then $\leqslant^{\mathrm{EM}}$ is a preorder on $\mathcal{P}(X)$, however, if $\leqslant$ is a partial order, then $\leqslant^{\mathrm{EM}}$ might not be a partial order: The following is an example of a Priestley space **X** such that $\leqslant^{\mathrm{EM}}$ is not antisymmetric on $K(\mathbf{X})$.

Example 28. Let us consider a four element chain $0 < a < b < 1$, which is a finite (distributive) lattice. By Example 4, this chain is a Priestley space if it is endowed with the discrete topology. The subsets $F = \{0, a, 1\}$ and $G = \{0, b, 1\}$ are distinct closed subsets which share the maximum and the minimum, and so $F \leqslant^{\mathrm{EM}} G$ and $G \leqslant^{\mathrm{EM}} F$.

Therefore $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$ is not in general a Priestley space for every Priestley space **X**, and the only condition that fails is the antisymmetry of $\leqslant^{\mathrm{EM}}$. For every preorder $\langle X, \leqslant \rangle$, we can consider the equivalence relation $\equiv \subseteq \mathcal{P}(X) \times \mathcal{P}(X)$ defined as follows: For every

$Y, Z \subseteq X$,

$$Y \equiv Z \quad \text{iff } Y \leqslant^{\mathrm{EM}} Z \text{ and } Z \leqslant^{\mathrm{EM}} Y.$$

The Vietoris endofunctor **V** on **Pri** will associate every Priestley space **X** with the $\equiv$-quotient space of $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.

3.3. The action of **V** *on the objects of* **Pri**

Definition 29 ($\mathbf{V}(\mathbf{X})$). For every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, $\mathbf{V}(\mathbf{X}) = \langle K(\mathbf{X})_{\equiv}, \leqslant^{\mathrm{EM}}_{\equiv}, \tau_{V\equiv}\rangle$, where:

$$K(\mathbf{X})_{\equiv} = \{[F] \mid F \in K(\mathbf{X})\},$$

where for every $F \in K(\mathbf{X})$, $[F] = \{G \in K(\mathbf{X}) \mid F \equiv G\}$.

For every $[F], [G] \in K(\mathbf{X})_{\equiv}$,

$$[F] \leqslant^{\mathrm{EM}}_{\equiv} [G] \quad \text{iff} \quad F' \leqslant^{\mathrm{EM}} G' \text{ for some } F' \in [F] \text{ and } G' \in [G].$$

$$\tau_{V\equiv} = \{\mathcal{X} \subseteq K(\mathbf{X})_{\equiv} \mid \pi^{-1}[\mathcal{X}] \in \tau_V\},$$

where $\pi : K(\mathbf{X}) \longrightarrow K(\mathbf{X})_{\equiv}$ is the canonical projection.

Item (3) of the next lemma is the main ingredient in the proof of the total order-disconnectedness of $\mathbf{V}(\mathbf{X})$ (Lemma 31), and it is a consequence of the fact that $\leqslant^{\mathrm{EM}}$ is a closed subset of $K(\mathbf{X}) \times K(\mathbf{X})$ with the product topology (see Lemma 26).

Lemma 30. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$,
(1) *for every* $F, G \in K(\mathbf{X})$, $[F] \leqslant^{\mathrm{EM}}_{\equiv} [G]$ *iff* $F \leqslant^{\mathrm{EM}} G$, *hence* $\leqslant^{\mathrm{EM}}_{\equiv}$ *is a partial order.*
(2) *The canonical projection* $\pi : \mathbf{K}^{\mathrm{EM}}(\mathbf{X}) \longrightarrow \mathbf{V}(\mathbf{X})$ *is a continuous and order-preserving map.*
(3) *For every* $F \in K(\mathbf{X})$, $[F]$ *is a closed subset of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.
(4) *For every U clopen increasing or clopen decreasing subset of X,*

$$\pi^{-1}[\pi[t(U)]] = t(U) \text{ and } \pi^{-1}[\pi[m(U)]] = m(U),$$

hence $\pi[t(U)]$ *and* $\pi[m(U)]$ *are clopen increasing subsets of* $\mathbf{V}(\mathbf{X})$.
(5) *If* $U_i, V_j \subseteq X$ *are clopen increasing* $i = 1, \ldots, n$ *and* $j = 1, \ldots, m$ *and* $A = (\bigcap_{i=1}^{n} m(U_i)) \cap (\bigcap_{j=1}^{m} t(V_j))$, *then* $\pi^{-1}[\pi[A]] = A$, *hence* $\pi[A]$ *is a clopen increasing subset of* $\mathbf{V}(\mathbf{X})$.

For every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, let us denote

$$\mathcal{B}_{\mathbf{X}} = \left\{ \pi\left[\left(\bigcap_{i=1}^{n} m(U_i)\right) \cap \left(\bigcap_{j=1}^{m} t(V_j)\right)\right] \,\middle|\, U_i, V_j \subseteq X \text{ clopen increasing} \right\}.$$

Lemma 31. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$,
(1) *for every* $[F], [G] \in K(\mathbf{X})_{\equiv}$, *if* $[F] \not\leqslant^{\mathrm{EM}}_{\equiv} [G]$, *then* $[F] \in B$ *and* $[G] \notin B$ *for some* $B \in \mathcal{B}_{\mathbf{X}}$.

(2) $\mathcal{B}_{\mathbf{X}} \cup \{(K(\mathbf{X})_{\equiv} \backslash \mathcal{U}) \mid \mathcal{U} \in \mathcal{B}_{\mathbf{X}}\}$ *is a subbase of the topology of* $\mathbf{V}(\mathbf{X})$.
(3) $\mathbf{V}(\mathbf{X})$ *is totally order-disconnected.*

Proposition 32. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau \rangle$, $\mathbf{V}(\mathbf{X})$ *is a Priestley space.*

Proof. The relation $\leqslant^{EM}_{\equiv}$ is a partial order (item (1) of Lemma 30). As $\mathbf{X}$ is compact, then $\mathbf{K}(\mathbf{X})\langle K(\mathbf{X}), \tau_V \rangle$ is compact, so $\mathbf{V}(\mathbf{X})$ is compact, for it is the quotient space of a compact space, moreover $\mathbf{V}(\mathbf{X})$ is totally order-disconnected (item (3) of Lemma 31). □

3.4. *The action of* V *on the morphisms of* Pri

Definition 33 ($\mathbf{V}(f)$). Let $\mathbf{X}_i = \langle X_i, \leqslant_i, \tau_i \rangle$ be Priestley spaces, $i = 1, 2$. For every continuous and order-preserving map $f : X_1 \longrightarrow X_2$, the map $\mathbf{V}(f) : K(\mathbf{X}_1)_{\equiv_1} \longrightarrow K(\mathbf{X}_2)_{\equiv_2}$ is given by the assignment $[F] \mapsto [f[F]]$ for every $F \in K(\mathbf{X}_1)$.

Some technical facts are listed in the following lemma, which are used in the proof of Proposition 35. All the omitted details can be found in [24].

Lemma 34. *For every continuous and order-preserving map* $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ *of Priestley spaces, and for every U clopen increasing subset of* X_2, *if* $\pi : K(\mathbf{X}_1) \longrightarrow K(\mathbf{X}_1)_{\equiv_1}$ *is the canonical projection, then*
(1) $\mathbf{V}(f)^{-1}[\pi[m(U)]] = \pi[\mathbf{K}(f)^{-1}[m(U)]]$.
(2) $\pi^{-1}[\pi[\mathbf{K}(f)^{-1}[m(U)]]] = \mathbf{K}(f)^{-1}[m(U)]$, *hence* $\pi[\mathbf{K}(f)^{-1}[[m(U)]] \subseteq \mathbf{V}(\mathbf{X}_2)$ *is clopen.*
(3) $\mathbf{V}(f)^{-1}[\pi[t(U)]] = \pi[\mathbf{K}(f)^{-1}[t(U)]]$.
(4) $\pi^{-1}[\pi[\mathbf{K}(f)^{-1}[t(U)]]] = \mathbf{K}(f)^{-1}[t(U)]$, *hence* $\pi[\mathbf{K}(f)^{-1}[t(U)]] \subseteq \mathbf{V}(\mathbf{X}_2)$ *is clopen.*

Proposition 35. *Let* $\mathbf{X}_i = \langle X_i, \leqslant_i, \tau_i \rangle$ *be Priestley spaces,* $i = 1, 2$. *For every continuous and order-preserving map* $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$, *the map* $\mathbf{V}(f) : K(\mathbf{X}_1)_{\equiv_1} \longrightarrow K(\mathbf{X}_2)_{\equiv_2}$, *given by the assignment* $[F] \mapsto [f[F]]$ *for every* $F \in K(\mathbf{X}_1)$, *is continuous and order-preserving.*

4. The equivalence between $\mathbf{K}^+$ and Coalg(V)

4.1. *From* $\mathbf{K}^+$ *to* Coalg(V)

Let $\mathcal{G} = \langle X, \leqslant, R, \mathcal{A} \rangle$ be a $\mathbf{K}^+$-space, so the space $\mathbf{X}_{\mathcal{G}}$ associated with $\mathcal{G}$ is a Priestley space by definition. Then we can consider the following map:

$$\begin{aligned} \rho_{\mathcal{G}} : \mathbf{X}_{\mathcal{G}} &\longrightarrow K(\mathbf{X}_{\mathcal{G}})_{\equiv} \\ x &\longmapsto \pi(R[x]). \end{aligned}$$

As $\mathcal{G}$ is a $\mathbf{K}^+$-space, then $R[x] \in K(\mathbf{X})$ for every $x \in X$, so $\rho_{\mathcal{G}}$ is of the right type.

Lemma 36. *For every* $\mathbf{K}^{+}$*-space* $\mathcal{G} = \langle X, \leqslant, R, \mathcal{A}\rangle$ *and every clopen increasing subset* $U \in \tau_{\mathcal{A}}$,

$$\rho_{\mathcal{G}}^{-1}[\pi[t(U)]]\Box_R(U) \text{ and } \rho_{\mathcal{G}}^{-1}[\pi[m(U)]] = \Diamond_R(U).$$

Proposition 37. *For every* $\mathbf{K}^{+}$*-space* $\mathcal{G} = \langle X, \leqslant, R, \mathcal{A}\rangle$ *the map* $\rho_{\mathcal{G}}$ *is a continuous and order-preserving map between Priestley spaces.*

Proof. Let us show that $\rho_{\mathcal{G}}$ is order preserving, so assume that $x \leqslant y$. As $\mathcal{G}$ is a $\mathbf{K}^{+}$-space, then by Lemma 17 $(\leqslant \circ R) \subseteq (R \circ \leqslant)$ and $(\geqslant \circ R) \subseteq (R \circ \geqslant)$, hence by Lemma 23, $R[x] \leqslant^{\mathrm{EM}} R[y]$, and as π is order-preserving (see item 2 of Lemma 30), then $\rho_{\mathcal{G}}(x) = \pi(R[x]) \leqslant^{\mathrm{EM}}_{\equiv} \pi(R[y]) = \rho_{\mathcal{G}}(y)$.

In order to show that $\rho_{\mathcal{G}}$ is continuous, by item 2 of Lemma 31 it is sufficient to show that for every $B \in \mathcal{B}_V$, $\rho_{\mathcal{G}}^{-1}[B]$ is a clopen subset of $\mathbf{X}_{\mathcal{G}}$. If $B \in \mathcal{B}_V$, then $B = \pi[(\bigcap_{i=1}^{n} m(U_i)) \cap (\bigcap_{j=1}^{m} t(V_j))]$ for some $U_i, V_j \subseteq X$ clopen increasing. Then using Lemma 36, one can see that $\rho_{\mathcal{G}}^{-1}[B] = (\bigcap_{i=1}^{n} \Diamond_R(U_i)) \cap (\bigcap_{j=1}^{m} \Box_R(V_j))$. As $U_i, V_j \subseteq X$ clopen increasing and $\mathcal{G}$ is a $\mathbf{K}^{+}$-space, then the collection of clopen increasing subsets of $\mathbf{X}_{\mathcal{G}}$ coincides with $\mathcal{A}$, and $\mathcal{A}$ is closed under $\Box_R$ and $\Diamond_R$, hence $\Diamond_R(U_i)$ and $\Box_R(V_j)$ are clopen increasing, and so $\rho_{\mathcal{G}}^{-1}[B]$ is clopen. $\Box$

Proposition 38. *For every bounded morphism of* $\mathbf{K}^{+}$*-spaces* $f : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, *f is a* $\mathbf{V}$*-coalgebra morphism between* $\rho_{\mathcal{G}_1}$ *and* $\rho_{\mathcal{G}_2}$.

Proof. By Lemma 19, $f : \mathbf{X}_{\mathcal{G}_1} \longrightarrow \mathbf{X}_{\mathcal{G}_2}$ is continuous and order preserving, and B2 and B3 imply the commutativity of the diagram. $\Box$

4.2. *The Egli–Milner order on convex subsets*

In order to establish the converse direction of the equivalence, we will rely on the remarks listed in the following lemma, which say that for every Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$, the points in $\mathbf{V}(\mathbf{X})$ (i.e. the $\equiv$-equivalence classes of closed subsets of $\mathbf{X}$) are in one-to-one correspondence with the closed and convex subsets of $\mathbf{X}$, and that this correspondence is canonical, because each closed and convex subset is the greatest element of its equivalence class.

Lemma 39. *For every Priestley space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$,

(1) *the restriction of* $\leqslant^{\mathrm{EM}}$ *to* $(\mathbf{Kcv}(\mathbf{X}) \times \mathbf{Kcv}(\mathbf{X}))$ *is antisymmetric, hence if* $F, F' \in \mathbf{Kcv}(\mathbf{X})$ *and* $F \equiv F'$, *then* $F = F'$.
(2) *For every* $F \in K(\mathbf{X})$, $F^{+} = \bigcup_{x,y \in F}(x\uparrow \cap\, y\downarrow) \in \mathbf{Kcv}(\mathbf{X})$ *and* $F \equiv F^{+}$.
(3) *For every* $F \in K(\mathbf{X})$, *there exists a unique* $F' \in \mathbf{Kcv}(\mathbf{X})$ *such that* $F \equiv F'$.
(4) *For every* $F \in \mathbf{Kcv}(\mathbf{X})$, $G \subseteq F$ *for every* $G \in [F]$.

4.3. *From* **Coalg**(**V**) *to* $\mathbf{K}^+$

Let $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$ be a **V**-coalgebra, so $\mathbf{X} = \langle X, \leqslant, \tau \rangle$ is a Priestley space, and the collection $\mathcal{A}_\tau$ of the clopen increasing subsets of τ is a sublattice of $\langle \mathcal{P}_\leqslant (X), \cap, \cup, \emptyset, X \rangle$. So far we have three of the four ingredients of a $\mathbf{K}^+$-space, namely the carrier X, the order $\leqslant$, and the algebra $\mathcal{A}_\tau$. Now we have to use the coalgebra map ρ in order to define a relation R_ρ on X that satisfies conditions D3 and D4 of Definition 15, i.e. such that, for every $x \in X$, $R_\rho[x]$ is a closed and convex subset of **X** (see Remark 16). By definition of **V**, it holds that for every $x \in X$, $\rho(x) \in K(\mathbf{X})_\equiv$, i.e.

$$\rho(x) = \pi(F) = [F] = \{G \in K(\mathbf{X}) \mid G \equiv F\}$$

for some $F \in K(\mathbf{X})$. By item 3 of Lemma 39, there exists a unique closed and convex subset F^+ such that $F^+ \in [F] = \rho(x)$. Let us define $R_\rho \subseteq X \times X$ by putting $R_\rho[x] = F^+$ for every $x \in X$.

Then we can associate ρ with $\mathcal{G}_\rho = \langle X, \leqslant, R_\rho, \mathcal{A}_\tau \rangle$.

Lemma 40. *For every* **V**-*coalgebra* $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$,
(1) *for every* $x \in X$, $\rho(x) = [R_\rho[x]]$.
(2) *For every open increasing* $U \subseteq \mathbf{X}$, $\Box_{R_\rho}(U)\rho^{-1}[\pi[t(U)]]$.
(3) *For every open* $U \subseteq \mathbf{X}$, $\Diamond_{R_\rho}(U) = \rho^{-1}[\pi[m(U)]]$.

Proposition 41. *For every* **V**-*coalgebra* $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$, $\mathcal{G}_\rho = \langle X, \leqslant, R_\rho, \mathcal{A}_\tau \rangle$ *is a* $\mathbf{K}^+$-*space*.

Proof. By construction, $\mathcal{A}_\tau$ is a sublattice of $\langle \mathcal{P}_\leqslant (X), \cap, \cup, \emptyset, X \rangle$, and for every $x \in X$, $R_\rho[x] \in \mathbf{Kcv}(\mathbf{X}_\mathcal{G})$, which implies, by Remark 16, that R_ρ verifies conditions D3 and D4 of Definition 15. So the only thing we have to show is that $\mathcal{A}_\tau$ is closed under $\Box_{R_\rho}$ and $\Diamond_{R_\rho}$, i.e. that for every clopen increasing $U \subseteq \mathbf{X}$, $\Box_{R_\rho}(U)$ and $\Diamond_{R_\rho}(U)$ are clopen increasing. By items (2) and (3) of Lemma 40, $\Box_{R_\rho}(U)\rho^{-1}[\pi[t(U)]]$, and $\Diamond_{R_\rho}(U)\rho^{-1}[\pi[m(U)]]$. As ρ is a **V**-coalgebra, then ρ is a continuous and order-preserving map, and as, by item (4) of Lemma 30, $\pi[t(U)]$ and $\pi[m(U)]$ are clopen increasing subsets of $\mathbf{V}(\mathbf{X})$, then $\rho^{-1}[\pi[t(U)]]$ and $\rho^{-1}[\pi[m(U)]]$ are clopen increasing subsets of **X**. $\Box$

Proposition 42. *For every* **V**-*coalgebra morphism* $f : \rho_1 \longrightarrow \rho_2$, f *is a bounded morphism between* $\mathcal{G}_{\rho_1}$ *and* $\mathcal{G}_{\rho_2}$.

Proof. Let $\rho_i : \mathbf{X}_i \longrightarrow \mathbf{V}(\mathbf{X}_i)$, $i = 1, 2$. By assumption, $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$ is a continuous and order-preserving map, such that the following diagram commutes:

$$\begin{tikzcd} \mathbf{X}_1 \arrow[r, "f"] \arrow[d, "\rho_1"'] & \mathbf{X}_2 \arrow[d, "\rho_2"'] \\ \mathbf{V}(\mathbf{X}_1) \arrow[r, "\mathbf{V}(f)"] & \mathbf{V}(\mathbf{X}_2). \end{tikzcd}$$

Let $\mathcal{G}_{\rho_i} = \langle X_i, \leqslant_i, R_{\rho_i}, \mathcal{A}_i \rangle$, $i = 1, 2$. By Lemma 19, f satisfies conditions B1 and B4.

For every $x \in X_i$, $\rho_i(x) = [R_{\rho_i}[x]]$, $i = 1, 2$, so the commutativity of the diagram implies that $[R_{\rho_2}[f(x)]] = \rho_2(f(x)) = \mathbf{V}(f)(\rho_1(x))[f[R_{\rho_1}[x]]]$, hence $R_{\rho_2}[f(x)] \equiv_2 f[R_{\rho_1}[x]]$. Let us show B3: If $y' \in R_{\rho_2}[f(x)]$, then, as $R_{\rho_2}[f(x)] \leqslant_2^{\mathrm{EM}} f[R_{\rho_1}[x]]$, there exist $z_1, z_2 \in R_{\rho_1}[x]$ such that $f(z_1) \leqslant_2 y' \leqslant_2 f(z_2)$. Finally, let us show B2: As $R_{\rho_2}[f(x)] \in \mathbf{Kcv}(\mathbf{X}_2)$ and $f[R_{\rho_1}[x]] \equiv_2 R_{\rho_2}[f(x)]$, then by item (4) of Lemma 39, $f[R_{\rho_1}[x]] \subseteq R_{\rho_2}[f(x)]$. Hence, if $y \in R_{\rho_1}[x]$, then $f(y) \in f[R_{\rho_1}[x]] \subseteq R_{\rho_2}[f(x)]$, and so $f(x) R_{\rho_2} f(y)$. $\square$

4.4. *Isomorphism of categories*

Proposition 43. *For every $\mathbf{K}^+$-space $\mathcal{G}$ and every $\mathbf{V}$-coalgebra ρ, $\mathcal{G}_{\rho_{\mathcal{G}}} = \mathcal{G}$ and $\rho_{\mathcal{G}_\rho} = \rho$.*

Proof. If $\mathcal{G} = \langle X, \leqslant, R, \mathcal{A}\rangle$, then by spelling out the definitions involved, we have that $\mathcal{G}_{\rho_{\mathcal{G}}} = \langle X, \leqslant, R_{\rho_{\mathcal{G}}}, \mathcal{A}\rangle$, and for every $x \in X$ $R_{\rho_{\mathcal{G}}}[x] \in \rho_{\mathcal{G}}(x) = [R[x]]$, hence $R_{\rho_{\mathcal{G}}}[x] \equiv R[x]$, and since both sets are closed and convex, then by item (1) of Lemma 39 $R_{\rho_{\mathcal{G}}}[x] = R[x]$.

If $\rho : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$, then by spelling out the definitions involved we have that $\mathbf{X}_{\mathcal{G}_\rho} = \mathbf{X}$, hence $\rho_{\mathcal{G}_\rho} : \mathbf{X} \longrightarrow \mathbf{V}(\mathbf{X})$, and for every $x \in X$ $\rho_{\mathcal{G}_\rho}(x) = [R_\rho[x]] = \rho(x)$. $\square$

The results of Sections 4.1 and 4.3 and the proposition above yield:

Theorem 44. *The category $\mathbf{K}^+$ of $\mathbf{K}^+$-space and their bounded morphisms is isomorphic to the category $\mathbf{Coalg}(\mathbf{V})$ of the coalgebras for the Vietoris endofunctor on Priestley spaces.*

5. A remark on intuitionistic propositional logic

Intuitionistic propositional logic is a paradigmatic example of an algebraizable logic, and Heyting algebras (see Definition 45 below) form its associated class of algebras. Heyting algebras and their homomorphisms form a category $\mathbf{H}$, that is dually equivalent [11] to the category $\mathbf{E}$ of *Esakia spaces* and continuous and *strongly isotone* maps (see Definitions 46 and 47 below). From these definitions, one immediately sees that the objects of $\mathbf{E}$ are ordered Stone spaces $\langle X, \leqslant, \tau\rangle$ such that the assignment $x \mapsto x\!\uparrow$ defines a coalgebra of the Vietoris endofunctor $\mathbf{K}$ on Stone spaces (see Section 2.4), and the arrows of $\mathbf{E}$ are the corresponding coalgebra morphisms. In other words, $\mathbf{E}$ is isomorphic to the full subcategory of $\mathbf{Coalg}(\mathbf{K})$ whose objects are those coalgebras ρ such that the associated relation R_ρ, defined as $x R_\rho y$ iff $y \in \rho(x)$, is a partial order.

The category $\mathbf{E}$ can be also characterized as a subcategory of Priestley spaces (see Proposition 51 below), and actually the duality between $\mathbf{H}$ and $\mathbf{E}$ can be obtained as the restricted Priestley duality (see [7] for details). So a natural question that can be asked is whether for every space in $\mathbf{E}$ the assignment $x \mapsto \pi(x\!\uparrow)$ defines a coalgebra of the endofunctor $\mathbf{V}$ on Priestley spaces, so that $\mathbf{E}$ can be also characterized as a subcategory of $\mathbf{Coalg}(\mathbf{V})$. We will give a negative answer to this question.

5.1. Heyting algebras and Esakia spaces

Definition 45 (*Heyting algebra*). An algebra $\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a *Heyting algebra* iff $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice and $\rightarrow$ is *the relative pseudocomplementation of* $\wedge$, i.e. it is a binary operation such that for every $a, b, c \in A$, $(a \wedge c) \leqslant b$ iff $c \leqslant (a \rightarrow b)$.

Definition 46 (*Esakia space, cf. Esakia [11, Definition 1]*). An *Esakia space* $\mathbf{X}=\langle X, \leqslant, \tau\rangle$ is an ordered Stone space (i.e. $\langle X, \tau\rangle$ is a Stone space, and $\leqslant$ is a partial order on X) such that the assignment $x \mapsto x\uparrow$ defines a continuous map $\rho : \langle X, \tau\rangle \rightarrow \langle K(\mathbf{X}), \tau_V\rangle$.

Definition 47 (*Strongly isotone map, cf. Esakia [11, Definition 2]*). Let $\langle X, \leqslant\rangle$ and $\langle Y, \leqslant'\rangle$ be pre-ordered sets. A map $f : X \rightarrow Y$ is *strongly isotone* iff

$$\forall x \in X\ \forall y \in Y(f(x) \leqslant' y \ \Leftrightarrow\ \exists x' \in X(x \leqslant x' \ \&\ f(x') = y)).$$

Clearly, if f is strongly isotone then it is monotone. It is easy to see that the composition of strongly isotone maps is strongly isotone, so $\mathbf{E}$ is indeed a category. A strongly isotone map can be thought of as a bounded morphism between Kripke frames such that the relations are preorders, which in turn, as it is well known, can be seen as coalgebra morphisms between the associated $\mathcal{P}$-coalgebras. This is the content of the next lemma, which provides the connection with the coalgebraic presentation of $\mathbf{E}$ when topology is added to the picture:

Lemma 48. *Let* $\mathbf{X}_i = \langle X_i, \leqslant_i\rangle$ *be preorders,* $i = 1, 2$. *The following are equivalent for every map* $f : X_1 \longrightarrow X_2$:
(1) *f is strongly isotone.*
(2) $f[Y\uparrow]f[Y]\uparrow$ *for every* $Y \subseteq X_1$.
(3) *f is a morphism between the $\mathcal{P}$-coalgebras ρ_i associated with* $\mathbf{X}_i$.

As Heyting algebras are particular bounded distributive lattices, the duality stated in the following theorem can be obtained as a restricted Priestley duality, although this is not the proof strategy adopted by Esakia [11]. See [7] for a discussion and a detailed proof.

Theorem 49 (*cf. Esakia [11, Theorem 3]*). *The category of Esakia spaces and strongly isotone and continuous maps is dually equivalent to the category of Heyting algebras and their homomorphisms.*

Lemma 50. *For every ordered space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$ *such that* $x\uparrow \in K(\mathbf{X})$ *for every* $x \in X$ *and every open subset* A, $\Diamond_{\leqslant}(A) = A\downarrow = \rho^{-1}[m(A)]$, *where* $\rho(x) = x\uparrow$ *for every* $x \in X$.

The next proposition is considered folklore, however, its proof can now be found in [7,24].

Proposition 51. *The following are equivalent for every ordered topological space* $\mathbf{X} = \langle X, \leqslant, \tau\rangle$:
(1) $\mathbf{X}$ *is an Esakia space.*
(2) $\mathbf{X}$ *is a Priestley space such that for every clopen subset U of* $\mathbf{X}$, $U\downarrow$ *is clopen.*

The next Proposition characterizes those Priestley spaces that can be seen as **V**-coalgebras in a natural way (see item (3) in particular):

Proposition 52. *The following are equivalent for every ordered topological space* $\mathbf{X} = \langle X, \leqslant, \tau \rangle$:

(1) $\mathbf{X}$ *is a Priestley space such that for every clopen increasing subset* U, $U\!\downarrow$ *is clopen increasing*.

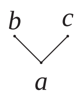
(2) *The general frame* $\mathcal{G}_{\mathbf{X}} = \langle X, \leqslant, \leqslant, \mathcal{A}_\tau \rangle$, *where* $\mathcal{A}_\tau$ *is the algebra of the clopen increasing subsets of* $\mathbf{X}$, *is a* $\mathbf{K}^+$-*space*.

(3) $\mathbf{X}$ *is a Priestley space such that the map* $\rho : \mathbf{X} \to \mathbf{V}(\mathbf{X})$ *given by* $\rho(x) = \pi[x\!\uparrow]$ *is a* $\mathbf{V}$-*coalgebra*.

(4) $\mathbf{X}$ *is a Priestley space such that the map* $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau_V \rangle$ *given by* $\rho'(x) = x\!\uparrow$ *is continuous and order-preserving*.

Now we are in a position to give negative answer to the question that we posed in the discussion at the beginning of this section. Clearly, if a space $\mathbf{X}$ satisfies condition (4) (and therefore any of the conditions) of the proposition above, then it is an Esakia space. On the other hand, the equivalence between items (3) and (4) of the Proposition above implies that not for every Esakia space $\mathbf{X}$ the map $\rho : \mathbf{X} \to \mathbf{V}(\mathbf{X})$ given by $\rho(x) = \pi[x\!\uparrow]$ is a $\mathbf{V}$-coalgebra, because the map $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau_V \rangle$ given by $\rho'(x) = x\!\uparrow$ might not be order-preserving:

Example 53. Let us consider the space $\mathbf{X} = \langle X, \leqslant, \tau \rangle$, where $X = \{a, b, c\}$, τ is the discrete topology, and $\leqslant$ is the partial order associated with the following Hasse diagram:



It is easy to see that $\mathbf{X}$ is a Priestley space such that for every clopen subset U, $U\!\downarrow$ is clopen, and so $\mathbf{X}$ is an Esakia space. By Lemma 23, the map $\rho' : \mathbf{X} \to \langle K(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau_V \rangle$ given by $\rho'(x) = x\!\uparrow$ is order-preserving iff $(\leqslant \circ \geqslant) \subseteq (\geqslant \circ \leqslant)$, i.e. for every $x, y \in X$ such that $z \leqslant x$ and $z \leqslant y$ for some $z \in X$, there exists $z' \in X$ such that $x \leqslant z'$ and $y \leqslant z'$. Clearly, this condition does not hold for $b, c \in X$.

6. The Vietoris endofunctor V on Esakia spaces

As we saw, Esakia spaces and strongly isotone and continuous maps form a subcategory **E** of the category **Pri** of Priestley spaces and monotone and continuous maps, so a natural question that arises is whether the restriction of the Vietoris endofunctor **V** to **E** is an endofunctor on **E**. In this section, we are going to show that this is the case, namely, that for every Esakia space $\mathbf{X}$, $\mathbf{V}(\mathbf{X})$ is an Esakia space, and for every continuous and strongly isotone map f between Esakia spaces, $\mathbf{V}(f)$ is continuous and strongly isotone. All the omitted details of proofs can be found in [25].

6.1. The action of **V** *on the objects of* **E**

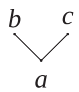
For every Esakia space $\mathbf{X}$ and every clopen subsets U, V of X, let

$$m(U)\downarrow = \{F \in K(\mathbf{X}) \mid F \leqslant^{\mathrm{EM}} G \text{ for some } G \in m(U)\},$$

$$t(V)\downarrow = \{F \in K(\mathbf{X}) \mid F \leqslant^{\mathrm{EM}} G \text{ for some } G \in t(V)\}.$$

In general, $(m(U) \cap t(V))\downarrow \neq m(U)\downarrow \cap t(V)\downarrow$, as the next example shows:

Example 54. Consider the partial order associated with the following Hasse diagram:



This partial order is an Esakia space when endowed with the discrete topology (see Example 53). Let $U\{a, b\}$ and $V = \{c\}$. As $V \cap U = \emptyset$, then $t(V) \cap m(U) = \emptyset$, and so $(m(U) \cap t(V))\downarrow\emptyset$. On the other hand, $\{a\} \in m(U)\downarrow \cap t(V)\downarrow$.

However, there are special cases in which the operator $\downarrow$ behaves well w.r.t. intersection, as it is stated in item (3) of the next Lemma. This is used to show item (4), which is what we need to prove Corollary 56.

Lemma 55. *For every Esakia space* $\mathbf{X}$ *and all clopen subsets* $U, V, U_i \subseteq X$, $i = 1, \ldots, n$,
(1) $m(U)\downarrow = m(U\downarrow)$ *and* $t(U)\downarrow t(U\downarrow)$, *hence* $m(U)\downarrow$ *and* $t(U)\downarrow$ *are clopen subsets of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.
(2) $t(V) \cap \bigcap_{i=1}^{n} m(U_i) = t(V) \cap \bigcap_{i=1}^{n} m(V \cap U_i)$.
(3) $(t(V) \cap \bigcap_{i=1}^{n} m(V \cap U_i))\downarrow t(V)\downarrow \cap \bigcap_{i=1}^{n}(m(V \cap U_i)\downarrow)$, *hence it is a clopen subset of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.
(4) $(t(V) \cap \bigcap_{i=1}^{n} m(U_i))\downarrow$ *is a clopen subset of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.

For every subset $\mathcal{U}$ of $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$, let $\mathcal{U}\downarrow = \{F \in K(\mathbf{X}) \mid F \leqslant^{\mathrm{EM}} G \text{ for some } G \in \mathcal{U}\}$.

Corollary 56. *For every Esakia space* $\mathbf{X}$ *and every clopen subset* $\mathcal{U}$ *of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$, $\mathcal{U}\downarrow$ *is a clopen subset of* $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$.

Proposition 57. *For every Esakia space* $\mathbf{X}$, $\mathbf{V}(\mathbf{X})$ *is an Esakia space*.

Proof. By Proposition 51, it is enough to show that if $\mathcal{U}$ is a clopen subset of $\mathbf{V}(\mathbf{X})$, then $\mathcal{U}\downarrow = \{[F] \in \mathbf{V}(\mathbf{X}) \mid [F] \leqslant^{\mathrm{EM}}_{\equiv} [G] \text{ for some } [G] \in \mathcal{U}\}$ is clopen. It holds that $\pi^{-1}[\mathcal{U}]$ is a clopen subset of $\mathbf{K}^{\mathrm{EM}}(\mathbf{X})$, and so by Corollary 56, $(\pi^{-1}[\mathcal{U}])\downarrow$ is clopen, hence $\mathcal{U}\downarrow\pi[(\pi^{-1}[\mathcal{U}])\downarrow]$ is a clopen subset of $\mathbf{V}(\mathbf{X})$. $\square$

6.2. The action of **V** *on the arrows of* **E**

Proposition 58. *Let* $\mathbf{X}_i$ *be Esakia spaces*, $i = 1, 2$. *For every continuous and strongly isotone map* $f : \mathbf{X}_1 \longrightarrow \mathbf{X}_2$, *the map* $\mathbf{V}(f) : \mathbf{V}(\mathbf{X}_1) \longrightarrow \mathbf{V}(\mathbf{X}_2)$, *given by the assignment* $[F] \mapsto [f[F]]$ *for every* $F \in K(\mathbf{X}_1)$, *is continuous and strongly isotone*.

Proof. By item (1) of Lemma 30, in order to show that $\mathbf{V}(f)$ is strongly isotone, it is enough to show that for every $F \in K(\mathbf{X}_1)$, $G \in K(\mathbf{X}_2)$, $f[F] \leqslant^{\mathrm{EM}} G$ iff there exists $F' \in K(\mathbf{X}_1)$ such that $F \leqslant^{\mathrm{EM}} F'$ and $f[F'] \equiv G$. As for the 'only if' part, take $F' = F\!\uparrow \cap\, h^{-1}[G]$, and use that $f[F]\!\uparrow = f[F\!\uparrow]$ (see Lemma 48) in order to show that $f[F']G$. $\square$

7. Related and further work

7.1. Closed and convex subsets

In order to be able to define the correspondence from $\mathbf{Coalg}(\mathbf{V})$ to $\mathbf{K}^+$, we relied on the fact that the $\equiv$-equivalence classes of any Priestley space $\mathbf{X} = \langle X, \leqslant, \tau\rangle$ can be identified with the closed and convex subsets of $\mathbf{X}$ (see Lemma 39). So a natural alternative way of defining $\mathbf{V}(\mathbf{X})$ would be to consider the space $\langle \mathbf{Kcv}(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau'_V\rangle$, where $\mathbf{Kcv}(\mathbf{X})$ is the set of the closed and convex subsets of $\mathbf{X}$, $\leqslant^{\mathrm{EM}}$ is the Egli–Milner power order restricted to $\mathbf{Kcv}(\mathbf{X}) \times \mathbf{Kcv}(\mathbf{X})$, and τ'_V is the topology defined by taking all the subsets of the form $m(A) = \{F \in \mathbf{Kcv}(\mathbf{X}) \mid F \cap A \neq \emptyset\}$, $t(A) = \{F \in \mathbf{Kcv}(\mathbf{X}) \mid F \subseteq A\}$ for every $A \in \tau$, as a subbase. This definition would be more desirable in many respects, for example it would make the connection with analogous constructions on spectral spaces more transparent, but at the moment we do not have proof that, for every Priestley space $\mathbf{X}$, the space $\langle \mathbf{Kcv}(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau'_V\rangle$ is compact. A sufficient condition for the compactness of this space is that the set $\mathbf{Kcv}(\mathbf{X})$ is a closed subset of $\langle K(\mathbf{X}), \tau_V\rangle$. Notice that this condition is not implied by the facts stated in Lemma 39, however these facts would imply that the $\equiv$-quotient space $\mathbf{V}(\mathbf{X})$ is homeomorphic to $\langle \mathbf{Kcv}(\mathbf{X}), \leqslant^{\mathrm{EM}}, \tau'_V\rangle$ under the hypothesis that $\mathbf{Kcv}(\mathbf{X})$ is a closed subset of $\langle K(\mathbf{X}), \tau_V\rangle$.

7.2. The old and the new semantics

Coalgebras of the Vietoris endofunctor on **Pri** are endowed with a notion of bisimulation. The relation between this notion and the standard one, and more in general, the specific features of $\mathbf{Coalg}(\mathbf{V})$ as a semantics for PML will be matter of further investigation.

7.3. Priestley coalgebras

In [15], a special class of endofunctors on **Set** is defined, namely the class of *Kripke polynomial functors*. This class of functors is inductively defined from products, coproducts and the covariant powerset functor $\mathcal{P}$, and a soundness and completeness theorem is given for the coalgebraic modal logics associated with coalgebras of Kripke polynomial functors. In [18], an analogous class of endofunctors on the category of Stone spaces is defined from products, coproducts and the Vietoris endofunctor $\mathbf{K}$, and the coalgebras for functors of this class are there called *Stone coalgebras*. It interesting to remark that, although Jacobs [15] does not mention Abramsky's work of [1], this connection is enlightened in Kupke et al. [18], which is meant to extend Jacob's framework to Stone spaces. An interesting line of investigation would be to define an analogous class of endofunctors on **Pri**, in which the role of $\mathcal{P}$ or $\mathbf{K}$ would be played by the endofunctor $\mathbf{V}$, and to study the associated coalgebraic (positive) modal logics. A further step in this research project would be to use the isomorphism between Priestley spaces and *spectral spaces* (see *coherent spaces in* [17]), in order to study the connections between such constructions and the framework presented by Abramsky in [1].

7.4. Dual equivalence

Given an endofunctor **T** on a category $\mathcal{C}$, the category $\mathbf{Alg}(\mathbf{T})$ of the **T**-algebras is dually equivalent to the category $\mathbf{Coalg}(\mathbf{T}^{\mathrm{op}})$ of the $\mathbf{T}^{\mathrm{op}}$-coalgebras. As **Pri** is equivalent to $\mathbf{BDL}^{\mathrm{op}}$ (**BDL** being the category of bounded distributive lattices and their homomorphisms) and the category **PMA** of positive modal algebras and their homomorphisms is dually equivalent to $\mathbf{K}^+$, then, as a consequence of the equivalence of categories established in Section 4, the following chain of categorical equivalences holds for some endofunctor **T** on **BDL**:

$$\mathbf{PMA}^{\mathrm{op}} \simeq \mathbf{K}^+ \simeq \mathbf{Coalg}(\mathbf{V}) \simeq \mathbf{Coalg}(\mathbf{T}^{\mathrm{op}}) \simeq \mathbf{Alg}(\mathbf{T})^{\mathrm{op}},$$

hence $\mathbf{PMA} \simeq \mathbf{Alg}(\mathbf{T})$ for some endofunctor **T** on **BDL**. This is analogous to the case treated in [18] (i.e. the category **BAO** of Boolean algebras with operators is equivalent to the category $\mathbf{Alg}(\mathbf{G})$ of the **G**-algebras, for some endofunctor **G** on Boolean algebras), and from the existence of the initial object in $\mathbf{Alg}(\mathbf{T})$ we can deduce the existence of the final object in $\mathbf{Coalg}(\mathbf{V})$. The equivalence between **PMA** and $\mathbf{Alg}(\mathbf{T})$ is worth further investigation.

7.5. Intuitionistic modal logics

As we saw in Section 6, the category of Esakia spaces can serve as well as a base category for Vietoris endofunctors. Besides **V**, other endofunctors can be defined on **E** using alternative Vietoris constructions (see [25]), and in particular one of these constructions is ‘canonical’ for Esakia spaces, in the sense that it characterizes Esakia spaces within Priestley spaces. This lays the grounds of investigation on coalgebraic semantics of intuitionistic modal logics such as $\mathbf{IntK}_{\Box}$, $\mathbf{IntK}_{\Diamond}$, **FS** and **MIPC** (see [28]).

Note. The proofs of some statements that appear in this paper are sketched or omitted. All the omitted details and proofs can be found in [24,25].

Acknowledgements

This work has been carried out during my stay as a visitor to the CWI in Amsterdam. I would like to thank Jan Rutten and the SEN3 group for their warm hospitality, and for providing me with a very friendly and stimulating environment. I would like to thank Clemens Kupke and Alexander Kurz for explaining me their ongoing work, and for many invaluable conversations. I would also like to thank Marcello Bonsangue and Yde Venema for many useful suggestions and comments about this work.

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