A finite element approach to Burgers' equation

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Because of its similarity to the Navier-Stokes equation Burgers' equation often arises in the mathematical modelling used to solve problems in fluid dynamics involving turbulence. Difficulties have been experienced in the past in the numerical solution of Burgers' equation for small values of the parameter \( \nu \). It is therefore useful to attempt a piecewise polynomial approximation (i.e. finite element) where the size of the elements are chosen to take into account the nature of the solution. The aim is to 'chase the peak' by altering the size of the elements at each stage using information from the previous step.

Introduction

Historically Burgers' equation:

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}
\]

(1)

where \( u = u(x, t) \) in some domain and \( \nu \) is a parameter, first appeared in a paper by Bateman\(^4\) when he mentioned it as worthy of study and gave a special solution. Since then the equation has found applications in fields as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. The complete and explicit solution of the equation became known in 1950 (see Hopf)\(^2\).

In a remarkable series of papers from 1939 to 1965 Burgers investigated various aspects of turbulence and developed a mathematical model illustrating the theory and also studied statistical and spectral aspects of the equation and related systems of equations. Due to his extensive work with problems involving this equation it gradually became known as Burgers' equation.

Cole\(^3\) studied the general properties of the equation and outlined some of its various applications. He pointed out that it shows the typical features of shock wave theory, a nonlinear term tending to steepen the wave fronts and produce complete dissipation and a viscous term of higher order which prevents formation of actual discontinuities which tends to diffuse any differences in velocity. Burgers' equation is related to turbulence theory as a mathematical model and this is largely due to its similarity to the Navier-Stokes equation. Both contain nonlinear terms of the type unknown functions multiplied by a first derivative and both contain higher order terms multiplied by a small parameter.

More recently, many other authors have used a variety of numerical techniques in attempting to solve the equation particularly for small values of \( \nu \) which correspond to steep fronts in the propagation of dynamic waveforms. A Fourier series approach which involves the method of lines has been used by Caldwell and Wanless.\(^1\) Although the coefficients of the harmonics are small for large values of \( \nu \) (\( \nu = 1 \) say), they become increasingly dominant as \( \nu \) becomes smaller. This indicates the development of a wave front which restricts the value of \( \nu \) because the number of necessary Fourier terms becomes very large. For \( \nu < 0.1 \) the method is unsuitable and the solutions indicate that a piecewise polynomial approximation (i.e. finite element) should be attempted where the size of the elements are chosen to take into account the nature of the solution.

In this paper, a finite element method is used to solve Burgers' equation with the aim of 'chasing the peak' by altering the size of the elements at each stage using information from the previous step. The numerical results are discussed and prove to be very satisfactory.

Finite element method

The two main reasons for considering a finite element approach are:

(i) it is a relatively simple method to formulate
(ii) we have the possibility of choosing the size of the elements appropriate to our problem.

Usually the elements in the solution domain are found by educated guesswork and are chosen to be independent of the solution. However, one of our aims is to allow the solution to control the element size.

The method is tested by solving Burgers' equation in an open rectangle where the boundary conditions are:

\[
\begin{align*}
    u(0, t) &= u(1, t) = 0 \\
    t &> 0
\end{align*}
\]
A finite element approach to Burgers’ equation: J. Caldwell et al.

and the initial condition is:

\[ u(x, 0) = f(x) \quad 0 < x < 1 \]

The solution is required in \( 0 < x < 1 \) for \( t > 0 \) and as an illustration \( f(x) \) is taken to be \( 4x(1 - x) \) in the first instance and \( \sin \pi x \) in the second instance.

For simplicity we choose two elements, one on each side of the ‘peak’. The geometry is shown in Figure 1.

Using the notation \( U(x, t_r) = U(x, rk) = U_r(x) \) and the approximation:

\[
\frac{\partial u}{\partial t} \approx \frac{U_{r+1}(x) - U_r(x)}{k}
\]

equation (1) can be approximated by:

\[
\frac{U_{r+1} - U_r}{k} = \frac{\nu}{\nu k} \frac{d^2 U_{r+1}}{d x^2} - \frac{\nu}{\nu k} \frac{d U_{r+1}}{d x}
\]

and hence we have:

\[
\frac{d^2 U_{r+1}}{d x^2} - \frac{U_{r+1} + U_r}{\nu} \frac{d U_{r+1}}{d x} = -\frac{U_r}{\nu k}
\]

On replacing:

\[
\frac{U_{r+1} - U_r}{\nu} \frac{d U_{r+1}}{d x}
\]

by

\[
\left( \frac{U_{r+1} - U_r}{\nu} + \frac{U_r}{2 \nu} \frac{d U_{r+1}}{d x} \right)
\]

this nonlinear equation reduces to the linear equation:

\[
\frac{d^2 U_{r+1}}{d x^2} - \frac{U_{r+1} + U_r}{\nu} \frac{d U_{r+1}}{d x} - \frac{U_r}{\nu k} = -\frac{U_r}{\nu k}
\]

Taking \( \alpha = 1/2\nu \) and \( \beta = 1/\nu k \) and setting \( Y = U_r, Z = U_{r+1} \) leads to the equation:

\[
Z'' - \alpha YZ' - (\alpha Y' + \beta) Z + \beta Y = 0
\]

subject to the conditions \( Z(0) = 0, Z(1) = 0, Y \) defined.

We define the residue \( R \) as follows:

\[
R = \int_a^b \{ Z'' - \alpha YZ' - (\alpha Y' + \beta) Z + \beta Y \}^2 \, dx
\]

where \( R = R_1 + R_2, \) and \( R_1, R_2 \) refer to the residues in each element on elements 1 and 2 respectively. The basic idea is to minimize the residue \( R. \) Both residues \( R_1 \) and \( R_2 \) are dependent upon the element size which is in turn dependent on the ‘peak’ (i.e. the value of \( a \)). We would expect \( R_1 \) to increase and \( R_2 \) to decrease as \( a \) increases, \( a \) itself being a function of time.

The order of approximation must be piecewise cubic as it must satisfy the conditions of continuity in the function and, at least, the first derivative at the element boundary. This leads to the choice of a Hermite interpolant using shape functions.

We set the solution at time \( t = 0 \) to be \( Y \) and at time \( (t + \delta t) \) to be \( Z \) and then we have:

\[
Y = Y^{(1)} + Y^{(2)}
\]

\[
Z = Z^{(1)} + Z^{(2)}
\]

where \( Y^{(1)}, Z^{(1)} \) are defined on element 1 and \( Y^{(2)}, Z^{(2)} \) on element 2. In Figure 2 the first coordinate is the \( u \) value and the second coordinate is the gradient at that point.

\[
Y \quad \text{and} \quad Z \quad \text{can be expressed in terms of shape functions as follows:}
\]

\[
Y^{(1)} = N_1z_1 + N_2z_1' + N_3y_2 + N_4y_2' \quad 0 < x < a
\]

\[
Y^{(2)} = N_5z_2 + N_6z_2' + N_7y_2 + N_8y_2' \quad a < x < 1
\]

\[
Z^{(1)} = N_1z_1 + N_2z_1' + N_3z_2 + N_4z_2' \quad 0 < x < b
\]

\[
Z^{(2)} = N_5z_2 + N_6z_2' + N_7z_3 + N_8z_3' \quad b < x < 1
\]

Since \( y_1, z_1, z_2, y_2 \) and \( z_3 \) are all zero, both \( N_1 \) and \( N_6 \) must be zero. The remaining six shape functions can be determined by assuming each is equal to a general cubic in \( x \) with unknown coefficients and then evaluating each shape function and its derivative at both ends of the element to which it applies. This results in four simultaneous equations in the four unknown coefficients which can be evaluated in terms of \( b \). This is repeated for all six shape functions which were found to be:

\[
N_2 = (b^3 x - 2bx^2 + x^3)/b^2
\]

\[
N_3 = (3bx^2 - 2x^3)/b^3
\]

\[
N_4 = (-bx^2 + x^3)/b^2
\]

\[
N_5 = (3b - 1 - 6bx + 3(b + 1)x^2 - 2x^3)/(b - 1)^3
\]
\[ N_1 = \left( -b + (2b + 1) x - (b + 2)x^2 + x^3 \right)/\left( b - 1 \right)^2 \]  
(14)

\[ N_2 = \left( -b^2 + b(2b + 2)x - 2b(2b + 1)x^2 + x^3 \right)/\left( b - 1 \right)^2 \]  
(15)

The residuals \( R_1, R_2 \) both satisfy equation (5) with \( Z \) replaced by \( Z(1) \), \( Z(2) \) respectively. The following stationary conditions are sought:

\[ \frac{\partial R}{\partial z_1} = \frac{\partial R}{\partial z_2} = \frac{\partial R}{\partial z_3} = 0 \]
leading to a system of four simultaneous equations in \( z_1, z_2, z_3 \).

Using equation (8) with \( N_1 = 0 \) the first stationary condition \( \partial R_1/\partial z_1 = 0 \) leads to the equation:

\[ b \left[ \int_0^b \left( F_2 + F_3 \right) \, dx \right] + z_0 \left[ \int_0^b \left( F_2 + F_4 \right) \, dx \right] = -\beta \int_0^b F_2 Y \, dx \]  
(16)

respectively. The analogous conditions:

\[ \frac{\partial R_2}{\partial z_2} = 0 \quad \frac{\partial R_3}{\partial z_3} = 0 \quad \frac{\partial R_4}{\partial z_4} = 0 \]

lead to:

\[ b \left[ \int_0^b \left( F_2 + F_3 \right) \, dx \right] + z_0 \left[ \int_0^b \left( F_2 + F_4 \right) \, dx \right] = -\beta \int_0^b F_2 Y \, dx \]  
(17)

\[ b \left[ \int_0^b \left( F_2 + F_3 \right) \, dx \right] + z_0 \left[ \int_0^b \left( F_2 + F_4 \right) \, dx \right] = -\beta \int_0^b F_2 Y \, dx \]  
(18)

Note that the coefficient matrix \( A \) is symmetric.

**Results and discussion**

A computer program was used to solve the matrix equation (22), the integrals being evaluated by a seven-point Simpson's rule. For purposes of verification this finite element approach was used on the special case of the diffusion equation:

\[ \frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} \]  
(23)

with:

\[ u(x, 0) = 4x(1-x) \]

This simplification means that the solution can also be calculated independently by hand. The solution for all \( t \) is clearly symmetric about \( x = 0.5 \).

The \( uuX \) term in equation (1) was removed in the general program by setting \( b = 0 \). The four values used initially and computed at each time step are the gradients at \( x = 0 \) and \( x = 1 \), taken as \(+4 \) and \(-4 \) respectively because of the initial condition, and the height of the peak in the centre and the gradient at that point. The computed results are presented for \( t = 0 \) (0.01) 0.25 in Table 1.
Table 2 Computed results for case \( u(x, 0) = \sin \pi x \), where \( \nu = 1 \) and \( k = 0.01 \) obtained by 'chasing the peak' 

<table>
<thead>
<tr>
<th>Time, t</th>
<th>( a_1 )</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( z_1' )</th>
<th>( z_2' )</th>
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</tr>
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</tr>
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<td>0.1669</td>
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</tbody>
</table>

In this case we only have the two stationary conditions: 
\[
\frac{\partial R}{\partial z_1} = 0 \quad \text{and} \quad \frac{\partial R}{\partial z_2} = 0 
\]

to satisfy. All the integrals were calculated by hand by making use of change of variable wherever possible. This gives at \( t = 0.01 \), \( z_1 = 3.3312 \) and \( z_2 = 0.9269 \) which agree to three significant figures with the computed values in Table 1.

For the second case \( u(x, 0) = \sin \pi x \) the program was rerun with the values of the slope at \( x = 0, 1 \) being changed to \( -\pi, \pi \) respectively. The first few time steps for the case \( \nu = 1 \) give:

<table>
<thead>
<tr>
<th>Time, t</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( z_2' )</th>
<th>( z_3 )</th>
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<td>2.3831</td>
<td>0.7530</td>
<td>0.0000</td>
<td>-2.3831</td>
</tr>
</tbody>
</table>

The analytical solution of equation (23) is given by:
\[
u(x, t) = e^{-\pi^2 t} \sin \pi x 
\]

which leads to the following results when \( \nu = 1 \) and \( t = 0.01 \):

\[
x \quad 0 \quad 0.5 \quad 1.0 \\
u \quad 0 \quad 0.9060 \quad 0 \\
\frac{\partial u}{\partial x} \quad 2.8463 \quad 0 \quad -2.8463
\]

We may also compare these results with the finite element method using the exact shape function. Using equation (4) this gives:
\[
Z'' - \beta Z - \beta Y = 0 
\]

where \( Z(0) = 0 \), \( Z(1) = 0 \) and \( Y = \sin \pi x \). Because of the boundary conditions the complementary function makes no contribution. Trying \( Z = \gamma \sin \pi x \) for the particular integral leads to:
\[
Z = \frac{\beta \sin \pi x}{(\beta + \pi^2)} 
\]

Of course, the same result is obtained by substituting the same trial function into the residual:
\[
R = \int_0^1 (Z'' - \beta Z + \beta Y)^2 \, dx 
\]

and solving for the stationary condition \( \partial R / \partial \gamma = 0 \).

For the case \( \nu = 1, k = 0.01 \) we have:
\[
Z = 0.9102 \sin \pi x 
\]

which leads to the following results:

\[
x \quad 0 \quad 0.5 \quad 1.0 \\
u \quad 0 \quad 0.9102 \quad 0 \\
\frac{\partial u}{\partial x} \quad 2.8594 \quad 0 \quad 2.8594
\]

The difference between the computed and exact values is approximately 0.8% at worst.
The program was then modified by setting \( \alpha = 0.5 \) (equivalent to \( \nu = 1 \)). When the program was rerun the peak clearly moved from \( x = 0.5 \) towards \( x = 1 \) and, as expected, the gradient in the first semi-interval \((0, 0.5)\) decreased quicker than that in the second semi-interval \((0.5, 1)\). We then required a means of selecting the size of the elements controlled by the 'peak'. To do this we modified the program to calculate the value of each of the residues \( R_1 \) and \( R_2 \) for a given value of \( a \) and for \( 1.01a \). As expected, \( R_1 \) increased and \( R \) decreased as \( a \) increased and this is illustrated in Figure 4. Similar triangles were used to determine where the straight lines joining \( R_1(a) \) to \( R_1(1.01a) \) and \( R_2(a) \) to \( R_2(1.01a) \) intersected. By equating the residues in this way after each step and using the \( x \) coordinate of the intercept as our value of \( a \) in the next step we were 'chasing the peak' and using it to control the size of our elements.

The results for \( \nu = 1 \) and \( k = 0.01 \) are presented in Table 2.

We considered the results as being very satisfactory and hence the program was rerun for values of \( \nu \) equal to \( 0.1 \), 0.01 and 0.001. We found it necessary to reduce the time step at the same time as the peak became more pronounced and moved away from the centre much quicker. The greater number of times steps allowed us to keep the boundary of the two elements nearer to the peak.

**Conclusions**

We feel that this type of solution which itself modifies the elements through feedback could be taken further. The first obvious extension would be to use more than two elements, again governed by the particular problem. This technique could also be applied to more difficult problems. Perhaps problems with developing boundary layers, such as the matching of inner and outer solutions could be solved in a similar way. We have restricted our attention to the use of piecewise polynomials, being the simplest form, but we would suggest that other functions appropriate to the problem could be used.

**References**


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**Table 1** Computed values of \( z_1', z_2, z_3', z_4' \) for the case \( u(x, 0) = 4x(1 - x) \)

<table>
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<tr>
<th>Time, ( t )</th>
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<th>( z_2 )</th>
<th>( z_3' )</th>
<th>( z_4' )</th>
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