Convergence of discrete delayed Hopfield neural networks

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A R T I C L E   I N F O

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A B S T R A C T

The discrete delayed Hopfield neural networks is an extension of the discrete Hopfield neural networks. In this paper, the convergence of discrete delayed Hopfield neural networks is mainly studied, and some results on the convergence are obtained by using Lyapunov function. Several new sufficient conditions for the delayed networks converging towards a limit cycle with period at most 2 are proved in parallel updating mode. Also, some conditions for the delayed networks converging towards a limit cycle with 2-period are investigated in parallel updating mode. All results established in this paper extend the previous results on the convergence of both the discrete Hopfield neural networks, and the discrete delayed Hopfield neural networks in parallel updating mode.

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1. Introduction

Recently, numerous models of nervous system and brain which are called artificial neural networks have been extensively studied and developed. From both theoretical and applied points of view, among the most popular models in literature are Hopfield-types neural networks, which have been studied. There are two Hopfield-types neural networks. One is the discrete Hopfield neural network, the other is the continuous Hopfield neural network. The discrete Hopfield neural network (DHNN) is one of the famous neural networks with a wide range of applications, such as content addressable memory, pattern recognition, and combinatorial optimization. Such applications heavily depend on the dynamic behavior of the networks. Therefore, the research on dynamic behavior are a necessary step for the design of the networks. Because the convergence of the DHNN is not only the foundation of the network’s applications, but also the most basic and important problem, the research on the convergence of the DHNN have attracted considerable interest [1–3].

The discrete delayed Hopfield neural network (DDHNN) is an extension of the DHNN. Also, the convergence of the DDHNN is an important problem. The convergence of the DDHNN in parallel updating mode is investigated and some results on the parallel convergence of the DDHNN are given in Refs. [4–6]. However, the previous research on the delayed networks assumed the interconnection matrix with strong restrict, such as symmetric or antisymmetric, etc. The aim of this paper is to provide some new results on the parallel convergence of the DDHNN. In this paper, we obtain some new sufficient convergence conditions for the DDHNN converging towards a limit cycle with period at most 2, and for the DDHNN converging towards a limit cycle with 2-period.

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The rest of this paper is organized as follows. Section 2 introduces some notations and definitions mostly used in the paper. Section 3 investigates the convergence of the DDHNN and gives some new results on the convergence of the DDHNN. Section 4 concludes the paper.

2. Basic model

The DDHNN with \( n \) neurons can be determined by two \( n \times n \) real matrices \( W^0 = (w_{ij}^0)_{n \times n} \), \( W^1 = (w_{ij}^1)_{n \times n} \), and an \( n \)-dimensional vector \( \theta = (\theta_1, \ldots, \theta_n)^T \), denoted by \( N = (W^0 \oplus W^1, \theta) \). There are two possible values for the state of each neuron: 1 or \(-1\) which denote the state of neuron \( i \) at time \( t \in \{0, 1, 2, \ldots\} \) as \( x_i(t) \), the vector \( X(t) = (x_1(t), \ldots, x_n(t))^T \) is the state of whole neurons at time \( t \), and the states set is \( B^0 \), where \( B = \{-1, 1\} \).

The updating mode of the DDHNN is determined by the following equation

\[
x_i(t + 1) = \text{sgn} \left( \sum_{j=1}^{n} w_{ij}^0 x_j(t) + \sum_{j=1}^{n} w_{ij}^1 x_j(t - 1) + \theta_i \right), \quad i \in \{1, 2, \ldots, n\},
\]

where \( t \in \{0, 1, 2, \ldots\} \), and the \text{sgn} function is defined as follows

\[
\text{sgn}(u) = \begin{cases} 1, & \text{if } u \geq 0, \\ -1, & \text{if } u < 0. \end{cases}
\]

We rewrite Eq. (2.1) in the compact form

\[
X(t + 1) = \text{sgn}(W^0X(t) + W^1X(t - 1) + \theta).
\]

If a state \( X^* \in B^0 \) satisfies the following condition

\[
X^* = \text{sgn}(W^0X^* + W^1X^* + \theta),
\]

then we call the state \( X^* \) being a stable state (or an equilibrium point).

For any states \( X^0 = (x_1^0, \ldots, x_n^0)^T, X^1 = (x_1^1, \ldots, x_n^1)^T, \) let

\[
H_i(X^0, X^1) = \sum_{j=1}^{n} w_{ij}^0 x_j^0 + \sum_{j=1}^{n} w_{ij}^1 x_j^0 + \theta_i \neq 0.
\]

If condition (2.4) is satisfied for all \( i \in I \), then we call the DDHNN (2.1) being strict. For any states \( X^0 = (x_1^0, \ldots, x_n^0)^T, X^1 = (x_1^1, \ldots, x_n^1)^T, \) there is at least one \( i \in I \) such that condition (2.4) holds, then we call the DDHNN (2.1) being weakly strict.

Let \( N = (W^0 \oplus W^1, \theta) \) start from any initial states \( X(0), X(1) \). For \( t \geq 2, \) if there exists time \( t_1 \in \{0, 1, 2, \ldots\} \) such that the updating sequence \( X(0), X(1), X(2), X(3), \ldots \) satisfies that \( X(t + T) = X(t) \) for all \( t \geq t_1, \) where \( T \) is the minimum value which satisfies the above condition, then we call that the initial states \( X(0), X(1) \) converges towards a limit cycle with \( T \)-period. If the network converges towards a limit cycle with \( T \)-period for any initial states \( X(0), X(1), \) then we call that the network converges towards a limit cycle with \( T \)-period. Obviously, a limit cycle with \( 1 \)-period is a stable state.

A matrix \( W = (w_{ij})_{i \neq j} \) is called to be row diagonally dominant, if the matrix \( W \) satisfies the following conditions for each neuron \( i \in I = \{1, 2, \ldots, n\} \)

\[
w_{ii} \geq \sum_{j \in \{1, 2, \ldots, n\}} |w_{ij}|.
\]

If \( Z^TWZ < 0 \) or \( Z^TWZ \leq 0 \) for each \( Z \in B^0 \), then matrix \( W = (w_{ij})_{i \neq j} \) is respectively called to be negative definite or nonpositive definite on the set \{-1, 1\}, where the matrix \( W \) is not necessarily symmetric.

3. The convergence of the DDHNN

**Theorem 1.** If there exists a positive diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \) \((d_i > 0, i = 1, \ldots, n)\) such that matrix \( DW^0 \) is symmetric, and matrix \( W^1 \) is row diagonally dominant, then the network \( N = (W^0 \oplus W^1, \theta) \) converges towards a limit cycle with period at most 2, i.e., for any initial states \( X(0), X(1), \) the network \( N = (W^0 \oplus W^1, \theta) \) converges towards a stable state or a limit cycle with 2-period.

**Proof.** Let

\[
\varepsilon_i = \max \left\{ \sum_{j=1}^{n} w_{ij}^0 x_j^0 + \sum_{j=1}^{n} w_{ij}^1 x_j^1 + \theta_i \left| \sum_{j=1}^{n} w_{ij}^0 x_j^0 + \sum_{j=1}^{n} w_{ij}^1 x_j^1 + \theta_i < 0, x_j^0, x_j^1 \in \{-1, 1\}, j \in I \right. \right\}.
\]
If there are no states as $X^0 = (x_1^0, \ldots, x_n^0)^T$, $X^1 = (x_1^1, \ldots, x_n^1)^T \in B^n$ which make that $\sum_{j=1}^{n} w_{ij}^0 x_j^0 + \sum_{j=1}^{n} w_{ij}^1 x_j^1 + \theta_i < 0$, then $\varepsilon_i$ can be chosen as any negative number. Set $\tilde{\theta}_i = \theta_i - \frac{\varepsilon_i}{2}$, $i = 1, 2, \ldots, n$, then

\[
\begin{align*}
    x_i(t + 1) &= \text{sgn} \left( \sum_{j=1}^{n} w_{ij}^0 x_j(t) + \sum_{j=1}^{n} w_{ij}^1 x_j(t - 1) + \theta_i \right) \\
    &= \text{sgn} \left( \sum_{j=1}^{n} w_{ij}^0 x_j(t) + \sum_{j=1}^{n} w_{ij}^1 x_j(t - 1) + \tilde{\theta}_i \right).
\end{align*}
\]

The convergence of the DDHNN (2.1) is equivalent to the convergence of the DDHNN (3.1). Obviously, the DDHNN (3.1) is strict, i.e., for any states $X^0 = (x_1^0, \ldots, x_n^0)^T$, $X^1 = (x_1^1, \ldots, x_n^1)^T$, we have

\[
\sum_{j=1}^{n} w_{ij}^0 x_j^0 + \sum_{j=1}^{n} w_{ij}^1 x_j^1 + \tilde{\theta}_i \neq 0.
\] (3.2)

Then we can easily prove that, whether $x_i(t + 1) = 1$ or $-1$, we have

\[
    x_i(t + 1) \left( \sum_{j=1}^{n} w_{ij}^0 x_j(t) + \sum_{j=1}^{n} w_{ij}^1 x_j(t - 1) + \tilde{\theta}_i \right) > 0, \quad i \in I.
\] (3.3)

We define energy function (Lyapunov function) of the DDHNN (3.1) as follows.

\[
E(t) = E(X(t), X(t - 1)) = -X^T(t)DW^0X(t - 1) - (X^T(t) + X^T(t - 1))D\tilde{\theta}.
\] (3.4)

Then

\[
\begin{align*}
    \Delta E(t) &= E(X(t + 1), X(t)) - E(X(t), X(t - 1)) \\
    &= -(X^T(t + 1) - X^T(t - 1))DW^0X(t) - (X^T(t + 1) - X^T(t - 1))D\tilde{\theta} \\
    &= -(X^T(t + 1) - X^T(t - 1))D(W^0X(t) + W^1X(t - 1) + \tilde{\theta}) + (X^T(t + 1) - X^T(t - 1))DW^1X(t - 1) \\
    &= -p(t) - q(t)
\end{align*}
\]

where

\[
\begin{align*}
    p(t) &= (X^T(t + 1) - X^T(t - 1))D(W^0X(t) + W^1X(t - 1) + \tilde{\theta}), \\
    q(t) &= -(X^T(t + 1) - X^T(t - 1))DW^1X(t - 1), \\
    \tilde{\theta} &= (\tilde{\theta}_1, \ldots, \tilde{\theta}_n)^T.
\end{align*}
\] (3.5)

We first prove that, if $x_i(t + 1) \neq x_i(t - 1)$, then

\[
x_i(t + 1) - x_i(t - 1) = 2x_i(t + 1) = -2x_i(t - 1).
\] (3.6)

Further more, for the case of $p(t) \geq 0$ and $q(t) \geq 0$ in (3.5), let

\[
I^+(t) = \{ i \in I | x_i(t + 1) \neq x_i(t - 1) \}.
\]

By Eq. (3.3) and the condition $W^1$ being row diagonally dominant, we can easily prove that

\[
\begin{align*}
    p(t) &= (X^T(t + 1) - X^T(t - 1))D(W^0X(t) + W^1X(t - 1) + \tilde{\theta}) \\
    &= \sum_{i \in I^+(t)} (x_i(t + 1) - x_i(t - 1))d_i \left( \sum_{j \in I} w_{ij}^0 x_j(t) + \sum_{j \in I} w_{ij}^1 x_j(t - 1) + \tilde{\theta}_i \right) \\
    &= 2 \sum_{i \in I^+(t)} x_i(t + 1)d_i \left( \sum_{j \in I} w_{ij}^0 x_j(t) + \sum_{j \in I} w_{ij}^1 x_j(t - 1) + \tilde{\theta}_i \right) \geq 0,
\end{align*}
\] (3.7)

\[
\begin{align*}
    q(t) &= -(X^T(t + 1) - X^T(t - 1))DW^1X(t - 1) \\
    &= 2 \sum_{i \in I^+(t)} x_i(t - 1)d_i \sum_{j \in I} w_{ij}^1 x_j(t - 1) \\
    &= 2 \sum_{i \in I^+(t)} d_i \left( w_{ii}^1 + \sum_{j \in I \backslash \{i\}} w_{ij}^1 x_j(t - 1) \right) \geq 0.
\end{align*}
\] (3.8)
Hence, \((3.5)\) then implies \(\Delta E(t) \leq 0\). Furthermore, \(\Delta E(t) = 0\) if and only if \(X(t) = \phi\), i.e., \(X(t + 1) = X(t - 1)\).

We easily calculate that the number of elements in set \(((X, Y)|X, Y \in B^n\) is \(2^{2n}\). For any initial states \(X(0), X(1)\), the updating sequence \(X(0), X(1), X(2), X(3), \ldots\) of the DDHNN \((2.1)\) satisfies \(E(t) = E(X(t), X(t - 1)) \geq E(X(t + 1), X(t)) = E(t + 1)\) for all \(t \in \{1, 2, 3, \ldots\}\), and the possible value of energy function \(E(X(t), X(t - 1))\) is only finite. This then implies that the minimum value of \(E(t)\) exists and the limit as \(t\) gets larger and larger of \(E(t)\) is the minimum value of \(E(t)\).

From all above, we obtain that, for any initial states \(X(0), X(1)\), there exists time \(t_1\) such that when \(t \geq t_1\), the states of the DDHNN \((2.1)\) satisfy \(X(t + 1) = X(t - 1)\).

Without loss generality, we assume when \(t \geq t_1\)
\[
x_i(t) = x_i(t + 1) = x_i(t - 1), \quad i = 1, 2, \ldots, i_0, \quad 0 \leq i_0 \leq n,
\]
\[
x_i(t) \neq x_i(t + 1) = x_i(t - 1), \quad i = i_0 + 1, i_0 + 2, \ldots, n.
\]

When \(i_0 = n\), the DDHNN \((2.1)\) converges towards a stable state. When \(i_0 = 0\), the DDHNN \((2.1)\) converges towards a limit cycle with 2-period. Consequently, the DDHNN \((2.1)\) converges towards a stable state or a limit cycle with 2-period for all initial states \(X(0), X(1)\).

The proof of Theorem 1 is completed. \(\square\)

**Remark 1.** If matrix \(W^0\) is symmetric, matrix \(W^1 = 0\), and the conditions in Theorem 1 are satisfied, then the DDHNN converges towards a limit cycle with period at most 2 for each initial states \(X(1) = X(0)\). This is equivalent to the results of the networks without delay in [1]. If matrix \(W^0\) is symmetric, and matrix \(W^1\) is row diagonally dominant, then Theorem 1 is the results in [6]. So, the Theorem 1 is an extension of the networks with delay and without delay.

**Example 1.** Consider the convergence of \(N = (W^0 \oplus W^1, \theta)\), the expressions of matrices \(W^0, W^1\) and \(\theta\) are respectively in the following
\[
W^0 = \begin{pmatrix} -2 & 1 & 2 \\ 3 & 1 & 2 \\ 3 & 1 & -4 \end{pmatrix}, \quad W^1 = \begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 1 \\ -2 & -1 & 3 \end{pmatrix}, \quad \theta = \begin{pmatrix} -1 \end{pmatrix}.
\]

Obviously, matrix \(W^1\) is row diagonally dominant, and there exists a positive diagonal matrix \(D = \text{diag}(3, 1, 2)\) such that matrix \(DW^0\) is symmetric. This implies that the conditions of Theorem 1 are satisfied. Consequently, the network \(N = (W^0 \oplus W^1, \theta)\) converges towards a stable state or a limit cycle with 2-period for all initial states \(X(0), X(1)\). Actually, we can validate the result. Since state \(X^* = (1, 1, 1)^T\) satisfies \(2.3\), then state \(X^* = (1, 1, 1)^T\) is a stable state. For initial states \(X(0) = (-1, -1, -1)^T, X(1) = (1, 1, 1)^T\), the states updating process can be interpreted as follows
\[
X(0) = (-1, -1, -1)^T, \quad X(1) = (1, 1, 1)^T \rightarrow X(2) = (1, 1, 1)^T \rightarrow X(3) = (1, 1, 1)^T.
\]

This means that, for initial states \(X(0) = (-1, -1, -1)^T, X(1) = (1, 1, 1)^T\), the DDHNN \((2.1)\) converges towards a stable state \(X^* = (1, 1, 1)^T\).

For initial states \(X(0) = (1, 1, 1)^T\) or \(X(0) = (-1, -1, -1)^T, X(1) = (-1, 1, -1)^T\), then the states updating process can be interpreted as follows
\[
X(0) = (1, 1, 1)^T \quad \text{or} \quad X(0) = (-1, -1, -1)^T,
\]
\[
X(1) = (-1, -1, -1)^T \rightarrow X(2) = (-1, -1, -1)^T \rightarrow X(3) = (1, 1, 1)^T \rightarrow X(4) = (-1, -1, -1)^T = X(2) \rightarrow X(5) = (1, 1, 1)^T = X(3).
\]

This means that, for initial states \(X(0) = (1, 1, 1)^T\) or \(X(0) = (-1, -1, -1)^T, X(1) = (-1, -1, -1)^T\), the DDHNN converges towards a limit cycle \((X(2), X(3))\) with 2-period.

Also, for other 61 initial states, we can test that the network \(N = (W^0 \oplus W^1, \theta)\) converges towards a stable state or a limit cycle with 2-period.

**Example 2.** Consider the convergence of \(N = (W^0 \oplus W^1, \theta)\), the expressions of matrices \(W^0, W^1\) and \(\theta\) are respectively in the following
\[
W^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad W^1 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

Obviously, the conditions in Theorem 1 are satisfied, and this means that the delayed network converges towards a stable state or a limit cycle with 2-period for all initial states \(X(0), X(1)\). Since each state satisfies \(2.3\), each state is a stable state.

But, for initial states \(X(0) = (-1, -1)^T, X(1) = (1, 1)^T\), the delayed network converges towards a limit cycle \((X(0), X(1))\) with 2-period. This shows that, even if matrices \(W^0\) and \(W^1\) are all a positive diagonal, and all states are stable states, the delayed network converging towards a stable state is not guaranteed.

From Examples 1 and 2, we know that, for some initial states \(X(0), X(1)\), the delayed network converges towards a stable state, and for some initial states \(X(0), X(1)\), the delayed network converges towards a limit cycle with 2-period. Then, what
conditions can guarantee the delayed network converging towards a stable state, and what conditions can guarantee the delayed network converging a limit cycle with 2-period for all initial states $X(0), X(1)$. In the following, we only give some results on the DDHNN (2.1) converging towards a limit cycle with 2-period.

**Theorem 2.** Suppose there exists a positive diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ ($d_i > 0$, $i = 1, \ldots, n$) such that matrix $DW^0$ is symmetric, matrix $W^1$ is row diagonally dominant, and there exists a neuron $i_0 \in I$ such that the following condition is satisfied

$$w_{i_0i_0}^0 + w_{i_0i_0}^1 \leq - \sum_{j \in I(j \neq I)} |w_{i_0j}^0 + w_{i_0j}^1| - |\theta_{i_0}|. \quad (3.9)$$

If either the delayed network is strict or inequality (3.9) is strict, then the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$.

**Proof.** Obviously, the conditions in Theorem 1 are satisfied, then the network $N = (W^0 \oplus W^1, \theta)$ converges towards a stable state or a limit cycle with 2-period for all initial states $X(0), X(1)$. In the following, we mainly prove that the network $N = (W^0 \oplus W^1, \theta)$ has no one stable state. Proof by contradiction, we assume that there exists a stable state $X^* = (x_1^*, \ldots, x_n^*)^T$, then the stable state $X^* = (x_1^*, \ldots, x_n^*)^T$ satisfies (2.3). By (2.3), whether $x_i^* = 1$ or $-1$, we can easily prove that inequality (3.10) is true for neuron $i_0$

$$x_{i_0}^* \left( \sum_{j \in I} (w_{i_0j}^0 + w_{i_0j}^1)x_j^* + \theta_{i_0} \right) \geq 0. \quad (3.10)$$

This then implies

$$w_{i_0i_0}^0 + w_{i_0i_0}^1 + \sum_{j \in I(j \neq I)} |w_{i_0j}^0 + w_{i_0j}^1| + |\theta_{i_0}| \geq w_{i_0i_0}^0 + w_{i_0i_0}^1 + \sum_{j \in I(j \neq I)} (w_{i_0j}^0 + w_{i_0j}^1)x_j^* x_j^* + \theta_{i_0} x_j^*$$

$$= x_{i_0}^* \left( \sum_{j \in I} (w_{i_0j}^0 + w_{i_0j}^1)x_j^* + \theta_{i_0} \right) \geq 0. \quad (3.11)$$

If the delayed network is strict, then, we can easily prove that inequality (3.10) is strict for neuron $i_0$, and inequality (3.11) is strict. This conflicts (3.9), and means that the network $N = (W^0 \oplus W^1, \theta)$ has no one stable state. Also, if inequality (3.9) is strict, then (3.11) conflicts (3.9), and means that the network $N = (W^0 \oplus W^1, \theta)$ has no one stable state. By Theorem 1, we know that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$.

The proof of Theorem 2 is completed. $\square$

**Example 3.** Consider the convergence of $N = (W^0 \oplus W^1, \theta)$, the expressions of matrices $W^0, W^1$ and $\theta$ are respectively in the following

$$W^0 = \begin{pmatrix} 2 & -2 & 1 \\ -1 & -5 & 2 \\ 1 & 4 & 2 \end{pmatrix}, \quad W^1 = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 3 & -2 \\ -2 & 1 & 3 \end{pmatrix}, \quad \theta = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}.$$  

Since $w_{22}^0 + w_{12}^1 = -2 < -|w_{21}^0 + w_{21}^1| = -|w_{23}^0 + w_{23}^1| = -|\theta_2| = -1$, then inequality (3.9) is strict. Also, matrix $W^1$ is row diagonally dominant, and there exists a positive diagonal matrix $D = \text{diag}(1, 2, 1)$ such that matrix $DW^0$ is symmetric. Consequently, the conditions of Theorem 2 are satisfied. This then implies that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$. Actually, for initial states $X(0) = (1, 1, 1)^T, X(1) = (-1, -1, -1)^T$, the states updating process can be interpreted as follows

$$X(0) = (1, 1, 1)^T, X(1) = (-1, -1, -1)^T \rightarrow X(2) = (1, 1, -1)^T \rightarrow X(3) = (1, -1, -1)^T \rightarrow X(4) = (1, 1, 1)^T.$$  

This means that, for initial states $X(0) = (1, 1, 1)^T, X(1) = (-1, -1, -1)^T$, the DDHNN converges towards a limit cycle $(X(2), X(3))$ with 2-period.

For initial states $X(0) = X(1) = (1, 1, 1)^T$, the states updating process can be interpreted as follows

$$X(0) = X(1) = (1, 1, 1)^T \rightarrow X(2) = (1, -1, 1)^T \rightarrow X(3) = (1, 1, 1)^T = X(1) \rightarrow X(4) = (1, -1, 1)^T = X(2) \rightarrow X(5) = (1, 1, 1)^T = X(3).$$  

This means that, for initial states $X(0) = X(1) = (1, 1, 1)^T$, the DDHNN converges towards a limit cycle $(X(1), X(2))$ with 2-period.

Similarly, for other 62 initial states, we can test that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period.
Theorem 3. Suppose there exists a positive diagonal matrix $D = \text{diag}(d_1, \ldots, d_n) \ (d_i > 0, i = 1, \ldots, n)$ such that matrix $DW^0$ is symmetric, matrix $W^1$ is row-diagonally dominant, and for all states $X$, the condition (3.12) is satisfied,

$$X^T D (W^0 + W^1) X \leq - \sum_{i \in I} |\theta_i| d_i,$$

(3.12)

If either the delayed network is weakly strict or inequality (3.12) is strict, then the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$. Especially, either if the delayed network is weakly strict and one of inequalities (3.13) and (3.14) is satisfied, or, if one of inequalities (3.13) and (3.14) is strict, then the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$.

$$w_{ii}^0 + w_{ii}^1 \leq - \sum_{j \in I (j \neq i)} \frac{d_j}{d_i} (w_{ji}^0 + w_{ij}^0) - |\theta_i|, \quad i \in I.$$

(3.13)

$$w_{ii}^0 + w_{ii}^1 \leq - \frac{1}{2} \sum_{j \in I (j \neq i)} |w_{ij}^0 + w_{ji}^0 + \frac{d_j}{d_i} (w_{ij}^0 + w_{ji}^0)| - |\theta_i|, \quad i \in I.$$

(3.14)

Proof. For any state $X$, we have

$$X^T D (W^0 + W^1) X + \theta \leq X^T D (W^0 + W^1) X + \sum_{i \in I} |\theta_i| d_i.$$

(3.15)

Combining condition (3.12) and inequality (3.15), for any state $X$, we have

$$X^T D (W^0 + W^1) X + \theta \leq 0.$$

(3.16)

Suppose that the delayed network exists a stable state $X^* = (x_1^*, \ldots, x_n^*)^T$. Then,

$$X^T D (W^0 + W^1) X^* + \theta > 0.$$

(3.17)

Based on (3.16) and (3.17), we have

$$X^T D (W^0 + W^1) X^* + \theta = 0.$$

(3.18)

If the delayed network is weakly strict, then, for the stable state $X^*$ we have

$$X^T D (W^0 + W^1) X^* + \theta > 0.$$

(3.19)

This means that (3.19) conflicts (3.18), and implies that the network $N = (W^0 \oplus W^1, \theta)$ has no one stable state. Similarly, if inequality (3.12) is strict, then, for the stable state $X^*$, inequality (3.16) is strict, which conflicts (3.18). This means that the network $N = (W^0 \oplus W^1, \theta)$ has no one stable state. By Theorem 1, we know that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$.

Especially, if one of inequalities (3.13) and (3.14) is satisfied, we can easily prove that inequality (3.12) is true. Also, if one of inequalities (3.13) and (3.14) is strict, we can easily prove that inequality (3.12) is strict. Consequently, Theorem 3 is true.

The proof of Theorem 3 is completed. \qed

Corollary 1. Suppose there exists a positive diagonal matrix $D = \text{diag}(d_1, \ldots, d_n) \ (d_i > 0, i = 1, \ldots, n)$ such that matrix $DW^0$ is symmetric, matrix $W^1$ is row-diagonally dominant, and $\theta = 0$. If the delayed network is weakly strict and matrix $D(W^0 + W^1)$ is nonpositive definite on the set $[-1, 1]$, then the network $N = (W^0 \oplus W^1, 0)$ converges towards a limit cycle with length 2 for all initial states $X(0), X(1)$.

Proof. Since matrix $D(W^0 + W^1)$ is nonpositive definite on the set $[-1, 1]$, and $\theta = 0$, then for any state $X$, we have

$$X^T D (W^0 + W^1) X \leq 0.$$

(3.20)

Inequality (3.20) means that inequality (3.12) holds when $\theta = 0$. Because of the delayed network being weakly strict, the conditions in Theorem 3 are satisfied. Based on Theorem 3, we know that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$. \qed

Corollary 2. Suppose there exists a positive diagonal matrix $D = \text{diag}(d_1, \ldots, d_n) \ (d_i > 0, i = 1, \ldots, n)$ such that matrix $DW^0$ is symmetric, matrix $W^1$ is row-diagonally dominant, and $\theta = 0$. If matrix $D(W^0 + W^1)$ is negative definite on the set $[-1, 1]$, then the network $N = (W^0 \oplus W^1, 0)$ converges towards a limit cycle with length 2 for all initial states $X(0), X(1)$.

Proof. Since matrix $D(W^0 + W^1)$ is negative definite on the set $[-1, 1]$, and $\theta = 0$, then for any state $X$, inequality (3.20) is strict. This then means that inequality (3.12) is strict when $\theta = 0$. By Theorem 3, we know that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period for all initial states $X(0), X(1)$. \qed
**Corollary 1.** If matrix $W^0$ is symmetric and negative definite on the set $\{-1, 1\}$, matrix $W^1 = 0$, and $\theta = 0$, then the network $N = (W^0 \oplus W^1, 0)$ converges towards a limit cycle with length 2 for all initial states $X(1) = X(0)$. This is equivalent to the results of the networks without delay in other paper. The results of the networks without delay is that, if matrix $W$ is symmetric and negative definite on the set $\{-1, 1\}$, then the network $N = (W, 0)$ converges towards a limit cycle with length 2 for all initial states $X(0)$. If matrix $W^0$ is symmetric, and matrix $W^1$ is row diagonally dominant, then Theorems 2 and 3 and it’s corollaries are generalization of the results in [6]. So, Theorems 2 and 3 and it’s corollaries are extension of the corresponding results on the networks with delay and without delay.

**Example 4.** Consider the convergence of $N = (W^0 \oplus W^1, \theta)$, the expressions of matrices $W^0$, $W^1$ and $\theta$ are respectively in the following

$$W^0 = \begin{pmatrix} -3 & 0 & -2 \\ 0 & -4 & -1 \\ -4 & -2 & -8 \end{pmatrix}, \quad W^1 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 6 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Since $w^0_{11} + w^0_{12} + w^0_{13} + w^1_{11} + w^1_{12} + w^1_{13} = -1$, we know that, for any states $X^0, X^1, H_1(X^0, X^1) \neq 0$ holds, this then means that the DDDHNN (2.1) is weakly strict. Obviously, matrix $W^1$ is row diagonally dominant, and there exists a positive diagonal matrix $D = \text{diag}(1, 1, 0.5)$ such that matrix $DW^0$ is symmetric and matrix $D(W^0 + W^1)$ is nonpositive definite on the set $\{-1, 1\}$, in which the expressions of matrix $D(W^0 + W^1)$ as follows

$$D(W^0 + W^1) = \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}.$$ 

Then, the conditions of Corollary 1 are satisfied. By Corollary 1, the network $N = (W^0 \oplus W^1, 0)$ converges towards a limit cycle with length 2 for all initial states $X(0), X(1)$. Actually, we can test that the network $N = (W^0 \oplus W^1, 0)$ always converges towards a limit cycle with length 2 for all initial states $X(0), X(1)$.

For initial states $X(0) = X(1) = (1, 1)^T$, the states updating process can be interpreted as follows

$$X(0) = X(1) = (1, 1, 1)^T \rightarrow X(2) = (-1, 1, -1)^T \rightarrow X(3) = (1, 1, 1)^T \rightarrow X(4) = (-1, -1, -1)^T \rightarrow X(5) = (1, 1, 1)^T \rightarrow X(6) = (-1, -1, -1)^T = X(4).$$

This means that, for initial states $X(0) = X(1) = (1, 1)^T$, the DDDHNN converges towards a limit cycle $(X(3), X(4))$ with 2-period.

Similarly, for other 63 initial states, we can test that the network $N = (W^0 \oplus W^1, \theta)$ converges towards a limit cycle with 2-period.

**Example 5.** Consider the convergence of $N = (W^0 \oplus W^1, \theta)$, the expressions of matrices $W^0$, $W^1$ and $\theta$ are respectively in the following

$$W^0 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}, \quad W^1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

For $X^0 = X^1 = (1, 1)^T$, we have $H_1(X^0, X^1) = H_2(X^0, X^1) = 0$. Consequently, the delayed network is not weak strict. Obviously, matrix $W^1$ is row diagonally dominant, matrix $W^0$ is symmetric, and matrix $W^0 + W^1$ is nonpositive definite on the set $\{-1, 1\}$ but not negative definite on the set $\{-1, 1\}$. Obviously, if neither the condition the delayed network being weakly strict in Corollary 1 nor the condition matrix $W^0 + W^1$ being negative definite on the set $\{-1, 1\}$ in Corollary 2 is satisfied, we can validate the corresponding results in Corollaries 1 and 2 do not hold. Actually, state $X^* = (1, 1)^T$ satisfies (2.3), then state $X^* = (1, 1)^T$ is a stable state. For initial states $X(0) = X(1) = (1, 1)^T$, the delayed network $N = (W^0 \oplus W^1, 0)$ converges towards a stable state $X^* = (1, 1)^T$. This shows that, if neither the condition the delayed network being weakly strict in Corollary 1 nor the condition matrix $D(W^0 + W^1)$ being negative definite on the set $\{-1, 1\}$ in Corollary 2 is satisfied, the corresponding results in Corollaries 1 and 2 can not be guaranteed.

4. Conclusion

In this paper, the convergence of the DDDHNN is mainly studied and some results on the convergence are obtained. The conditions for the DDDHNN converging towards a limit cycle with length at most 2, and with length 2 are respectively obtained. We not only prove these results, but also give some examples to explain the results. From the examples and remarks, we know that these conditions can not be weakened and these conditions are only sufficient conditions but not necessary conditions. Obviously, the established results here partially generalize the existing results on convergence of networks with delay and without delay.

References