Note

A note on the strength and minimum color sum of bipartite graphs

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A B S T R A C T

The strength of a graph $G$ is the smallest integer $s$ such that there exists a minimum sum coloring of $G$ using integers $\{1, \ldots, s\}$, only. For bipartite graphs of maximum degree $\Delta$ we show the following simple bound: $s \leq \lceil \Delta \rceil + 1$. As a consequence, there exists a quadratic time algorithm for determining the strength and minimum color sum of bipartite graphs of maximum degree $\Delta \leq 4$.

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1. Introduction

For a simple undirected graph $G = (V, E)$, a (proper) vertex coloring $c$ is an assignment $c : V \rightarrow \mathbb{N}$ such that for all edges $\{u, v\} \in E$, $c(u) \neq c(v)$. Given a coloring $c$, we define its color sum $\Sigma_c = \sum_{v \in V} c(v)$, and its span $\chi_c = \max_{v \in V} c(v)$. The minimum color sum $\Sigma(G)$ is the minimum value of the color sum taken over all colorings of $G$, the chromatic number $\chi(G)$ is the minimum value of span taken over all colorings of $G$, and the strength $s(G)$ is the minimum value of span taken over those colorings of $G$ which have a color sum equal to $\Sigma(G)$. The maximum vertex degree in $G$ is denoted by $\Delta(G)$, whereas the minimum vertex degree is denoted by $\delta(G)$.

The problem of bounding or determining the exact values of $\Sigma(G)$ and $s(G)$ for different graph classes has been given a lot of attention due to the importance of the sum coloring problem in task scheduling (see e.g. [4] for a nice survey of results). The following upper bound on $s(G)$ was shown in [2] and holds for all graphs:

$$s(G) \leq \left\lfloor \frac{\Delta(G) + \text{col}(G)}{2} \right\rfloor,$$

(1)

where $\text{col}(G) = 1 + \max_{H \subseteq G} \delta(H)$ is the so-called coloring number of $G$. It is known that $\chi(G) \leq \text{col}(G) \leq \Delta(G) + 1$, and the authors of [2] have conjectured that bound (1) can in fact be strengthened as follows:

Conjecture 1 (Mehrabadi’s Conjecture [2,1]). For any graph $G$, $s(G) \leq \left\lfloor \frac{\Delta(G) + \chi(G)}{2} \right\rfloor$.

The bound in Mehrabadi’s conjecture has been proved to hold and be tight for the class of trees [3]. In this note we point out that the conjecture is in fact true for all bipartite graphs (i.e. whenever $\chi(G) = 2$), and remark on some algorithmic consequences of this observation.

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2. A proof of Mehrabadi’s conjecture for bipartite graphs

Theorem 1. For any bipartite graph \( G \), \( s(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \).

Proof. Let \( G \) be a bipartite graph with bipartite partitions \( V = V_1 \cup V_2 \), and let \( c \) be a coloring of \( G \) with \( \Sigma_c = \Sigma(G) \). To complete the proof it is enough to show a procedure which constructs a proper coloring \( c' \) of \( G \) such that \( \Sigma_{c'} \leq \Sigma_c \) and \( \chi_{c'} \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \). Initially, for all \( v \in V \), we put \( c'(v) := \min\{c(v), \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \} \). At this point coloring \( c' \) may be improper due to the existence of neighboring vertices sharing color \( \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \). We will proceed to modify coloring \( c' \) to eliminate these conflicts, in such a way that at every step the color sum of \( c' \) does not increase, and that \( c' \) restricted to vertices having colors \( \{1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil \} \) always remains proper. The condition \( \chi_{c'} \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) will be fulfilled throughout the process.

Let \( V_C \subseteq V_2 \) be the subset of nodes \( v \in V_2 \) such that \( c'(v) = \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) and \( v \) has at least one neighbor in \( V_1 \) colored with the same color as \( v \). As long as \( V_C \) is non-empty, at each step we arbitrarily choose a vertex \( v \in V_C \). Since \( V_C \) has at least one neighbor in \( V_1 \) also colored with color \( \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \), \( v \) can have at most \((\Delta - 1)\) neighbors colored with colors from the range \( \{1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil \} \), and by the pigeon-hole principle there must exist a color value \( a \in \{1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil \} \) such that \( v \) has at most one neighbor colored with color \( a \). If \( v \) has no neighbor colored with color \( a \), we simply put \( c'(v) := a \), thus decreasing the color sum of \( c' \) without creating any new conflicts. Otherwise, let \( u \in V_1 \) be the unique neighbor of \( v \) such that \( c'(u) = a \).

We now modify coloring \( c' \) by switching the color values of \( u \) and \( v \), i.e. \( c'(u) := \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) and \( c'(v) = a \). This does not change the color sum of \( c' \), and moreover \( c' \) restricted to vertices having colors \( \{1, \ldots, \left\lceil \frac{\Delta(G)}{2} \right\rceil \} \) remains proper since \( u \) was the unique neighbor of \( v \) originally having color \( a \).

The above procedure is iterated until set \( V_C \) is empty. It terminates after at most a linear number of steps because at each step the number of vertices in \( V_2 \) having color \( \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) decreases by exactly 1. (In some steps the size of set \( V_C \) may increase, but this is irrelevant.) When the procedure terminates, since set \( V_C \) is empty and the graph is bipartite, \( c' \) is a proper coloring. Recalling that \( \Sigma_{c'} \leq \Sigma_c \) and \( \chi_{c'} \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \) completes the proof. \( \square \)

3. Sum coloring of bipartite graphs with \( \Delta \leq 4 \)

The problem of determining the color sum \( \Sigma(G) \) and strength \( s(G) \) of a graph is known to be computationally hard even when restricted to special graph classes. For example, the problem “is \( s(G) \leq 2? \)” is coNP-complete even for bipartite graphs [6], whereas determining the exact value of \( \Sigma(G) \) is NP-hard for bipartite graphs for any value of maximum degree \( \Delta(G) \geq 5 \) [5]. On the other hand, it was shown in [5] that it is possible to determine \( \Sigma(G) \) precisely in polynomial time for bipartite graphs with \( \Delta(G) \leq 3 \), while the question of the complexity of determining \( \Sigma(G) \) for bipartite graphs of maximum degree \( \Delta(G) = 4 \) was posed as the main open problem. Taking into account the proof of Theorem 1, we can now provide a positive answer to this question.

Theorem 2. For any bipartite graph \( G \) of maximum degree \( \Delta(G) \leq 4 \), the values of \( \Sigma(G) \) and \( s(G) \) can be exactly determined in \( O(|V|^2) \) time.

Proof. In order to find \( \Sigma(G) \), we take advantage of an advanced routine from [5, Thm. 3], which finds in \( O(|V||E|) \) time an improper coloring \( c \) of any bipartite graph with colors \( \{1, 2, 3\} \), such that \( c \) restricted to colors \( \{1, 2\} \) is proper (though vertices having color 3 can be adjacent), and moreover \( \Sigma_c \leq \Sigma(G) \). Observing that for \( \Delta \leq 4 \) we have \( \left\lceil \frac{\Delta(G)}{3} \right\rceil + 1 \leq 3 \), by applying the procedure from the proof of Theorem 1 to modify coloring \( c \), we obtain in \( O(|V||E|) = O(|V|^2) \) time a proper coloring \( c' \) of \( G \) such that \( \Sigma_{c'} \leq \Sigma_c \leq \Sigma(G) \). Obviously, \( c' \) is an optimal sum coloring of \( G \) and \( \Sigma(G) = \Sigma_{c'} \).

In order to determine \( s(G) \), we note that by Theorem 1 for \( \Delta \leq 4 \), \( s(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1 \leq 3 \). Assuming that \( G \) is non-empty, this means that either \( s(G) = 2 \), or \( s(G) = 3 \). So, it suffices to check whether \( s(G) = 2 \), and this holds if and only if for each connected component \( H \) of \( G \) we have \( \Sigma(H) = \min\{\Sigma_{c_1}, \Sigma_{c_2}\} \), where \( c_1 \) and \( c_2 \) are the only two distinct colorings of bipartite graph \( H \) using 2 colors. Since the parameters \( \Sigma(H) \), \( \Sigma_{c_1} \), \( \Sigma_{c_2} \) can be determined in \( O(|V|^2) \) time, this completes the proof. \( \square \)

It is interesting to ask whether any of the simple techniques presented here, especially the proof of Theorem 1, can be generalized to non-bipartite graphs. A direct application of the proposed construction only removes color conflicts with respect to one independent set of the graph.

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