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Numerical algorithms and simulations for reflected backward stochastic differential equations with two continuous barriers

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ABSTRACT

In this paper we study different algorithms for reflected backward stochastic differential equations (BSDE in short) with two continuous barriers based on the framework of using a binomial tree to simulate 1-d Brownian motion. We introduce numerical algorithms by the penalization method and the reflected method, respectively. In the end simulation results are also presented.

1. Introduction

Non-linear backward stochastic differential equations (BSDEs in short) were firstly studied in [1], who proved the existence and uniqueness of the adapted solution, under smooth square integrability assumptions on the coefficient and the terminal condition, plus that the coefficient $g(t, \omega, y, z)$ is (t, ω)-uniformly Lipschitz in (y, z). Then El Karoui et al. introduced the notion of reflected BSDE (RBSDE in short) [2], with one continuous lower barrier. More precisely, a solution for such an equation associated to a coefficient g, a terminal value ξ, a continuous barrier L_t, is a triplet $(Y_t, Z_t, A_t)_{0\leq t\leq T}$ of adapted processes valued in R^{1+d+1}, which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + A_T - A_t + \int_t^T Z_s \mathrm{d}B_s, \quad 0 \leq t \leq T, \text{ a.s.,}$$

and $Y_t \geq L_t$ a.s. for any $0 \leq t \leq T$. A_t is non-decreasing continuous, and B_t is a d-dimensional Brownian motion. The role of A_t is to push upward the process Y in a minimal way, in order to keep it above L. In this sense it satisfies $\int_0^T (Y_s - L_s)\mathrm{d}A_s = 0$.

Following this paper, Cvitanic and Karatzas [3], introduced the notion of reflected BSDE with two continuous barriers. In this case a solution of such an equation associated to a coefficient g, a terminal value ξ, a continuous lower barrier L_t and a continuous upper barrier U_t, with $L_t \leq U_t$ and $L_T \leq \xi \leq U_T$ a.s., is a quadruple $(Y_t, Z_t, A_t, K_t)_{0\leq t\leq T}$ of adapted processes, valued in R^{1+d+1}, which satisfies

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s + A_T - A_t - (K_T - K_t) - \int_t^T Z_s \mathrm{d}B_s, \quad 0 \leq t \leq T, \text{ a.s.,}$$

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and $L_t \leq Y_t \leq U_t$, a.s. for any $0 \leq t \leq T$. Here A_t and K_t are increasing continuous process, whose roles are to keep the process Y between L and U in such a way that

$$\int_0^T (Y_s - L_s)\mathrm{d}A_s = 0 \quad \text{and} \quad \int_0^T (Y_s - U_s)\mathrm{d}K_s = 0.$$

Aiming to prove the existence and uniqueness of a solution, the method is based on a Picard-type iteration procedure, which requires at each step the solution of a Dynkin game problem. Furthermore, the authors proved the existence result by the penalization method when the coefficient g does not depend on z. In 2004 [4], Lepeltier and San Martin relaxed in some sense the condition on the barriers, proved by a penalization method an existence result, without any assumption other than the square integrability one on L and U, but only when there exists a continuous semi-martingale with terminal value ξ, between L and U. More recently, Lepeltier and Xu [5] studied the case when the barriers are right continuous and left limit (RCLL in short), and proved the existence and uniqueness of a solution in both Picard iteration and penalization method. In 2005, Peng and Xu [6] considered the most general case when barriers are just $\mathbf{L}^2$-processes by the penalization method, and studied a special penalization BSDE, which penalized with two barriers at the same time, and proved that the solutions of these equations converge to the solution of reflected BSDE.

The calculation and simulation of BSDEs is essentially different from those of SDEs (see [7]). When g is linear in y and z, we may solve the solution of BSDE by considering its dual equation, which is a forward SDE. However for nonlinear cases of g, we cannot find the solution explicitly. Here our numerical algorithms are based on approximating Brownian motion by random walk. This method was first considered by Peng and his students in 2000 (cf. Introduction in [8]). The convergence of this type of numerical algorithms is proved by Briand et al. in 2000 [9] and 2002 [10]. In 2002, Mémin et al. studied the algorithms for reflected BSDE with one barrier and proved its convergence (cf. [11]). Peng and Xu in [8] studied the convergence results of an explicit scheme based on this kind of algorithm, which is efficient in programming. Recently Chassagneux also studied discrete-time approximation of doubly reflected BSDE in [12] from another point of view.

In this paper, we consider different numerical algorithms for a reflected BSDE with two continuous barriers. The basic idea is to approximate a Brownian motion by random walks based on the binary tree model. Compared with the one barrier case (cf. [11]), the additive barrier brings more difficulties in proving the convergence of the algorithm, which requires us to get a finer estimation. If we combine it with a diffusion process, and assume the coefficient to be a deterministic function, then with the help of non-linear Feymann–Kac formulae, we know that the solution of BSDE equals the solution of variational inequality. In such a case, our algorithm will coincide with the finite difference method of variational inequality (cf. [13,14]). The algorithm studied in this paper is in a stochastic point of view, in fact the coefficient of reflected BSDE can be a random function. When the Brownian motion is 1-dimensional, our algorithms have advantages in computer programming. In fact we developed a software package based on these algorithms for a BSDE with two barriers. Furthermore it also contains programs for classical BSDEs and reflected BSDEs with one barrier. One significant advantage of this package is that the users have a very convenient user–machine interface. Any user who knows the basics of BSDE can run this package without difficulty. Meanwhile, we can also generalize algorithms in this paper to multi-dimensional Brownian motion, which will require a huge amount of calculation. So we will not discuss this subject in this paper.

This paper is organized as follows. In Section 2, we recall some classical results of reflected BSDE with two continuous barriers, and discretization for reflected BSDE. In Section 3, we introduce implicit and implicit–explicit penalization schemes and prove their convergence. In Section 4, we study implicit and explicit reflected schemes, and get their convergence. In Section 5, we present some simulations for reflected BSDE with two barriers. The proof of the convergence of penalization solutions is in Appendix.

We should point out that recently there have been many different algorithms for computing solutions of BSDEs and the related results in numerical analysis, for example [15,9,16–20]. In contrast to these results, our methods can easily be realized by computing the 1-dimensional Brownian motion case. In the multi-dimensional case, the algorithms are still suitable, however to realize them by computation is difficult, since it will require larger amount of calculation than the 1-dimensional case.

2. Preliminaries: reflected BSDEs with two barriers and basic discretization

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $(B_t)_{t\geq 0}$ a 1-dimensional Brownian motion defined on a fixed interval $[0, T]$, with a fixed $T > 0$. We denote by $\{\mathcal{F}_t\}_{0\leq t\leq T}$ the natural filtration generated by the Brownian motion B, i.e., $\mathcal{F}_t = \sigma\{B_s; 0 \leq s \leq t\}$ augmented with all P-null sets of $\mathcal{F}$. Here we mainly consider the 1-dimensional case, since the solution of reflected BSDE is 1-dimensional. In fact, we can also generalize algorithms in this paper to multi-dimensional Brownian motion, which will require a huge amount of calculation. So we will not discuss this subject in this paper. Now we introduce the following spaces for $p \in [1, \infty)$:

- $\mathbf{L}^p(\mathcal{F}_t) := \{\mathbb{R}\text{-valued}\mathcal{F}_t\text{-measurable random variables } X \text{ s.t. } E[|X|^p] < \infty\}$;
- $\mathbf{L}^p_{\mathcal{F}}(0, t) := \{\mathbb{R}\text{-valued and } \mathcal{F}_t\text{-adapted processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E\int_0^t |\varphi_s|^p \mathrm{d}s < \infty\}$;
- $\mathbf{S}^p(0, t) := \{\mathbb{R}\text{-valued and } \mathcal{F}_t\text{-adapted continuous processes } \varphi \text{ defined on } [0, t], \text{ s.t. } E[\sup_{0\leq t\leq T} |\varphi_t|^2] < \infty\}$;
- $\mathbf{A}^p(0, t) := \{\text{increasing processes in } \mathbf{S}^p(0, t) \text{ with } A(0) = 0\}$.

We are especially interested in the case $p = 2$.

2.1. Reflected BSDE: definition and convergence results

The random variable ξ is considered as a terminal value, satisfying $\xi \in \mathbf{L}^2(\mathcal{F}_T)$. Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a (t,ω)-uniformly Lipschitz function in (y,z), i.e., there exists a fixed $\mu > 0$ such that

$$|g(t,y_1,z_1) - g(t,y_2,z_2)| \le \mu(|y_1-y_2| + |z_1-z_2|) \quad \forall t \in [0,T],\ \forall (y_1,z_1),(y_2,z_2) \in \mathbb{R}\times\mathbb{R}. \tag{1}$$

And $g(\cdot,0,0)$ is a progressively measurable square integrable process.

The solution of our BSDE with two barriers is reflected between a lower barrier L and an upper barrier U, which are supposed to satisfy

Assumption 2.1. L and U are $\mathcal{F}_t$-progressively measurable continuous processes valued in $\mathbb{R}$, such that

$$E\left[\sup_{0\le t\le T}((L_t)^+)^2 + \sup_{0\le t\le T}((U_t)^-)^2\right] < \infty \tag{2}$$

and there exists a continuous process $X_t = X_0 - \int_0^t \sigma_s \mathrm{d}B_s + V_t^+ - V_t^-$ where $\sigma \in \mathbf{L}^2_{\mathcal{F}}(0,T)$, V^+ and V^- are $(\mathcal{F}_t)$-adapted continuous increasing processes with $E[|V_T^+|^2] + E[|V_T^-|^2] < \infty$ such that

$$L_t \le X_t \le U_t, \quad P\text{-a.s. for } 0 \le t \le T.$$

Remark 2.1. Condition (2) permits us to treat situations when $U_t \equiv +\infty$ or $L_t \equiv -\infty$, $t \in [0,T]$, in such cases the corresponding reflected BSDE with two barriers becomes a reflected BSDE with a single lower barrier L or a single upper barrier U, respectively.

Definition 2.1. The solution of a reflected BSDE with two continuous barriers is a quadruple $(Y,Z,A,K) \in \mathbf{S}^2(0,T) \times \mathbf{L}^2_{\mathcal{F}}(0,T) \times \mathbf{A}^2(0,T) \times \mathbf{A}^2(0,T)$ defined on $[0,T]$ satisfying the following equations

$$\begin{aligned} &-\mathrm{d}Y_t = g(t,Y_t,Z_t)\mathrm{d}t + \mathrm{d}A_t - \mathrm{d}K_t - Z_t\mathrm{d}B_t, \quad Y_T = \xi \\ &L_t \le Y_t \le U_t, \quad \mathrm{d}A_t \ge 0, \qquad \mathrm{d}K_t \ge 0, \qquad \mathrm{d}A_t \cdot \mathrm{d}K_t = 0 \end{aligned} \tag{3}$$

and the reflecting conditions

$$\int_0^T (Y_t - L_t)\mathrm{d}A_t = \int_0^T (Y_t - U_t)\mathrm{d}K_t = 0. \tag{4}$$

To prove the existence of the solution, the penalization method is important. Thanks to the convergence results of the penalization solution in [4,21] for the continuous barriers' case and methods in [6], we have the following results, especially it gives the convergence speed of penalization solutions.

Theorem 2.1. (a) *There exists a unique solution (Y,Z,A,K) of reflected BSDE, i.e. it satisfies (3), (4). Moreover it is the limit of penalization solutions $(\widehat{Y}_t^{m,p}, \widehat{Z}_t^{m,p}, \widehat{A}_t^{m,p}, \widehat{K}_t^{m,p})$ as $m \to \infty$ then $p \to \infty$, or equivalent as $q \to \infty$ then $m \to \infty$. Here the penalization solution $(\widehat{Y}_t^{m,p}, \widehat{Z}_t^{m,p}, \widehat{A}_t^{m,p}, \widehat{K}_t^{m,p})$ with respect to two barriers L and U is defined, for $m \in \mathbb{N}$, $p \in \mathbb{N}$, as the solution of a classical BSDE*

$$\begin{aligned} &-\mathrm{d}\widehat{Y}_t^{m,p} = g(t,\widehat{Y}_t^{m,p},\widehat{Z}_t^{m,p})\mathrm{d}t + m(\widehat{Y}_t^{m,p} - L_t)^-\mathrm{d}t - p(\widehat{Y}_t^{m,p} - U_t)^+\mathrm{d}t - \widehat{Z}_t^{m,p}\mathrm{d}B_t, \\ &\widehat{Y}_T^{m,p} = \xi. \end{aligned} \tag{5}$$

And we set $\widehat{A}_t^{m,p} = m\int_0^t (\widehat{Y}_s^{m,p} - L_s)^-\mathrm{d}s$, $\widehat{K}_t^{m,p} = p\int_0^t (\widehat{Y}_s^{m,p} - U_s)^+\mathrm{d}s$.

(b) *Consider a special penalized BSDE for the reflected BSDE with two barriers: for any $p \in \mathbb{N}$,*

$$\begin{aligned} &-\mathrm{d}Y_t^p = g(t,Y_t^p,Z_t^p)\mathrm{d}t + p(Y_t^p - L_t)^-\mathrm{d}t - p(Y_t^p - U_t)^+\mathrm{d}t - Z_t^p\mathrm{d}B_t, \\ &Y_T^p = \xi, \end{aligned} \tag{6}$$

with $A_t^p = \int_0^t p(Y_s^p - L_s)^-\mathrm{d}s$ and $K_t^p = \int_0^t p(Y_s^p - U_s)^+\mathrm{d}s$. Then we have, as $p \to \infty$, $Y_t^p \to Y_t$ in $\mathbf{S}^2(0,T)$, $Z_t^p \to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$ and $A_t^p \to A_t$ weakly in $\mathbf{S}^2(0,T)$ as well as $K_t^p \to K_t$. Moreover there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$E\left[\sup_{0\le t\le T}|Y_t^p - Y_t|^2 + \int_0^T |Z_t^p - Z_t|^2\mathrm{d}t + \sup_{0\le t\le T}[(A_t - K_t) - (A_t^p - K_t^p)]^2\right] \le \frac{C}{\sqrt{p}}. \tag{7}$$

The proof is based on the results in [4,6], we put it in Appendix.

Remark 2.2. In the following, we focus on the penalized BSDE as (7), which considers the penalization with respect to the two barriers at the same time. And p in superscribe always stands for the penalization parameter.

Now we consider a special case: Assume that

Assumption 2.2. L and U are Itô processes of the following form

$$L_t = L_0 + \int_0^t l_s \mathrm{d}s + \int_0^t \sigma_s^l \mathrm{d}B_s, \qquad (8)$$
$$U_t = U_0 + \int_0^t u_s \mathrm{d}s + \int_0^t \sigma_s^u \mathrm{d}B_s.$$

Suppose that l_s and u_s are right continuous with left limits (RCLL in short) processes, σ_s^l and σ_s^u are predictable with $E\int_0^T [|l_s|^2 + |\sigma_s^l|^2 + |u_s|^2 + |\sigma_s^u|^2]\mathrm{d}s < \infty$.

It is easy to check that if $L_t \le U_t$, then Assumption 2.1 is satisfied. We may just set $X = L$ or U, with $\sigma_s = \sigma_s^l$ or σ_s^u and $V^{\pm} = \int_0 l_s^{\pm}\mathrm{d}s$ or $\int_0 u_s^{\pm}\mathrm{d}s$. Here $l_s^{\pm}$ (resp. $u_s^{\pm}$) is the positive or the negative part of l (resp. u). As Proposition 4.2 in [2], we have following proposition for two increasing processes, which can give us the integrability of the increasing processes by barriers.

Proposition 2.1. *Let (Y, Z, A, K) be a solution of reflected BSDE (3). Then $Z_t = \sigma_t^l$, a.s.-$\mathrm{d}P \times \mathrm{d}t$ on the set $\{Y_t = L_t\}$, $Z_t = \sigma_t^u$, a.s.-$\mathrm{d}P \times \mathrm{d}t$ on the set $\{Y_t = U_t\}$. And*

$$0 \le \mathrm{d}A_t \le 1_{\{Y_t = L_t\}}[g(t, L_t, \sigma_t^l) + l_t]^- \mathrm{d}t,$$
$$0 \le \mathrm{d}K_t \le 1_{\{Y_t = U_t\}}[g(t, U_t, \sigma_t^u) + u_t]^+ \mathrm{d}t.$$

So there exist positive predictable processes α and β, with $0 \le \alpha_t, \beta_t \le 1$, such that $\mathrm{d}A_t = \alpha_t 1_{\{Y_t = L_t\}}[g(t, L_t, \sigma_t^l) + l_t]^- \mathrm{d}t$, $\mathrm{d}K_t = \beta_t 1_{\{Y_t = U_t\}}[g(t, U_t, \sigma_t^u) + u_t]^+ \mathrm{d}t$.

Proof. We can prove these results easily by using similar techniques as in Proposition 4.2 in [2], in view that on the set $\{L_t = U_t\}$, we have $\sigma_t^l = \sigma_t^u$ and $l_t = u_t$. So we omit the details of the proof here. □

In the following, we will assume Assumption 2.2 hold for two barriers.

2.2. Approximation of Brownian motion and parameters of reflected BSDE

We use random walk to approximate the Brownian motion. Consider for each $j = 1, 2, \ldots,$

$$B_t^n := \sqrt{\delta}\sum_{j=1}^{[t/\delta]} \varepsilon_j^n, \quad \text{for all } 0 \le t \le T,\ \delta = \frac{T}{n},$$

where $\{\varepsilon_j^n\}_{j=1}^n$ is a $\{1, -1\}$-valued i.i.d. sequence with $P(\varepsilon_j^n = 1) = P(\varepsilon_j^n = -1) = 0.5$, i.e., it is a Bernoulli sequence. We consider the discrete filtration $\mathcal{G}_j^n := \sigma\{\varepsilon_1^n, \ldots, \varepsilon_j^n\}$. Set $t_j = j\delta$, for $j = 0, 1, \ldots, n$. We denote by $\mathbf{D}_t$ the space of RCLL functions from $[0, t]$ in $\mathbb{R}$, endowed with the topology of uniform convergence, and we assume that:

Assumption 2.3. $\Gamma : \mathbf{D}_T \to \mathbf{R}$ is K-Lipschitz. We consider $\xi = \Gamma(B)$, which is $\mathcal{F}_T$-measurable and $\xi^n = \Gamma(B^n)$, which is $\mathcal{G}_n^n$-measurable, such that

$$E[|\xi|^2] + \sup_n E[|\xi^n|^2] < \infty.$$

For the coefficient $g(y, z)$, we also need to consider its approximation $(g_j^n(y, z))_{0 \le j \le n}$, which is $\mathcal{G}_j^n$-adapted, and satisfies the following assumption:

Assumption 2.4. $g^n(y, z)$ is Lipschitz in (y, z) with the same μ for all n, and there exists a constant C_g such that for all $n > 1 + 2\mu + 2\mu^2$,

$$E\left[\sum_{j=0}^{n-1} |g_j^n(0, 0)|^2 \frac{1}{n}\right] < C_g.$$

And if we set $g^n(t, y, z) = g_{[t/\delta]}^n(y, z)$, then $g^n(t, y, z)$ converges to $g(t, y, z)$ in $\mathbf{S}^2(0, T)$.

Remark 2.3. If the coefficient takes the form as $g = g(t, (B_s)_{0 \le s \le t}, y, z)$, then its natural candidate of approximation is $g_j^n(y, z) = g(t_j, (B_s^n)_{0 \le s \le t}, y, z)$. Assume that g is Lipschitz in B and $t \to g(t, B, y, z)$ is continuous, we know that g_j^n satisfies Assumption 2.4.

Remark 2.4. When g is a deterministic function, $g_j^n(y,z)=g(t_j,y,z)$ is an approximation of g satisfying Assumption 2.4, if $t\to g(t,y,z)$ is continuous.

Now we consider the approximation of the barriers L and U. Notice that L and U are progressively measurable with respect to the filtration $(\mathcal{F}_t)$, which is generated by Brownian motion. So they can be presented as a functional of Brownian motion, i.e. for each $t\in[0,T]$, $L_t=\Psi_1(t,(B_s)_{0\le s\le t})$ and $U_t=\Psi_2(t,(B_s)_{0\le s\le t})$, where $\Psi_1(t,\cdot)$ and $\Psi_2(t,\cdot):\mathbf{D}_t\to\mathbf{R}$. And we assume that $\Psi_1(t,\cdot)$ and $\Psi_2(t,\cdot)$ are Lipschitz. Then we get the discretizaton of the barriers $L_j^n=\Psi_1(t_j,(B_s^n)_{0\le s\le t})$ and $U_t^n=\Psi_2(t_j,(B_s^n)_{0\le s\le t})$. If $L_t\le U_t$, then $L_j\le U_j$. On the other hand, we can consider barriers which are Itô processes and satisfy Assumption 2.2. So we have a natural approximation: for $j=1,2,\ldots,n$,

$$L_j^n=L_0+\delta\sum_{i=0}^{j-1}l_i+\sum_{i=0}^{j-1}\sigma_i^l\varepsilon_{i+1}^n\sqrt{\delta},$$
$$U_j^n=U_0+\delta\sum_{i=0}^{j-1}u_i+\sum_{i=0}^{j-1}\sigma_i^u\varepsilon_{i+1}^n\sqrt{\delta}$$

where $l_i=l_{t_i},\sigma_i^l=\sigma_{t_i}^l,u_i=u_{t_i},\sigma_i^u=\sigma_{t_i}^u$. Then L_j^n and U_j^n are discrete versions of L and U, with $\sup_n E[\sup_j((L_j^n)^+)^2+\sup_j((U_j^n)^-)^2]<\infty$ and $L_j^n\le U_j^n$ still hold. In the following, we may use both approximations, depending on different situations.

In this paper, we study two different types of numerical schemes. The first one is based on the penalization approach, whereas the second is to obtain the solution Y by reflecting it between L and U and get two reflecting processes A and K directly. Throughout this paper, n always stands for the discretization of the time interval. And process $(\phi_j^n)_{0\le j\le n}$ is a discrete process with $n+1$ values, for $\phi=L,U,y^p,y$, etc.

3. Algorithms based on penalization BSDE and their convergence

3.1. Discretization of penalization BSDE and penalization schemes

First we consider the discretization of penalized BSDE (7) with respect to two discrete barriers L^n and U^n. After the discretization of time interval and approximating parameters, we get the following discrete backward equation on the small interval $[t_j,t_{j+1}]$, for $j=0,1,\ldots,n-1$,

$$\begin{aligned}y_j^{p,n}&=y_{j+1}^{p,n}+g_j^n(y_j^{p,n},z_j^{p,n})\delta+a_j^{p,n}-k_j^{p,n}-z_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}^n,\\ a_j^{p,n}&=p\delta(y_j^{p,n}-L_j^n)^-,\qquad k_j^{p,n}=p\delta(y_j^{p,n}-U_j^n)^+.\end{aligned}\tag{9}$$

The terminal condition is $y_n^{p,n}=\xi^n$. Since for a large fixed $p>0$, (6) is in fact a classical BSDE. By numerical algorithms for BSDEs (cf. [22]), explicit scheme gives $z_j^{p,n}=\frac{1}{\delta}E[y_{j+1}^{p,n}\varepsilon_{j+1}^n|\mathcal{G}_j^n]=\frac{1}{2\sqrt{\delta}}(y_{j+1}^{p,n}|_{\varepsilon_{j+1}=1}-y_{j+1}^{p,n}|_{\varepsilon_{j+1}=-1})$, and $y_j^{p,n}$ is solved from the inversion of the following mapping

$$y_j^{p,n}=(\Theta^p)^{-1}(E[y_{j+1}^{p,n}|\mathcal{G}_j^n]),$$
$$\text{where }\Theta^p(y)=y-g_j^n(y,z_j^{p,n})\delta-p\delta(y-L_j^n)^-+p\delta(y-U_j^n)^+,$$

by substituting $E[y_{j+1}^{p,n}|\mathcal{G}_j^n]=\frac{1}{2}(y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1}+y_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1})$ into it. And increasing processes $a_j^{p,n}$ and $k_j^{p,n}$ will be obtained from (9).

In many cases, the inversion of the operator Θ^p is not easy to solve. So we apply the implicit–explicit penalization scheme to (9), replacing $y_j^{p,n}$ in g by $E[y_{j+1}^{p,n}|\mathcal{G}_j^n]$, and get

$$\begin{aligned}\bar{y}_j^{p,n}&=\bar{y}_{j+1}^{p,n}+g_j^n(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta+\bar{a}_j^{p,n}-\bar{k}_j^{p,n}-\bar{z}_j^{p,n}\sqrt{\delta}\varepsilon_{j+1}^n\\ \bar{a}_j^{p,n}&=p\delta(\bar{y}_j^{p,n}-L_j^n)^-,\qquad \bar{k}_j^{p,n}=p\delta(\bar{y}_j^{p,n}-U_j^n)^+.\end{aligned}$$

In the same way, we get $\bar{z}_j^{p,n}=\frac{1}{\delta}E[\bar{y}_{j+1}^{p,n}\varepsilon_{j+1}^n|\mathcal{G}_j^n]=\frac{1}{2\sqrt{\delta}}(\bar{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=1}-\bar{y}_{j+1}^{p,n}|_{\varepsilon_{j+1}^n=-1})$ and

$$\bar{y}_j^{p,n}=E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g_j^n(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta+\bar{a}_j^{p,n}-\bar{k}_j^{p,n}.\tag{10}$$

Solving this equation, we obtain

$$\begin{aligned}\bar{y}_j^{p,n}=&\ E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g_j^n(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta+\frac{p\delta}{1+p\delta}(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g_j^n(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta-L_j^n)^-\\ &-\frac{p\delta}{1+p\delta}(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n]+g_j^n(E[\bar{y}_{j+1}^{p,n}|\mathcal{G}_j^n],\bar{z}_j^{p,n})\delta-U_j^n)^+\end{aligned}$$

with $E[\bar{y}^{p,n}_{j+1}|\mathcal{G}^n_j] = \frac{1}{2}(\bar{y}^{p,n}_{j+1}|_{\varepsilon^n_{j+1}=1} + \bar{y}^{p,n}_{j+1}|_{\varepsilon^n_{j+1}=-1})$. The increments of increasing processes are given by

$$\bar{a}^{p,n}_j = \frac{p\delta}{1+p\delta}(E[\bar{y}^{p,n}_{j+1}|\mathcal{G}^n_j] + g^n_j(E[\bar{y}^{p,n}_{j+1}|\mathcal{G}^n_j], \bar{z}^{p,n}_j)\delta - L^n_j)^-,$$
$$\bar{k}^{p,n}_j = \frac{p\delta}{1+p\delta}(E[\bar{y}^{p,n}_{j+1}|\mathcal{G}^n_j] + g^n_j(E[\bar{y}^{p,n}_{j+1}|\mathcal{G}^n_j], \bar{z}^{p,n}_j)\delta - U^n_j)^+.$$

3.2. Convergence of penalization schemes and estimations

First we give the following lemma, which is proved in [11]. This Gronwall type lemma is classical but here it is given with a more detailed formulation.

Lemma 3.1. *Let a, b and α be positive constants, $\delta b < 1$ and a sequence $(v_j)_{j=1,\ldots,n}$ of positive numbers such that, for every j*

$$v_j + \alpha \le a + b\delta \sum_{i=1}^{j} v_i. \tag{11}$$

Then

$$\sup_{j\le n} v_j + \alpha \le a\mathcal{E}_\delta(b),$$

where $\mathcal{E}_\delta(b) = 1 + \sum_{p=1}^{\infty} \frac{b^p}{p}(1+\delta)\cdots(1+(p-1)\delta)$, which is a convergent series.

Notice the $\mathcal{E}_\delta(b)$ is increasing in δ and $\delta < \frac{1}{b}$, so we can replace the right hand side of (11) by a constant depending on b. We define the discrete solutions, $(Y^{p,n}_t, Z^{p,n}_t, A^{p,n}_t, K^{p,n}_t)$ by the implicit penalization scheme

$$Y^{p,n}_t = y^{p,n}_{[t/\delta]}, \qquad Z^{p,n}_t = z^{p,n}_{[t/\delta]}, \qquad A^{p,n}_t = \sum_{m=0}^{[t/\delta]} a^{p,n}_m, \qquad K^{p,n}_t = \sum_{m=0}^{[t/\delta]} k^{p,n}_m,$$

or $(\bar{Y}^{p,n}_t, \bar{Z}^{p,n}_t, \bar{A}^{p,n}_t, \bar{K}^{p,n}_t)$ by the implicit–explicit penalization scheme,

$$\bar{Y}^{p,n}_t = \bar{y}^{p,n}_{[t/\delta]}, \qquad \bar{Z}^{p,n}_t = \bar{z}^{p,n}_{[t/\delta]}, \qquad \bar{A}^{p,n}_t = \sum_{m=0}^{[t/\delta]} \bar{a}^{p,n}_m, \qquad \bar{K}^{p,n}_t = \sum_{m=0}^{[t/\delta]} \bar{k}^{p,n}_m.$$

Let us notice that the laws of the solutions (Y^p, Z^p, A^p, K^p) and $(Y^{p,n}, Z^{p,n}, A^{p,n}, K^{p,n})$ or $(\bar{Y}^{p,n}, \bar{Z}^{p,n}, \bar{A}^{p,n}, \bar{K}^{p,n})$ to penalized BSDE depend only on $(\mathbf{P}_B, \Gamma^{-1}(\mathbf{P}_B), g, \Psi_1^{-1}(\mathbf{P}_B), \Psi_2^{-1}(\mathbf{P}_B))$ and $(\mathbf{P}_{B^n}, \Gamma^{-1}(\mathbf{P}_{B^n}), g, \Psi_1^{-1}(\mathbf{P}_{B^n}), \Psi_2^{-1}(\mathbf{P}_{B^n}))$ where $\mathbf{P}_B$ (resp. $\mathbf{P}_{B^n}$) is the probability introduced by B (resp. B^n), and $f^{-1}(\mathbf{P}_B)$ (resp. $f^{-1}(\mathbf{P}_{B^n})$) is the law of $f(B)$ (resp. $f(B^n)$) for $f = \Gamma, \Psi_1, \Psi_2$. So if we concern the convergence in law, we can consider these equations on any probability space.

By Donsker's theorem and Skorokhod's representation theorem, there exists a probability space, such that $\sup_{0\le t\le T}|B^n_t - B_t| \to 0$, as $n \to \infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. So we will work on this space with respect to the filtration generated by B^n and B, trying to prove the convergence of solutions. Thanks to the convergence of B^n, (L^n, U^n) also converges to (L, U). Then we have the following result, which is based on the convergence results of numerical solutions for BSDE (cf. [9,10]) and the penalization method for reflected BSDE (Theorem 2.1).

Remark 3.1. When g is a stochastic function, we do not have a result on convergence rate from the beginning paper [9,10]. When g is a deterministic function, we can put the equation into a Markovian framework as mentioned in the Introduction. In such a case this algorithm coincides with the difference method for PDE, where there is many convergence rate results (cf. [13,14]).

Proposition 3.1. *Assuming 2.3 and 2.4 hold, the sequence $(Y^{p,n}_t, Z^{p,n}_t)$ converges to (Y_t, Z_t) in the following sense*

$$\lim_{p\to\infty}\lim_{n\to\infty} E\left[\sup_{0\le t\le T}|Y^{p,n}_t - Y_t|^2 + \int_0^T |Z^{p,n}_s - Z_s|^2 ds\right] \to 0, \tag{12}$$

and for $0 \le t \le T$, $A^{p,n}_t - K^{p,n}_t \to A_t - K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, as $n \to \infty$, $p \to \infty$.

Remark 3.2. From later studies, we know that the order of taking limits is not important, by the convergence of reflected algorithm(s). In fact, in practice, we can choose p independent of n, i.e. we can choose p much larger than n. This is shown in the simulation.

Proof. Notice

$$E\left[\sup_{0\le t\le T}|Y_t^{p,n}-Y_t|^2+\int_0^T|Z_s^{p,n}-Z_s|^2ds\right]\le 2E\left[\sup_{0\le t\le T}|Y_t^{p,n}-Y_t^p|^2+\int_0^T|Z_s^{p,n}-Z_s^p|^2ds\right]$$
$$+2E\left[\sup_{0\le t\le T}|Y_t^p-Y_t|^2+\int_0^T|Z_s^p-Z_s|^2ds\right].$$

By the convergence results of numerical solutions for BSDE (cf. [9,10]), the first expectation tends to 0. For the second expectation, it is a direct application of Theorem 2.1 of the penalization method. So we get (12). For the increasing processes, we have

$$\begin{aligned}E[((A_t^{p,n}-K_t^{p,n})-(A_t-K_t))^2]&\le 2E[((A_t^{p,n}-K_t^{p,n})-(A_t^p-K_t^p))^2]+2E[((A_t^p-K_t^p)-(A_t-K_t))^2]\\&\le 2E[((A_t^{p,n}-K_t^{p,n})-(A_t^p-K_t^p))^2]+\frac{C}{\sqrt{p}},\end{aligned}$$

in view of (7). While for fixed p,

$$A_t^{p,n}-K_t^{p,n}=Y_0^{p,n}-Y_t^{p,n}-\int_0^t g(s,Y_s^{p,n},Z_s^{p,n})ds+\int_0^t Z_s^{p,n}dB_s^n,$$
$$A_t^p-K_t^p=Y_0^p-Y_t^p-\int_0^t g(s,Y_s^p,Z_s^p)ds+\int_0^t Z_s^p dB_s,$$

from Corollary 14 in [10], we know that $\int_0^\cdot Z_s^{p,n}dB_s^n$ converges to $\int_0^\cdot Z_s^p dB_s$ in $\mathbf{S}^2(0,T)$, as $n\to\infty$, then with the Lipschitz condition of g and the convergence of $Y^{p,n}$, we get $(A_t^{p,n}-K_t^{p,n})\to(A_t-K_t)$ in $\mathbf{L}^2(\mathcal{F}_t)$, as $n\to\infty$, $p\to\infty$. □

Now we consider the implicit–explicit penalization scheme. From Proposition 5 in [8], we know that for the implicit–explicit scheme, the difference between this solution and the totally implicit one depends on $\mu+p$ for fixed $p\in\mathbb{N}$. So we have

Proposition 3.2. *For any $p\in\mathbb{N}$, when $n\to\infty$,*

$$E\left[\sup_{0\le t\le T}|\overline{Y}_t^{p,n}-Y_t^{p,n}|^2\right]+\int_0^T|\overline{Z}_s^{p,n}-Z_s^{p,n}|^2ds\to 0,$$

with $(\overline{A}_t^{p,n}-\overline{K}_t^{p,n})-(A_t^{p,n}-K_t^{p,n})\to 0$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0\le t\le T$.

Proof. The convergence of $(\overline{Y}_t^{p,n},\overline{Z}_t^{p,n})$ to $(Y_t^{p,n},Z_t^{p,n})$ is a direct consequence of Proposition 5 in [8]. More precisely, there exists a constant C which depends only on μ and T, such that

$$E\left[\sup_{0\le t\le T}|\overline{Y}_t^{p,n}-Y_t^{p,n}|^2\right]+E\int_0^T|\overline{Z}_s^{p,n}-Z_s^{p,n}|^2ds\le C\delta^2.$$

The rest is to consider the convergence of the increasing processes, notice that for $0\le t\le T$,

$$\overline{A}_t^{p,n}-\overline{K}_t^{p,n}=\overline{Y}_0^{p,n}-\overline{Y}_t^{p,n}-\int_0^t g(s,\overline{Y}_s^{p,n},\overline{Z}_s^{p,n})ds+\int_0^t\overline{Z}_s^{p,n}dB_s^n,$$

compare with $A_t^{p,n}-K_t^{p,n}=Y_0^{p,n}-Y_t^{p,n}-\int_0^t g(s,Y_s^{p,n},Z_s^{p,n})ds+\int_0^t Z_s^{p,n}dB_s^n$, thanks to the Lipschitz condition of g and the convergence of $(\overline{Y}^{p,n},\overline{Z}^{p,n})$, we get $\overline{A}_t^{p,n}-\overline{K}_t^{p,n}\to A_t^{p,n}-K_t^{p,n}$, in $\mathbf{L}^2(\mathcal{F}_t)$, as $n\to\infty$, for fixed p. So the result follows. □

Remark 3.3. From this proposition and Proposition 3.1, we get the convergence of the implicit–explicit penalization scheme.

Before going further, we prove an *a-priori* estimation of $(y^{p,n},z^{p,n},a^{p,n},k^{p,n})$. This result will help us to get the convergence of the reflected scheme, which will be discussed in the next section. Throughout this paper, we use $C_{\phi,\psi,\ldots}$ to denote a constant which depends on $\phi,\psi,\ldots$. Here $\phi,\psi,\ldots$ can be random variables or stochastic processes.

Lemma 3.2. *For each $p\in\mathbb{N}$ and δ such that $\delta(1+2\mu+2\mu^2)<1$, there exists a constant c such that*

$$E\left[\sup_j|y_j^{p,n}|^2+\sum_{j=0}^n|z_j^{p,n}|^2\delta+\frac{1}{p\delta}\sum_{j=0}^n|a_j^{p,n}|^2+\frac{1}{p\delta}\sum_{j=0}^n|k_j^{p,n}|^2\right]\le cC_{\xi^n,g,L^n,U^n}.$$

Here C_{ξ^n,g^n,L^n,U^n} depends on ξ^n, g^n, $(L^n)^+$ and $(U^n)^-$, while c depends only on μ and T.

Proof. Recall (9), we apply the 'discrete Itô formula' (cf. [11]) for $(y_j^{p,n})^2$, and get

$$E\left[|y_j^{p,n}|^2+\sum_{i=j}^{n-1}|z_i^{p,n}|^2\delta\right]\le E[|\xi^n|^2]+2\left[\sum_{i=j}^{n-1}y_i^{p,n}|g_i^n(y_i^{p,n},z_i^{p,n})|\delta\right]+2E\sum_{i=j}^{n-1}(y_i^{p,n}\cdot a_i^{p,n}-y_i^{p,n}\cdot k_i^{p,n}).$$

Since $y_i^{p,n}\cdot a_i^{p,n}=-p\delta((y_i^{p,n}-L_i^n)^-)^2+p\delta L_i^n(y_i^{p,n}-L_i^n)^-=\frac{1}{p\delta}a_i^{p,n}+L_i^na_i^{p,n}$ and $y_i^{p,n}\cdot k_i^{p,n}=p\delta((y_i^{p,n}-U_i^n)^+)^2+U_i^np\delta(y_i^{p,n}-U_i^n)^+=\frac{1}{p\delta}k_i^{p,n}+U_i^nk_i^{p,n}$, we have

$$\begin{aligned}
&E\left[|y_j^{p,n}|^2+\frac{1}{2}\sum_{i=j}^{n-1}|z_i^{p,n}|^2\delta\right]+2E\left[\frac{1}{p\delta}\sum_{i=j}^{n-1}(a_i^{p,n})^2+\frac{1}{p\delta}\sum_{i=j}^{n-1}(k_i^{p,n})^2\right]\\
&\le E\left[|\xi^n|^2+\sum_{i=j}^{n-1}|g_i^n(0,0)|^2\delta+(1+2\mu+2\mu^2)\sum_{i=j}^{n-1}|y_i^{p,n}|^2\delta+2\sum_{i=j}^{n-1}(L_i^n)^+a_i^{p,n}+2\sum_{i=j}^{n-1}(U_i^n)^-k_i^{p,n}\right]\\
&\le E\left[|\xi^n|^2+\sum_{i=j}^{n-1}|g_i^n(0,0)|^2\delta\right]+(1+2\mu+2\mu^2)\delta E\sum_{i=j}^{n-1}|y_i^{p,n}|^2+\frac{1}{\alpha}E\left(\sum_{i=j}^{n-1}a_i^{p,n}\right)^2\\
&\quad+\alpha E\sup_{j\le i\le n-1}((L_i^n)^+)^2+\frac{1}{\beta}E\left(\sum_{i=j}^{n-1}k_i^{p,n}\right)^2+\beta E\sup_{j\le i\le n-1}((U_i^n)^+)^2.
\end{aligned}$$

Since L^n and U^n are approximations of Itô processes, we can find a process X_j^n of the form $X_j^n=X_0-\sum_{i=0}^{j-1}\sigma_i\varepsilon_{i+1}^n\sqrt{\delta}+V_j^{+n}-V_j^{-n}$, where $V_j^{\pm n}$ are $\mathcal{G}_j^n$-adapted increasing processes with $E[|V_n^{+n}|^2+|V_n^{-n}|^2]<+\infty$, and $L_j^n\le X_j^n\le U_j^n$ holds. Then applying similar techniques of stopping times as in the proof of Lemma 2 in [4] for the discrete case with $L_j^n\le X_j^n\le U_j^n$, we can prove

$$E\left(\sum_{i=j}^{n-1}a_i^{p,n}\right)^2+E\left(\sum_{i=j}^{n-1}k_i^{p,n}\right)^2\le 3\mu\left(C_{\xi^n,g^n,X^n}+E\sum_{i=j}^{n-1}\left[|y_i^{p,n}|^2+|z_i^{p,n}|^2\right]\delta\right).$$

While X^n can be dominated by L^n and U^n, we can replace it by L^n and U^n. Set $\alpha=\beta=12\mu$ in the previous inequality, with Lemma 3.1, we get

$$\sup_j E\left[|y_j^{p,n}|^2\right]+E\left[\sum_{i=0}^{n-1}|z_i^{p,n}|^2\delta\right]+\frac{1}{p\delta}\sum_{i=0}^{n-1}(a_i^{p,n})^2+\frac{1}{p\delta}\sum_{i=0}^{n-1}(k_i^{p,n})^2\le cC_{\xi^n,g^n,L^n,U^n}.$$

We reconsider the Itô formula for $|y_j^{p,n}|^2$, and take $\sup_j$ before expectation. Using the Burkholder–Davis–Gundy inequality for martingale part $\sum_{i=0}^{j}y_i^{p,n}z_i^{p,n}\sqrt{\delta}\varepsilon_{i+1}^n$, with similar techniques, we get

$$E\left[\sup_j|y_j^{p,n}|^2\right]\le C_{\xi^n,g,L^n,U^n}+C_\mu E\left[\sum_{i=0}^{n-1}|y_i^{p,n}|^2\delta\right]\le C_{\xi^n,g^n,L^n,U^n}+C_\mu T\sup_j E[|y_j^{p,n}|^2].$$

It follows the desired results. □

4. Reflected algorithms and their convergence

4.1. Reflected schemes

This type of numerical schemes is based on reflecting the solution y^n between two barriers by a^n and k^n directly. In such a way the discrete solution y^n really stays between two barriers L^n and U^n. Compared with the penalized method, the numerical solution of the reflected algorithm is truly controlled between two barriers and we can see better how the increasing processes work during the time interval. After discretization of the time interval, our discrete reflected BSDE with two barriers on small interval $[t_j,t_{j+1}]$, for $j=0,1,\dots,n-1$, is

$$y_j^n=y_{j+1}^n+g_j^n(y_j^n,z_j^n)\delta+a_j^n-k_j^n-z_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \tag{13}$$

with terminal condition $y_n^n=\xi^n$, and constraint and discrete integral conditions hold:

$$\begin{aligned}
&a_j^n\ge 0,\ k_j^n\ge 0,\quad a_j^n\cdot k_j^n=0,\\
&L_j^n\le y_j^n\le U_j^n,\quad (y_j^n-L_j^n)a_j^n=(y_j^n-U_j^n)k_j^n=0.
\end{aligned}\tag{14}$$

Note that, all terms in (13) are $\mathcal{G}_j^n$-measurable except y_{j+1}^n and ε_{j+1}^n.

Since our calculation is backward, the key point of our numerical schemes is how to solve $(y_j^n, z_j^n, a_j^n, k_j^n)$ from (13) using the $\mathcal{G}_{j+1}^n$-measurable random variable y_{j+1}^n obtained in the preceding step. First z_j^n is obtained by

$$z_j^n = E[y_{j+1}^n \varepsilon_{j+1}^n | \mathcal{G}_j^n] = \frac{1}{2\sqrt{\delta}}(y_{j+1}^n|_{\varepsilon_{j+1}^n=1} - y_{j+1}^n|_{\varepsilon_{j+1}^n=-1}).$$

Then (13) with (14) becomes

$$\begin{aligned} y_j^n &= E[y_{j+1}^n|\mathcal{G}_j^n] + g_j^n(y_j^n, z_j^n)\delta + a_j^n - k_j^n, \quad a_j^n \geq 0,\ k_j^n \geq 0, \\ L_j^n &\leq y_j^n \leq U_j^n, \quad (y_j^n - L_j^n)a_j^n = (y_j^n - U_j^n)k_j^n = 0. \end{aligned} \tag{15}$$

Set $\Theta(y) := y - g(t_j, y, z_j^n)\delta$. In view of $\langle \Theta(y) - \Theta(y'), y - y' \rangle \geq (1 - \delta\mu)|y - y'|^2 > 0$, for δ small enough, we get that in such a case $\Theta(y)$ is strictly increasing in y. So

$$\begin{aligned} y \geq L_j^n &\iff \Theta(y) \geq \Theta(L_j^n), \\ y \leq U_j^n &\iff \Theta(y) \leq \Theta(U_j^n). \end{aligned}$$

Then the implicit reflected scheme gives the results with $E[y_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2}(y_{j+1}^n|_{\varepsilon_{j+1}^n=1} + y_{j+1}^n|_{\varepsilon_{j+1}^n=-1})$ as follows

$$\begin{aligned} y_j^n &= \Theta^{-1}(E[y_{j+1}^n|\mathcal{G}_j^n] + a_j^n - k_j^n), \\ a_j^n &= (E[y_{j+1}^n|\mathcal{G}_j^n] + g_j^n(L_j^n, z_j^n)\delta - L_j^n)^-, \\ k_j^n &= (E[y_{j+1}^n|\mathcal{G}_j^n] + g_j^n(U_j^n, z_j^n)\delta - U_j^n)^+, \end{aligned}$$

on the set $\{L_j^n < U_j^n\}$, then we know that $\{y_j^n - L_j^n = 0\}$ and $\{y_j^n - U_j^n = 0\}$ are disjoint. So with $(y_j^n - L_j^n)a_j^n = (y_j^n - U_j^n)k_j^n = 0$, we have $a_j^n \cdot k_j^n = 0$. On the set $\{L_j^n = U_j^n\}$, we get $a_j^n = (I_j^n)^+$ and $k_j^n = (I_j^n)^-$ by definition, where $I_j^n := E[y_{j+1}^n|\mathcal{G}_j^n] + g_j^n(L_j^n, z_j^n)\delta - L_j^n$. So automatically $a_j^n \cdot k_j^n = 0$.

In many cases, the inverse of mapping Θ is not easy to solve directly, e.g. g is not a linear function on y, like $g(y) = \sin(y)$. So we introduce the explicit reflected scheme to handle such a situation and improve efficiency. The key point is of the explicit reflected scheme to replace y_j^n in g_j^n by $E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]$ in (15). So we get the following equation,

$$\begin{aligned} \bar{y}_j^n &= E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n, \quad \bar{a}_j^n \geq 0,\ \bar{k}_j^n \geq 0, \\ L_j^n &\leq \bar{y}_j^n \leq U_j^n, \quad (\bar{y}_j^n - L_j^n)\bar{a}_j^n = (\bar{y}_j^n - U_j^n)\bar{k}_j^n = 0. \end{aligned} \tag{16}$$

Then with $E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] = \frac{1}{2}(\bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=1} + \bar{y}_{j+1}^n|_{\varepsilon_{j+1}^n=-1})$, we get the solution

$$\begin{aligned} \bar{y}_j^n &= E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n, \\ \bar{a}_j^n &= (E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta - L_j^n)^-, \\ \bar{k}_j^n &= (E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta - U_j^n)^+. \end{aligned} \tag{17}$$

4.2. Convergence of reflected implicit schemes

Now we study the convergence of reflected schemes. For the implicit reflected scheme, we denote

$$Y_t^n = y_{[t/\delta]}^n, \qquad Z_t^n = z_{[t/\delta]}^n, \qquad A_t^n = \sum_{i=0}^{[t/\delta]} a_i^n, \qquad K_t^n = \sum_{i=0}^{[t/\delta]} k_i^n,$$

and for the explicit reflected scheme

$$\bar{Y}_t^n = \bar{y}_{[t/\delta]}^n, \qquad \bar{Z}_t^n = \bar{z}_{[t/\delta]}^n, \qquad \bar{A}_t^n = \sum_{i=0}^{[t/\delta]} \bar{a}_i^n, \qquad \bar{K}_t^n = \sum_{i=0}^{[t/\delta]} \bar{k}_i^n.$$

First we prove an estimation result for (y^n, z^n, a^n, k^n).

Lemma 4.1. *For δ such that $\delta(1 + 2\mu + 2\mu^2) < 1$, there exists a constant c depending only on μ and T such that*

$$E\left[\sup_j |y_j^n|^2 + \sum_{j=0}^{n-1} |z_j^n|^2\delta + \left|\sum_{j=0}^{n-1} a_j^n\right|^2 + \left|\sum_{j=0}^{n-1} k_j^n\right|^2\right] \leq cC_{\xi^n, g, L^n, U^n}.$$

Proof. First we consider the estimation of a_i^n and k_i^n. In view of $L_j^n \le Y_j^n \le U_j^n$, we have that

$$\begin{aligned} a_j^n &\le (E[L_{j+1}^n|\mathcal{G}_j^n] + g_j^n(L_j^n, z_j^n)\delta - L_j^n)^- = \delta(l_j + g_j^n(L_j^n, z_j^n))^-, \\ k_j^n &\le (E[U_{j+1}^n|\mathcal{G}_j^n] + g_j^n(U_j^n, z_j^n)\delta - U_j^n)^+ = \delta(u_j + g_j^n(U_j^n, z_j^n))^+. \end{aligned} \tag{18}$$

We consider the following discrete BSDEs with $\widehat{y}_n^n = \widetilde{y}_n^n = \xi^n$,

$$\begin{aligned} \widehat{y}_j^n &= \widehat{y}_{j+1}^n + [g_j^n(\widehat{y}_j^n, \widehat{z}_j^n) + (l_j)^- + g_j^n(L_j^n, \widehat{z}_j^n)^-]\delta - \widehat{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \\ \widetilde{y}_j^n &= \widetilde{y}_{j+1}^n + [g_j^n(\widetilde{y}_j^n, \widetilde{z}_j^n) - (u_j)^+ - g_j^n(U_j^n, \widetilde{z}_j^n)^+]\delta - \widetilde{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n. \end{aligned}$$

Thanks to the discrete comparison theorem in [11], we have $\widetilde{y}_j^n \le y_j^n \le \widehat{y}_j^n$, so

$$E\left[\sup_j |y_j^n|^2\right] \le \max\left\{E\left[\sup_j |\widehat{y}_j^n|^2\right], E\left[\sup_j |\widetilde{y}_j^n|^2\right]\right\} \le cC_{\xi^n, g, L^n, U^n}. \tag{19}$$

The last inequality follows from estimations of the discrete solution of classical BSDE $(\widehat{y}_j^n)^2$ and $(\widetilde{y}_j^n)^2$, which is obtained by Itô formulae and the discrete Gronwall inequality in Lemma 3.1. For z_j^n, we use the discrete Itô formula' (cf. [11]) again for $(y_j^n)^2$, and get

$$\begin{aligned} E|y_j^n|^2 + \sum_{i=j}^{n-1} |z_i^n|^2\delta &= E\left[|\xi^n|^2 + 2\sum_{i=j}^{n-1} y_i^n g_i^n(y_i^n, z_i^n)\delta + 2\sum_{i=j}^{n-1} y_i^n a_i^n - 2\sum_{i=j}^{n-1} y_i^n k_i^n\right] \\ &\le E\left[|\xi^n|^2 + \sum_{i=j}^{n-1} |g_i^n(0,0)|^2\delta + \delta(1 + 2\mu + 2\mu^2)\sum_{i=j}^{n-1} |y_i^n|^2 + \frac{1}{2}\sum_{i=j}^{n-1} |z_i^n|^2\delta\right] \\ &\quad + \alpha E\left[\sup_j((L_j^n)^+)^2 + \sup_j((U_j^n)^-)^2\right] + \frac{1}{\alpha}E\left[\left(\sum_{i=j}^{n-1} a_i^n\right)^2 + \left(\sum_{i=j}^{n-1} k_i^n\right)^2\right], \end{aligned}$$

using $(y_i^n - L_i^n)a_i^n = 0$ and $(y_i^n - U_i^n)k_i^n = 0$. And from (18), we have

$$\begin{aligned} E\left(\sum_{i=j}^{n-1} a_i^n\right)^2 &\le 4\delta E\sum_{i=j}^{n-1}[(l_i)^2 + g_i^n(0,0)^2 + \mu|L_i^n|^2 + \mu|z_i^n|^2], \\ E\left(\sum_{i=j}^{n-1} k_i^n\right)^2 &\le 4\delta E\sum_{i=j}^{n-1}[(u_i)^2 + g_i^n(0,0)^2 + \mu|U_i^n|^2 + \mu|z_i^n|^2]. \end{aligned}$$

Set $\alpha = 32\mu$, it follows

$$\begin{aligned} E\left[|y_j^n|^2 + \frac{1}{4}\sum_{i=j}^{n-1} |z_i^n|^2\delta\right] &\le E[|\xi^n|^2 + \left(1 + \frac{1}{8\mu^2}\right)\sum_{i=j}^{n-1} |g_i^n(0,0)|^2\delta] + \delta(1 + 2\mu + 2\mu^2)\sum_{i=j}^{n-1} |y_i^n|^2 \\ &\quad + 32\mu^2 E\left[\sup_j((L_j^n)^+)^2 + \sup_j((U_j^n)^-)^2\right] + \frac{1}{8\mu^2}E\sum_{i=j}^{n-1}[(l_i)^2 + (u_i)^2] \\ &\quad + \frac{1}{8}\delta E\sum_{i=j}^{n-1}\left[|L_i^n|^2 + |U_i^n|^2\right]. \end{aligned}$$

With (19), we obtain $\sum_{i=0}^{n-1} |z_i^n|^2\delta \le cC_{\xi^n, g^n, L^n, U^n}$. Then applying these estimations to (18), we obtain the desired results. □

With arguments similar to those preceding Proposition 3.1, the laws of the solutions (Y, Z, A, K) and (Y^n, Z^n, A^n, K^n) or $(\bar{Y}^n, \bar{Z}^n, \bar{A}^n, \bar{K}^n)$ to reflected BSDE depend only on $(\mathbf{P}_B, \Gamma^{-1}(\mathbf{P}_B), g, \Psi_1^{-1}(\mathbf{P}_B), \Psi_2^{-1}(\mathbf{P}_B))$ and $(\mathbf{P}_{B^n}, \Gamma^{-1}(\mathbf{P}_{B^n}), g, \Psi_1^{-1}(\mathbf{P}_{B^n}), \Psi_2^{-1}(\mathbf{P}_{B^n}))$ where $f^{-1}(\mathbf{P}_B)$ (resp. $f^{-1}(\mathbf{P}_{B^n})$) is the law of $f(B)$ (resp. $f(B^n)$) for $f = \Gamma, \Psi_1, \Psi_2$. So if we concern the convergence in law, we can consider these equations on any probability space.

From Donsker's theorem and Skorokhod's representation theorem, there exists a probability space satisfying $\sup_{0\le t\le T} |B_t^n - B_t| \to 0$, as $n \to \infty$, in $\mathbf{L}^2(\mathcal{F}_T)$, since ε_k is in $\mathbf{L}^{2+\delta}$. And it is sufficient for us to prove convergence results in this probability space. Our convergence result for the implicit reflected scheme is as follows:

Theorem 4.1. *Under Assumption 2.3 and suppose moreover that g satisfies Lipschitz condition (1), we have when* $n \to +\infty$,

$$E[\sup_t |Y_t^n - Y_t|^2] + E\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t \to 0, \tag{20}$$

and $A_t^n - K_t^n \to A_t - K_t$ *in* $\mathbf{L}^2(\mathcal{F}_t)$, *for* $0 \le t \le T$.

Proof. The proof is done in three steps.

In the first step, we consider the difference between discrete solutions of the reflecting implicit scheme and of penalization implicit scheme introduce in Sections 4.1 and 3.1, respectively. More precisely, we will prove that for each p,

$$E\left[\sup_j |y_j^n - y_j^{p,n}|^2\right] + \delta E\sum_{j=0}^{n-1} |z_j^n - z_j^{p,n}|^2 \le cC_{\xi^n,g^n,L^n,U^n}\frac{1}{\sqrt{p}}. \tag{21}$$

Here c only depends on μ and T. From (9) and (13), applying the'discrete Itô formula' (cf. [11]) to $(y_j^n - y_j^{p,n})^2$, we get

$$\begin{aligned} E|y_j^n - y_j^{p,n}|^2 + \delta E\sum_{i=j}^{n-1}|z_i^n - z_i^{p,n}|^2 &= 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(g_i^n(y_i^n, z_i^n) - g_i^n(y_i^{p,n}, z_i^{p,n}))\delta] \\ &\quad + 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n})] - 2E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n})]. \end{aligned}$$

From (14), we have

$$\begin{aligned} (y_i^n - y_i^{p,n})(a_i^n - a_i^{p,n}) &= (y_i^n - L_i^n)a_i^n - (y_i^{p,n} - L_i^n)a_i^n - (y_i^n - L_i^n)a_i^{p,n} + (y_i^{p,n} - L_i^n)a_i^{p,n} \\ &\le (y_i^{p,n} - L_i^n)^- a_i^n - ((y_i^{p,n} - L_i^n)^-)^2, \\ &\le (y_i^{p,n} - L_i^n)^- a_i^n. \end{aligned}$$

Similarly we have $(y_i^n - y_i^{p,n})(k_i^n - k_i^{p,n}) \ge -(y_i^{p,n} - U_i^n)k_i^n$. By (18) and the Lipschitz property of g, it follows

$$\begin{aligned} &E|y_j^n - y_j^{p,n}|^2 + \frac{\delta}{2}E\sum_{i=j}^{n-1}|z_i^n - z_i^{p,n}|^2 \\ &\le (2\mu + 2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})^2] + 2E\sum_{i=j}^{n-1}[(y_i^{p,n} - L_i^n)^- a_i^n + (y_i^{p,n} - U_i^n)^+ k_i^n] \\ &\le (2\mu + 2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})^2] + 2\left(\delta E\sum_{i=j}^{n-1}((y_i^{p,n} - L_i^n)^-)^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((l_j + g_i^n(L_j^n, z_j^n))^-)^2\right)^{\frac{1}{2}} \\ &\quad + 2\left(\delta E\sum_{i=j}^{n-1}((y_i^{p,n} - U_i^n)^+)^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((u_j + g_i^n(U_j^n, z_j^n))^+)^2\right)^{\frac{1}{2}} \\ &= (2\mu + 2\mu^2)\delta E\sum_{i=j}^{n-1}[(y_i^n - y_i^{p,n})^2] + \frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}E\sum_{i=j}^{n-1}(a_i^{p,n})^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((l_j + g_i^n(L_j^n, z_j^n))^-)^2\right)^{\frac{1}{2}} \\ &\quad + \frac{2}{\sqrt{p}}\left(\frac{1}{p\delta}E\sum_{i=j}^{n-1}(k_i^{p,n})^2\right)^{\frac{1}{2}}\left(\delta E\sum_{i=j}^{n-1}((u_j + g_i^n(U_j^n, z_j^n))^+)^2\right)^{\frac{1}{2}}. \end{aligned}$$

Then by estimation results in Lemmas 3.2 and 4.1 and the discrete Gronwall inequality in Lemma 3.1, we get

$$\sup_j E|y_j^n - y_j^{p,n}|^2 + \delta E\sum_{i=0}^{n-1}|z_i^n - z_i^{p,n}|^2 \le cC_{\xi^n,g^n,L^n,U^n}\frac{1}{\sqrt{p}}.$$

Apply the B–D–G inequality, we obtain (21).

In the second step, we want to prove (20). We have

$$E\left[\sup_t |Y_t^n - Y_t|^2\right] + E\left[\int_0^T |Z_t^n - Z_t|^2 \mathrm{d}t\right]$$

$$\leq 3E\left[\sup_t |Y_t^p - Y_t|^2 + \int_0^T |Z_t^p - Z_t|^2 dt\right] + 3E\left[\sup_t |Y_t^n - Y_t^{p,n}|^2 + \int_0^T |Z_t^n - Z_t^{p,n}|^2 dt\right]$$
$$+ 3E\left[\sup_t |Y_t^p - Y_t^{p,n}|^2 + \int_0^T |Z_t^p - Z_t^{p,n}|^2 dt\right]$$
$$\leq 3Cp^{-\frac{1}{2}} + cC_{\xi^n,g,L^n,U^n}p^{-\frac{1}{2}} + 3E\left[\sup_t |Y_t^p - Y_t^{p,n}|^2 + \int_0^T |Z_t^p - Z_t^{p,n}|^2 dt\right],$$

in view of (21) and Theorem 2.1. For fixed $p > 0$, by convergence results of numerical algorithms for BSDE, (Theorem 12 in [10] and Theorem 2 in [8]), we know that the last two terms converge to 0, as $\delta \to 0$. And when δ is small enough, C_{ξ^n,g^n,L^n,U^n} is dominated by ξ^n, g^n, L and U. This implies that we can choose a suitable δ such that the right hand side is as small as we want, so (20) follows.

In the last step, we consider the convergence of (A^n, K^n). Recall that for $0 \leq t \leq T$,

$$A_t^n - K_t^n = Y_0^n - Y_t^n - \int_0^t g(s, Y_s^n, Z_s^n)ds + \int_0^t Z_s^n dB_s^n,$$
$$A_t^{p,n} - K_t^{p,n} = Y_0^{p,n} - Y_t^{p,n} - \int_0^t g(s, Y_s^{p,n}, Z_s^{p,n})ds + \int_0^t Z_s^{p,n} dB_s^n.$$

By (21) and the Lipschitz condition of g, we get

$$E[|(A_t^n - K_t^n) - (A_t^{p,n} - K_t^{p,n})|^2] \leq cC_{\xi^n,g^n,L^n,U^n}\frac{1}{\sqrt{p}}.$$

Since

$$\begin{aligned} E[|(A_t^n - K_t^n) - (A_t - K_t)|^2] &\leq 3E[|(A_t^n - K_t^n) - (A_t^{p,n} - K_t^{p,n})|^2] + 3E[|(A_t^p - K_t^p) - (A_t - K_t)|^2] \\ &\quad + 3E[|(A_t^p - K_t^p) - (A_t^{p,n} - K_t^{p,n})|^2] \\ &\leq c(C_{\xi^n,g^n,L^n,U^n} + C_{\xi,g,L,U})\frac{1}{\sqrt{p}} + 3E[|(A_t^p - K_t^p) - (A_t^{p,n} - K_t^{p,n})|^2], \end{aligned}$$

with similar techniques, we obtain $E[|(A_t^n - K_t^n) - (A_t - K_t)|^2] \to 0$. Here the fact that $(A_t^{p,n} - K_t^{p,n})$ converges to $(A_t^p - K_t^p)$ for fixed p follows from the convergence results of $(Y_t^{p,n}, Z_t^{p,n})$ to (Y_t^p, Z_t^p). □

4.3. *Convergence of the reflected explicit scheme*

Then we study the convergence of the explicit reflected scheme, which is an efficient algorithm, when g is non-linear in y and z. Before going further, we need an estimation result for $(\bar{y}^n, \bar{z}^n, \bar{a}^n, \bar{k}^n)$.

Lemma 4.2. *For δ such that $\delta\left(\frac{9}{4} + 2\mu + 4\mu^2\right) < 1$, there exists a constant c depending only on μ and T, such that*

$$E\left[\sup_j |\bar{y}_j^n|^2\right] + E\left[\sum_{j=0}^{n-1} |\bar{z}_j^n|^2\delta + \left|\sum_{j=0}^{n-1} \bar{k}_j^n\right|^2 + \left|\sum_{j=0}^{n-1} \bar{a}_j^n\right|^2\right] \leq cC_{\xi^n,g,L^n,U^n}.$$

Proof. We recall the explicit reflected scheme, which is: For $0 \leq j \leq N-1$

$$\begin{aligned} &\bar{y}_j^n = \bar{y}_{j+1}^n + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n - \bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \quad \bar{a}_j^n \geq 0,\ \bar{k}_j^n \geq 0, \\ &L_j^n \leq \bar{y}_j^n \leq U_j^n, \quad (\bar{y}_j^n - L_j^n)\bar{a}_j^n = (\bar{y}_j^n - U_j^n)\bar{k}_j^n = 0. \end{aligned} \tag{22}$$

Then we have

$$\begin{aligned} |\bar{y}_j^n|^2 &= |\bar{y}_{j+1}^n|^2 - |\bar{z}_j^n|^2\delta + 2\bar{y}_{j+1}^n \cdot g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + 2\bar{y}_j^n \cdot \bar{a}_j^n - 2\bar{y}_j^n \cdot \bar{k}_j^n \\ &\quad + |g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)|^2\delta^2 - (\bar{a}_j^n)^2 - (\bar{k}_j^n)^2 - 2\bar{y}_j^n\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n \\ &\quad + 2g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\bar{z}_j^n\delta\sqrt{\delta}\varepsilon_{j+1}^n - 2(\bar{a}_j^n - \bar{k}_j^n)\bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n. \end{aligned} \tag{23}$$

In view of $(\bar{y}_j^n - L_j^n)\bar{a}_j^n = (\bar{y}_j^n - U_j^n)\bar{k}_j^n = 0$, $\bar{a}_j^n$ and $\bar{k}_j^n \geq 0$, and taking expectation, we have

$$\begin{aligned} E|\bar{y}_j^n|^2 + E|\bar{z}_j^n|^2\delta &\leq E|\bar{y}_{j+1}^n|^2 + 2E[\bar{y}_{j+1}^n \cdot g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)]\delta + 2E[(L_j^n)^+ \cdot \bar{a}_j^n] + E[(U_j^n)^- \cdot \bar{k}_j^n] \\ &\quad + E[|g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)|^2\delta^2] \end{aligned}$$

$$\leq E|\bar{y}^n_{j+1}|^2 + (\delta + 3\delta^2)E[|g^n_j(0,0)|^2] + \left(\frac{1}{4}\delta + 3\mu^2\delta^2\right)E[(\bar{z}^n_j)^2]$$
$$+ \delta(1 + 2\mu + 4\mu^2 + 3\mu^2\delta)E|\bar{y}^n_{j+1}|^2 + 2E[(L^n_j)^+ \cdot \bar{a}^n_j] + E[(U^n_j)^- \cdot \bar{k}^n_j].$$

Taking the sum for $j = i, \ldots, n-1$ yields

$$E|\bar{y}^n_i|^2 + \frac{1}{2}\sum_{j=i}^{n-1} E|\bar{z}^n_j|^2\delta \leq E|\xi^n|^2 + (\delta + 3\delta^2)E\sum_{j=i}^{n-1}[|g^n_j(0,0)|^2] + \delta(1 + 2\mu + 4\mu^2 + 3\mu^2\delta)E\sum_{j=i}^{n-1}|\bar{y}^n_{j+1}|^2$$
$$+ \alpha E\left[\sup_j((L^n_j)^+)^2 + \sup_j((U^n_j)^+)^2\right] + \frac{1}{\alpha}E\left[\left(\sum_{j=i}^{n-1}\bar{a}^n_j\right)^2 + \left(\sum_{j=i}^{n-1}\bar{k}^n_j\right)^2\right], \tag{24}$$

where α is a constant to be decided later. From (17), we have

$$\bar{a}^n_j \leq (E[L^n_{j+1}|\mathcal{G}^n_j] + g^n_j(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j], \bar{z}^n_j)\delta - L^n_j)^- = (l_j + g^n_j(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j], \bar{z}^n_j))^-\delta,$$
$$\bar{k}^n_j \leq (E[U^n_{j+1}|\mathcal{G}^n_j] + g^n_j(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j], \bar{z}^n_j)\delta - U^n_j)^+ = (u_j + g^n_j(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j], \bar{z}^n_j))^-\delta.$$

Then we get

$$E\left(\sum_{j=i}^{n-1}\bar{a}^n_j\right)^2 \leq 4\delta E\sum_{j=i}^{n-1}[(l_j)^2 + g^n_j(0,0)^2 + \mu^2(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j])^2 + \mu^2(\bar{z}^n_j)^2], \tag{25}$$
$$E\left(\sum_{j=i}^{n-1}\bar{k}^n_j\right)^2 \leq 4\delta E\sum_{j=i}^{n-1}[(u_j)^2 + g^n_j(0,0)^2 + \mu^2(E[\bar{y}^n_{j+1}|\mathcal{G}^n_j])^2 + \mu^2(\bar{z}^n_j)^2].$$

Set $\alpha = 32\mu^2$ in (24), it follows

$$E|\bar{y}^n_i|^2 + \frac{1}{4}\sum_{j=i}^{n-1} E|\bar{z}^n_j|^2\delta \leq E|\xi^n|^2 + \left(\delta + \frac{1}{4\mu^2}\delta + 3\delta^2\right)E\sum_{j=i}^{n-1}[|g^n_j(0,0)|^2] + 32\mu^2 E\left[\sup_j((L^n_j)^+)^2 + \sup_j((U^n_j)^+)^2\right]$$
$$+ \delta\left(\frac{5}{4} + 2\mu + 4\mu^2 + 3\mu^2\delta\right)E\sum_{j=i}^{n-1}|\bar{y}^n_{j+1}|^2 + \frac{1}{8\mu^2}\delta E\sum_{j=i}^{n-1}[(u^l_i)^2 + (u^u_i)^2].$$

Notice that $3\mu^2\delta < 1$, so $3\mu^2\delta^2 < \delta$. Then by applying the discrete Gronwall inequality in Lemma 3.1, and the estimation of $\bar{a}^n_j$ and $\bar{k}^n_j$ follows from (25), we get

$$\sup_j E\left[|\bar{y}^n_j|^2\right] + E\left[\sum_{j=0}^{n-1}|\bar{z}^n_j|^2\delta + \left|\sum_{j=0}^{n-1}\bar{k}^n_j\right|^2 + \left|\sum_{j=0}^{n-1}\bar{a}^n_j\right|^2\right] \leq cC_{\xi^n,g^n,L^n,U^n}.$$

We reconsider (23) as before, taking sum and $\sup_j$, then taking the expectation, using the Burkholder–Davis–Gundy inequality for the martingale part, with similar techniques, we get

$$E\left[\sup_j|\bar{y}^n_j|^2\right] \leq C_{\xi^n,g^n,L^n,U^n} + C_\mu E\sum_{j=0}^{n-1}|\bar{y}^n_j|^2\delta \leq E\left[\sup_j|\bar{y}^n_j|^2\right] \leq C_{\xi^n,g^n,L^n,U^n} + C_\mu T\sup_j E\left[|\bar{y}^n_j|^2\right],$$

which implies the final result. □

Then our convergence result for the explicit reflected scheme is

Theorem 4.2. *Under the same assumptions as in Theorem 4.1, when $n \to +\infty$,*

$$E\left[\sup_t|\overline{Y}^n_t - Y_t|^2\right] + E\int_0^T|\overline{Z}^n_t - Z_t|^2\mathrm{d}t \to 0. \tag{26}$$

And $\overline{A}^n_t - \overline{K}^n_t \to A_t - K_t$ in $\mathbf{L}^2(\mathcal{F}_t)$, for $0 \leq t \leq T$.

Proof. Thanks to Theorem 4.1, it is sufficient to prove that as $n \to +\infty$,

$$E\left[\sup_j|\bar{y}^n_j - y^n_j|^2\right] + E\sum_{j=0}^{n-1}|\bar{z}^n_j - z^n_j|^2\delta \to 0. \tag{27}$$

Since

$$y_j^n = y_{j+1}^n + g_j^n(y_j^n, z_j^n)\delta + a_j^n - k_j^n - z_j^n\sqrt{\delta}\varepsilon_{j+1}^n, \qquad (28)$$
$$\bar{y}_j^n = E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n - \bar{z}_j^n\sqrt{\delta}\varepsilon_{j+1}^n,$$

we get

$$\begin{aligned} E|y_j^n - \bar{y}_j^n|^2 &= E|y_{j+1}^n - \bar{y}_{j+1}^n|^2 - \delta E|z_j^n - \bar{z}_j^n|^2 + 2\delta E[(y_j^n - \bar{y}_j^n)(g_j^n(y_j^n, z_j^n) - g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n))] \\ &\quad - E[\delta(g_j^n(y_j^n, z_j^n) - g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)) + (a_j^n - \bar{a}_j^n) - (k_j^n - \bar{k}_j^n)]^2 \\ &\quad + 2E[(y_j^n - \bar{y}_j^n)(a_j^n - \bar{a}_j^n)] - 2E[(y_j^n - \bar{y}_j^n)(k_j^n - \bar{k}_j^n)] \\ &\le E|y_{j+1}^n - \bar{y}_{j+1}^n|^2 - \delta E|z_j^n - \bar{z}_j^n|^2 + 2\delta E[(y_j^n - \bar{y}_j^n)(g_j^n(y_j^n, z_j^n) - g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n))] \end{aligned}$$

in view of

$$\begin{aligned} (y_j^n - \bar{y}_j^n)(a_j^n - \bar{a}_j^n) &= (y_j^n - L_j^n)a_j^n + (\bar{y}_j^n - L_j^n)(\bar{a}_j^n) - (\bar{y}_j^n - L_j^n)a_j^n - (y_j^n - L_j^n)(\bar{a}_j^n) \\ &\le 0 \\ (y_j^n - \bar{y}_j^n)(k_j^n - \bar{k}_j^n) &= (y_j^n - U_j^n)k_j^n + (\bar{y}_j^n - U_j^n)\bar{k}_j^n - (y_j^n - U_j^n)\bar{k}_j^n - (\bar{y}_j^n - U_j^n)k_j^n \\ &\ge 0. \end{aligned}$$

We take sum over j from i to $n-1$, with $\xi^n - \bar{\xi}^n = 0$, then we get

$$\begin{aligned} E|y_j^n - \bar{y}_j^n|^2 + \delta\sum_{j=i}^{n-1} E|z_j^n - \bar{z}_j^n|^2 &\le 2\delta\sum_{j=i}^{n-1} E[(y_j^n - \bar{y}_j^n)(g_j^n(y_j^n, z_j^n) - g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n))] \\ &\le 2\mu^2\delta E\sum_{j=i}^{n-1} |y_j^n - \bar{y}_j^n|^2 + \frac{\delta}{2}\sum_{j=i}^{n-1} E|z_j^n - \bar{z}_j^n|^2 \\ &\quad + 2\mu\delta E\sum_{j=i}^{n-1} |y_j^n - \bar{y}_j^n| \cdot |y_j^n - E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]|. \end{aligned}$$

Since $\bar{y}_j^n - E[\bar{y}_{j+1}^n|\mathcal{G}_j^n] = g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], \bar{z}_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n$, we have

$$\begin{aligned} 2\mu\delta E\left[|y_j^n - \bar{y}_j^n| \cdot |y_j^n - E[\bar{y}_{j+1}^n|\mathcal{G}_j^n]|\right] &= 2\mu\delta E\left[|y_j^n - \bar{y}_j^n| \cdot |y_j^n - \bar{y}_j^n + g_j^n(E[\bar{y}_{j+1}^n|\mathcal{G}_j^n], z_j^n)\delta + \bar{a}_j^n - \bar{k}_j^n|\right] \\ &\le (2\mu + 1)\delta E\left[|y_j^n - \bar{y}_j^n|^2\right] + \mu^2\delta E\Big[3\delta^2|g_j^n(0,0)|^2 + \mu^2|\bar{y}_{j+1}^n|^2 \\ &\quad + \mu^2|z_j^n|^2 + (\bar{a}_j^n)^2 + (\bar{k}_j^n)^2\Big]. \end{aligned}$$

Then by Lemma 4.2, we obtain

$$E|y_j^n - \bar{y}_j^n|^2 + \frac{\delta}{2}\sum_{j=i}^{n-1} E|z_j^n - \bar{z}_j^n|^2 \le (2\mu^2 + 2\mu + 1)\delta\sum_{j=i}^{n-1} E|y_j^n - \bar{y}_j^n|^2 + \delta C_{\xi^n, g, L^n, U^n}. \qquad (29)$$

By the discrete Gronwall inequality in Lemma 3.1, we get

$$\sup_{j\le n} E|y_j^n - \bar{y}_j^n|^2 \le C\delta^2 e^{(2\mu+2\mu^2+1)T}.$$

With (29), it follows $E\left[\delta\sum_{j=0}^{n-1} E|z_j^n - \bar{z}_j^n|^2\right] \le C\delta^2$. Then we reconsider (28), this time we take expectation after taking square, sum and sup over j. Using the Burkholder–Davis–Gundy inequality for the martingale parts and similar techniques, it follows that

$$E\sup_{j\le n} |y_j^n - \bar{y}_j^n|^2 \le CE\left[\sum_{j=0}^{n-1} E|y_j^n - \bar{y}_j^n|^2\delta\right] \le CT\sup_{j\le n} E|y_j^n - \bar{y}_j^n|^2,$$

which implies (27).

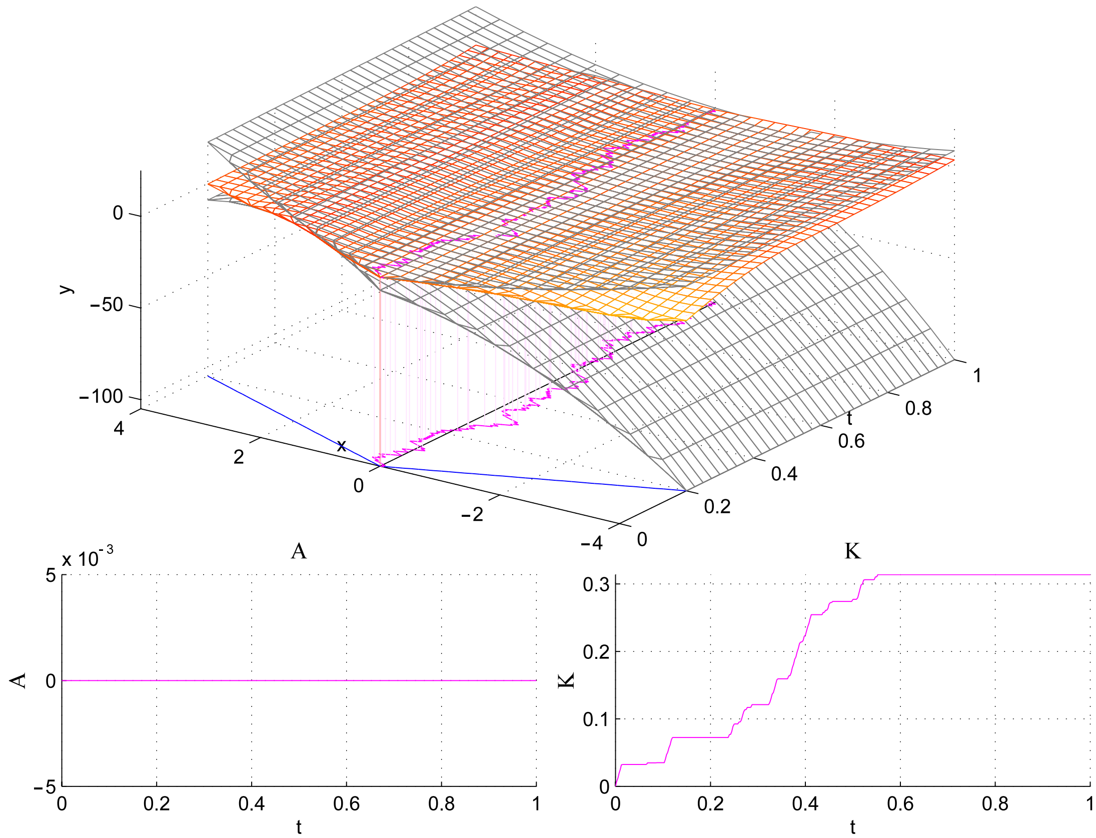

Fig. 1. A solution surface of reflected BSDE with two barriers.

For the convergence of $(\overline{A}^n, \overline{K}^n)$, we consider

$$\overline{A}^n_t - \overline{K}^n_t = \overline{Y}^n_0 - \overline{Y}^n_t - \int_0^t g(s, \overline{Y}^n_s, \overline{Z}^n_s)\mathrm{d}s + \int_0^t \overline{Z}^n_s \mathrm{d}B^n_s,$$

$$A^n_t - K^n_t = Y^n_0 - Y^n_t - \int_0^t g(s, Y^n_s, Z^n_s)\mathrm{d}s + \int_0^t Z^n_s \mathrm{d}B^n_s,$$

then the convergence results follow easily from the convergence of A^n, (26) and the Lipschitz condition of g. □

5. Simulations of reflected BSDE with two barriers

For computational convenience, we consider the case when $T = 1$. The calculation begins from $y^n_n = \xi^n$ and proceeds backward to solve $(y^n_j, z^n_j, a^n_j, k^n_j)$, for $j = n - 1, \ldots, 1, 0$. Due to the amount of computation, here we present a simple case: $\xi = \Phi(B_1), L_t = \Psi_1(t, B(t)), U_t = \Psi_2(t, B(t))$, where Φ, Ψ_1 and Ψ_2 are real analytic functions defined on $\mathbb{R}$ and $[0, 1] \times \mathbb{R}$ respectively. As mentioned in the Introduction, we have developed a Matlab toolbox for calculating and simulating solutions of reflected BSDEs with two barriers which has a well-designed interface to present both global solution surface and trajectories of solution. This toolbox can be downloaded from http://www.sciencenet.cn/u/xvmingyu or http://159.226.47.50:8080/iam/xumingyu/English/index.jsp, with clicking 'Preprint' on the left side.

We take the following example: $g(y, z) = -5|y + z| - 1$, $\Phi(x) = |x|$, $\Psi_1(t, x) = -3(x - 2)^2 + 3$, $\Psi_2(t, x) = (x + 1)^2 + 3(t - 1)$, and $n = 400$.

In Fig. 1, we can see both the global situation of the solution surface of y^n and its partial situation, i.e. its trajectory. In the upper portion of Fig. 1, it is 3-dimensional. The lower surface shows the lower barrier L, as well the upper one for the upper barrier U. The solution y^n is in the middle of them. Then we generate one trajectory of the discrete Brownian motion $(B^n_j)_{0 \le j \le n}$, which is drawn on the horizontal plane. The value of y^n_j with respect to this Brownian sample is showed on the solution surface. The remainder of the figure shows respectively the trajectory of the accumulating force $A^n_j = \sum_{i=0}^j a^n_i$ and $K^n_j = \sum_{i=0}^j k^n_i$.

The lower graphs shows clearly that A^n (respective K^n) increases only if y^n touches the lower barrier L^n, i.e. on the set $\{y^n = L^n\}$ (respective the upper barrier U, i.e. on the set $\{y^n = U^n\}$), and they never increase at the same time.

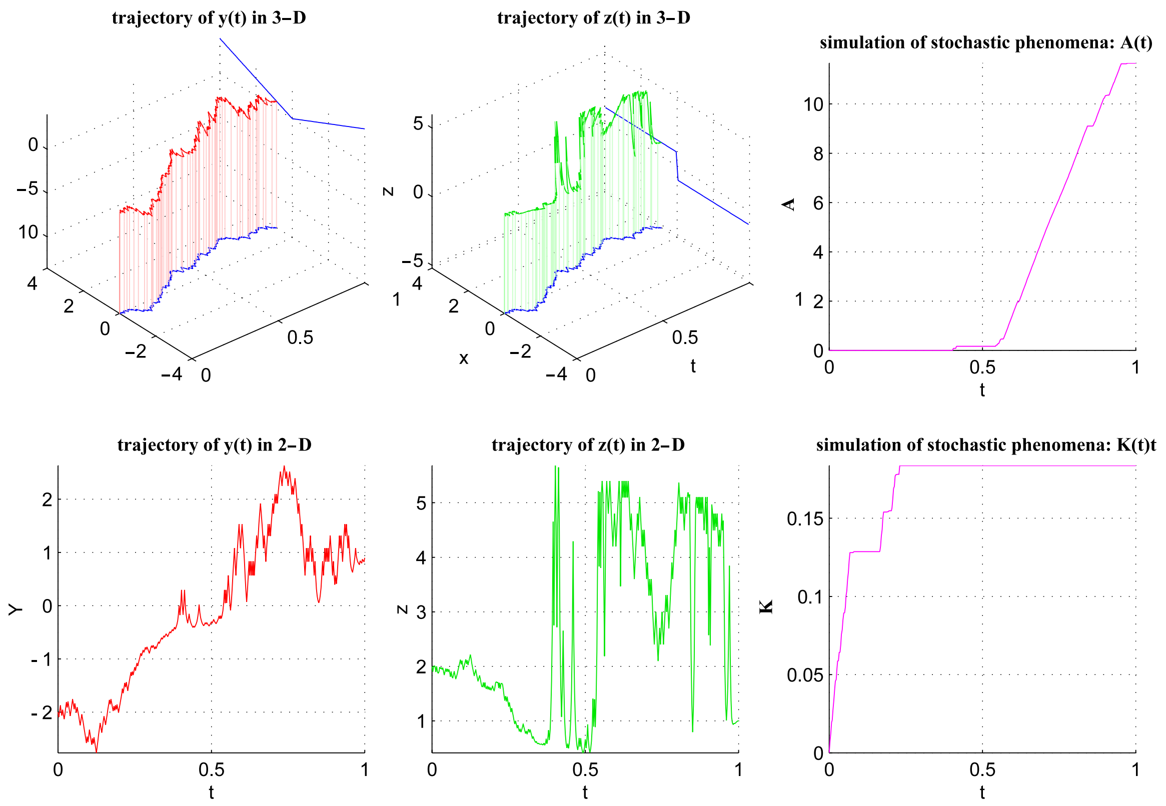

Fig. 2. The trajectories of solutions of (3).

In the upper portion we can see that there is an area, named Area I, (resp. Area II) where the solution surface and the lower barrier surface (resp. the solution surface and the upper barrier surface) stick together. When the trajectory of solution y_j^n goes into Area I (resp. Area II), the force A_j^n (resp. K_j^n) will push y_j^n upward (resp. downward). Indeed, if we don't have the barriers here, y_j^n intends to go up or down to cross the reflecting barrier L_j^n and U_j^n, so to keep y_j^n between L_j^n and U_j^n, the action of forces A_j^n and K_j^n are necessary. In Fig. 1, the increasing process A_j^n remains zero, while K_j^n increases from the beginning. Correspondingly in the beginning y_j^n goes into Area II, but always stays out of Area I. Since Area I and Area II are totally disjoint, so A_j^n and K_j^n never increase at same time.

About this point, let us have a look at Fig. 2. This figure shows a group of 3-dimensional dynamic trajectories (t_j, B_j^n, Y_j^n) and (t_j, B_j^n, Z_j^n), simultaneously, of 2-dimensional trajectories of (t_j, Y_j^n) and (t_j, Z_j^n). For the other sub-figures, the upper-right one is for the trajectories A_j^n, and while the lower-left one is for K_j^n, then comparing these two sub-figures, as in Fig. 1, $\{a_j^n \neq 0\}$ and $\{k_j^n \neq 0\}$ are disjoint, but their complement may be not.

Now we present some numerical results using the explicit reflected scheme and the implicit–explicit penalization scheme, respectively, with different discretization number. Consider the parameters: $g(y,z) = -5|y+z| - 1$, $\Phi(x) = |x|$, $\Psi_1(t,x) = -3(x-2)^2 + 3$, $\Psi_2(t,x) = (x+1)^2 + 3t - 2.5$:

$n = 400$, reflected explicit scheme: $y_0^n = -1.7312$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	−1.8346	−1.7476	−1.7329	−1.7314

$n = 1000$, reflected explicit scheme: $y_0^n = -1.7142$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	−1.8177	−1.7306	−1.7161	−1.7144

$n = 2000$, reflected explicit scheme: $y_0^n = -1.7084$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	−1.8124	−1.7250	−1.7103	−1.7068

$n = 4000$, reflected explicit scheme: $y_0^n = -1.7055$

penalization scheme:

p	20	200	2000	2×10^4
$y_0^{p,n}$	−1.8096	−1.7222	−1.7074	−1.7057

From this form, first we can see that as the penalization parameter p increases, the penalization solution $y_0^{p,n}$ tends increasingly to the reflected solution y_0^n. Second, as the discretaization parameter n increases, the differences of y_0^n with different n become smaller as well as that of $y_0^{p,n}$. An important fact is that the numerical solution is stable with respect to the penalization factor. The penalization parameter p can be chosen larger than the time discretization n. For the data, we can know $y^{p,n}$ is increasing in p, which is obvious from the comparison result of BSDE. Another phenomenon is that $y^{p,n}$ and y^n are both increasing in n, this is because of the choice of coefficient.

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Appendix. The proof of Theorem 2.1

To complete the paper, here we give the proof of Theorem 2.1.

Proof of Theorem 2.1. (a) is the main result in [4], so we omit its proof.

Now we consider (b). The convergence of (Y_t^p, Z_t^p) is a direct consequence of [6]. For the convergence speed, the proof is a combination of results in [4,6]. From [4], we know that for (5), as $m \to \infty$, $\widehat{Y}_t^{m,p} \nearrow \underline{Y}_t^p$ in $\mathbf{S}^2(0,T)$, $\widehat{Z}_t^{m,p} \to \underline{Z}_t^p$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, $\widehat{A}_t^{m,p} \to \underline{A}_t^p$ in $\mathbf{S}^2(0,T)$, where $(\underline{Y}_t^p, \underline{Z}_t^p, \underline{A}_t^p)$ is a solution of the following reflected BSDE with one lower barrier L

$$
\begin{aligned}
&-\mathrm{d}\underline{Y}_t^p = g(t, \underline{Y}_t^p, \underline{Z}_t^p)\mathrm{d}t + \mathrm{d}\underline{A}_t^p - p(\underline{Y}_t^p - U_t)^+\mathrm{d}t - \underline{Z}_t^p \mathrm{d}B_t, \qquad \underline{Y}_T^p = \xi, \\
&\underline{Y}_t^p \geq L_t, \quad \int_0^T (\underline{Y}_t^p - L_t)\mathrm{d}\underline{A}_t^p = 0.
\end{aligned} \tag{30}
$$

Set $\underline{K}_t^p = \int_0^t p(\underline{Y}_s^p - U_s)^+\mathrm{d}s$. Then letting $p \to \infty$, it follows that $\underline{Y}_t^p \searrow Y_t$ in $\mathbf{S}^2(0,T)$, $\underline{Z}_t^p \to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$. By the comparison theorem, $\mathrm{d}\underline{A}_t^p$ is increasing in p. So $\underline{A}_T^p \nearrow A_T$, and $0 \leq \sup_t[\underline{A}_t^{p+1} - \underline{A}_t^p] \leq \underline{A}_T^{p+1} - \underline{A}_T^p$. It follows that $\underline{A}_t^p \to A_t$ in $\mathbf{S}^2(0,T)$. Then with Lipschitz condition of g and convergence results, we get $\underline{K}_t^p \to K_t$ in $\mathbf{S}^2(0,T)$. Moreover from Lemma 4 in [4], we know that there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$
E\left[\sup_{0\leq t\leq T} |\underline{Y}_t^p - Y_t|^2 + \int_0^T (|\underline{Z}_t^p - Z_t|^2)\mathrm{d}t\right] \leq \frac{C}{\sqrt{p}}.
$$

Similarly for (5), first letting $p \to \infty$, we get $\widehat{Y}_t^{m,p} \searrow \overline{Y}_t^m$ in $\mathbf{S}^2(0,T)$, $\widehat{Z}_t^{m,p} \to \overline{Z}_t^m$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, $\widehat{K}_t^{m,p} \to \overline{K}_t^m$ in $\mathbf{S}^2(0,T)$, where $(\overline{Y}_t^m, \overline{Z}_t^m, \overline{K}_t^m)$ is a solution of the following reflected BSDE with one upper barrier U

$$
\begin{aligned}
&-\mathrm{d}\overline{Y}_t^m = g(t, \overline{Y}_t^m, \overline{Z}_t^m)\mathrm{d}t + m(L_t - \overline{Y}_t^m)^+\mathrm{d}t - \mathrm{d}\overline{K}_t^m - \overline{Z}_t^m \mathrm{d}B_t, \qquad \overline{Y}_T^m = \xi, \\
&\overline{Y}_t^m \leq U_t, \quad \int_0^T (\overline{Y}_t^p - U_t)\mathrm{d}\overline{K}_t^m = 0.
\end{aligned} \tag{31}
$$

In the same way, as $m \to \infty$, $\overline{Y}_t^m \nearrow Y_t$ in $\mathbf{S}^2(0,T)$, $\overline{Z}_t^m \to Z_t$ in $\mathbf{L}^2_{\mathcal{F}}(0,T)$, and $(\overline{A}_t^m, \overline{K}_t^m) \to (A_t, K_t)$ in $(\mathbf{S}^2(0,T))^2$, where $\overline{A}_t^m = \int_0^t m(L_s - \overline{Y}_s^m)^+\mathrm{d}s$. Also there exists a constant C depending on ξ, $g(t,0,0)$, μ, L and U, such that

$$
E\left[\sup_{0\leq t\leq T} |\overline{Y}_t^m - Y_t|^2 + \int_0^T (|\overline{Z}_t^m - Z_t|^2)\mathrm{d}t\right] \leq \frac{C}{\sqrt{m}}.
$$

Applying the comparison theorem to (6) and (30), (6) and (31) (let $m = p$), we have $\overline{Y}_t^p \leq Y_t^p \leq \underline{Y}_t^p$. Then we get

$$
E\left[\sup_{0\leq t\leq T} |Y_t^p - Y_t|^2\right] \leq \frac{C}{\sqrt{p}},
$$

for some constant C. To get the estimated results for Z^p, we apply Itô formula to $|Y_t^p - Y_t|^2$, and get

$$
\begin{aligned}
E|Y_0^p - Y_0|^2 + \frac{1}{2}E\int_0^T |Z_s^p - Z_s|^2\mathrm{d}s = {} & (\mu + 2\mu^2)E\int_0^T |Y_s^p - Y_s|^2\mathrm{d}s + 2E\int_0^T (Y_s^p - Y_s)\mathrm{d}A_s^p \\
& - 2E\int_0^T (Y_s^p - Y_s)\mathrm{d}A_s - 2E\int_0^T (Y_s^p - Y_s)\mathrm{d}K_s^p \\
& + 2E\int_0^T (Y_s^p - Y_s)\mathrm{d}K_s.
\end{aligned}
$$

Since

$$2E\int_0^T (Y_s^p - Y_s)\mathrm{d}A_s^p = 2E\int_0^T (Y_s^p - L_s)\mathrm{d}A_s^p - 2E\int_0^T (Y_s - L_s)\mathrm{d}A_s^p$$
$$\leq 2pE\int_0^T (Y_s^p - L_s)(Y_s^p - L_s)^-\mathrm{d}s \leq 0$$

and $2E\int_0^T (Y_s^p - Y_s)\mathrm{d}K_s^p \geq 2pE\int_0^T (Y_s^p - U_s)(Y_s^p - U_s)^+\mathrm{d}s \geq 0$, we have

$$E\int_0^T |Z_s^p - Z_s|^2\mathrm{d}s \leq \frac{C}{\sqrt{p}},$$

in view of the estimation of A and K and the convergence of Y^p.

Now we consider the convergence of A^p and K^p. Since

$$A_t - K_t = Y_0 - Y_t - \int_0^t g(s, Y_s, Z_s)\mathrm{d}s + \int_0^t Z_s\mathrm{d}B_s,$$
$$A_t^p - K_t^p = Y_0^p - Y_t^p - \int_0^t g(s, Y_s^p, Z_s^p)\mathrm{d}s + \int_0^t Z_s^p\mathrm{d}B_s,$$

from the Lipschitz condition of g and the convergence results of Y^p and Z^p, we have immediately

$$E[\sup_{0\leq t\leq T}[(A_t - K_t) - (A_t^p - K_t^p)]^2] \leq 8E\left[\sup_{0\leq t\leq T}|Y_t^p - Y_t|^2 + 4\mu\int_0^T |Y_s^p - Y_s|^2\mathrm{d}s + C\int_0^T |Z_s^p - Z_s|^2\mathrm{d}s\right] \leq \frac{C}{\sqrt{p}}.$$

Meanwhile we know $E[(A_T^p)^2 + (K_T^p)^2] < \infty$, so A^p and K^p admits weak limit $\widetilde{A}$ and $\widetilde{K}$ in $\mathbf{S}^2(0, T)$ respectively. By the comparison results of $\overline{Y}_t^p$, Y_t^p and $\underline{Y}_t^p$, we get

$$\mathrm{d}A_t^p = p(Y_t^p - L_t)^-\mathrm{d}t \leq p(\overline{Y}_t^p - L_t)^-\mathrm{d}t = \mathrm{d}\overline{A}_t^p,$$
$$\mathrm{d}K_t^p = p(Y_t^p - U_t)^+\mathrm{d}t \geq p(\underline{Y}_t^p - U_t)^+\mathrm{d}t = \mathrm{d}\underline{K}_t^p.$$

So $\mathrm{d}\widetilde{A}_t \leq \mathrm{d}A_t$ and $\mathrm{d}\widetilde{K}_t \geq \mathrm{d}K_t$, it follows that $\mathrm{d}\widetilde{A}_t - \mathrm{d}\widetilde{K}_t \leq \mathrm{d}A_t - \mathrm{d}K_t$. On the other hand, the limit of Y^p is Y, so $\mathrm{d}\widetilde{A}_t - \mathrm{d}\widetilde{K}_t = \mathrm{d}A_t - \mathrm{d}K_t$. Then there must be $\mathrm{d}\widetilde{A}_t = \mathrm{d}A_t$ and $\mathrm{d}\widetilde{K}_t = \mathrm{d}K_t$, which implies $\widetilde{A}_t = A_t$ and $\widetilde{K}_t = K_t$. □

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