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Order-unit quantum Gromov–Hausdorff distance

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Abstract

We introduce a new distance $\mathrm{dist}_{\mathrm{oq}}$ between compact quantum metric spaces. We show that $\mathrm{dist}_{\mathrm{oq}}$ is Lipschitz equivalent to Rieffel's distance $\mathrm{dist}_{\mathrm{q}}$, and give criteria for when a parameterized family of compact quantum metric spaces is continuous with respect to $\mathrm{dist}_{\mathrm{oq}}$. As applications, we show that the continuity of a parameterized family of quantum metric spaces induced by ergodic actions of a fixed compact group is determined by the multiplicities of the actions, generalizing Rieffel's work on noncommutative tori and integral coadjoint orbits of semisimple compact connected Lie groups; we also show that the θ-deformations of Connes and Landi are continuous in the parameter θ.

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1. Introduction

In [9] Connes initiated the study of metric spaces in noncommutative setting in the framework of his spectral triple [10]. The main ingredient of a spectral triple is a Dirac operator D. On the one hand, it captures the differential structure by setting $df = [D, f]$. On the other hand, it enables us to recover the Lipschitz seminorm

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L, which is usually defined as

$$L(f) := \sup\left\{\frac{|f(x) - f(y)|}{\rho(x, y)} : x \neq y\right\}, \tag{1}$$

where ρ is the geodesic metric on the Riemannian manifold, instead by means of $L(f) = \|[D, f]\|$, and then one recovers the metric ρ by

$$\rho(x, y) = \sup_{L(f)\leqslant 1} |f(x) - f(y)|. \tag{2}$$

In Section 2 of [9] Connes went further by considering the (possibly $+\infty$-valued) metric on the state space of the algebra defined by (2). Motivated by what happens to ordinary compact metric spaces, in [35,36,38] Rieffel introduced "compact quantum metric spaces" which requires the metric on the state space to induce the weak-$*$ topology. Many interesting examples of compact quantum metric spaces have been constructed [35,37,30,27]. Rieffel's theory of compact quantum metric space does not require C^*-algebras, and is set up on more general spaces, namely order-unit spaces. Also, one does not need Dirac operators, but only the seminorm L.

Motivated by questions in string theory, in [38] Rieffel also introduced a notion of quantum Gromov–Hausdorff distance for compact quantum metric spaces, as an analogue of the Gromov–Hausdorff distance $\mathrm{dist}_{\mathrm{GH}}$ [17] for ordinary compact metric spaces. This is defined as a modified ordinary Gromov–Hausdorff distance for the state spaces. This distance $\mathrm{dist}_{\mathrm{q}}$ is a metric on the set CQM of all isometry classes of compact quantum metric spaces, and has many nice properties. Two nontrivial examples of convergence with respect to $\mathrm{dist}_{\mathrm{q}}$ have been established by Rieffel. One is that the n-dimensional noncommutative tori T_θ's equipped with the quantum metrics induced from the canonical action of $\mathbb{T}^n$ are continuous, with the parameter θ as $n \times n$ real skew-symmetric matrices [38, Theorem 9.2]. The other one is that some natural matrices related to representations of a semisimple compact connected Lie group converge to integral coadjoint orbits of this group [39, Theorem 3.2]. In general, it is not easy to show the continuity of a parameterized family of compact quantum metric spaces. In particular, the methods used in these two examples are quite different.

In view of the principle of noncommutative geometry, it may be more natural to define the quantum distance as a modified Gromov–Hausdorff distance for the order-unit spaces (or C^*-algebras) directly. Under this guidance, we define an *order-unit quantum Gromov–Hausdorff distance*, $\mathrm{dist}_{\mathrm{oq}}$, as a modified ordinary Gromov–Hausdorff distance for certain balls in the order-unit spaces (Definition 4.2). We also introduce a variant $\mathrm{dist}^R_{\mathrm{oq}}$ for the compact quantum metric spaces with radii bounded above by R. Denote by CQM^R the set of all isometry classes of these compact quantum metric spaces. It turns out that these order-unit quantum distances are Lipschitz equivalent to Rieffel's quantum distance.

Theorem 1.1. $\mathrm{dist}_{\mathrm{q}}$ *and* $\mathrm{dist}_{\mathrm{oq}}$ *are Lipschitz equivalent metrics on* CQM, *that is*

$$\tfrac{1}{3}\mathrm{dist}_{\mathrm{oq}} \leqslant \mathrm{dist}_{\mathrm{q}} \leqslant 5\mathrm{dist}_{\mathrm{oq}};$$

while $\mathrm{dist_q}$ *and* $\mathrm{dist}^R_{\mathrm{oq}}$ *are Lipschitz equivalent metrics on* CQM^R, *that is*

$$\tfrac{1}{2}\mathrm{dist}^R_{\mathrm{oq}} \leqslant \mathrm{dist_q} \leqslant \tfrac{5}{2}\mathrm{dist}^R_{\mathrm{oq}}.$$

As an advantage of our approach, we can give criteria for when a parameterized family of compact quantum metric spaces is continuous with respect to the order-unit quantum distance. We introduce a notion of continuous fields of compact quantum metric spaces (Definition 6.4), as a concrete way of saying "a parameterized family". This is an analogue of continuous fields of Banach spaces [15, Section 10.1]. Roughly speaking, these criteria say that the family is continuous under quantum distances if and only if continuous sections are uniformly dense in the balls (the set $\mathcal{D}(A_t)$ in below) we use to define the order-unit quantum distance.

Theorem 1.2. *Let* $(\{(A_t, L_t)\}, \Gamma)$ *be a continuous field of compact quantum metric spaces over a locally compact Hausdorff space T. Let* $t_0 \in T$, *and let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence in* Γ, *the space of continuous sections, such that* $(f_n)_{t_0} \in \mathcal{D}(A_{t_0})$ *for each* $n \in \mathbb{N}$ *and the set* $\{(f_n)_{t_0} : n \in \mathbb{N}\}$ *is dense in* $\mathcal{D}(A_{t_0})$. *Then the following are equivalent*:

(i) $\mathrm{dist_{oq}}(A_t, A_{t_0}) \to 0$ *as* $t \to t_0$;
(ii) $\mathrm{dist_{GH}}(\mathcal{D}(A_t), \mathcal{D}(A_{t_0})) \to 0$ *as* $t \to t_0$;
(iii) *for any* $\varepsilon > 0$, *there is an* N *such that the open* ε*-balls in* A_t *centered at* $(f_1)_t, \ldots, (f_N)_t$ *cover* $\mathcal{D}(A_t)$ *for all* t *in some neighborhood* $\mathcal{U}$ *of* t_0.

Similar criteria are also given for convergence with respect to $\mathrm{dist}^R_{\mathrm{oq}}$ (Theorem 7.1), which is useful when the radii of the compact quantum metric spaces are known to be bounded above by R.

An important class of compact quantum metric spaces come from ergodic actions of compact groups [35]. Let G be a compact group with a fixed length function l given by $l(x) = d(x, e_G)$, where $x \in G$ and d is a left-invariant metric on G and e_G is the identity. For an ergodic action α of G on a unital C^*-algebra $\mathcal{A}$ (i.e. the only α-invariant elements are the scalar multiples of the identity of $\mathcal{A}$), Rieffel proved that the seminorm $L(a) = \sup\{\frac{\|\alpha_x(a)-a\|}{l(x)} : x \in G, x \neq e_G\}$ makes $\mathcal{A}$ into a compact quantum metric space [35, Theorem 2.3]. This includes the examples of noncommutative tori and coadjoint integral orbits mentioned above. In general, one can talk about ergodic actions of G on complete order-unit spaces $\overline{A}$. When the action α is finite in the sense that the multiplicity $\mathrm{mul}(\overline{A}^{\mathbb{C}}, \gamma)$ of every equivalence class of irreducible representations $\gamma \in \hat{G}$ in the induced action $\alpha \otimes I$ on $\overline{A}^{\mathbb{C}} = \overline{A} \otimes \mathbb{C}$ is finite (which is always true in C^*-algebra case [19, Proposition 2.1]), the same construction also makes $\overline{A}$ into a compact quantum metric spaces. Using our criteria for quantum distance convergence (Theorem 7.1) we give a unified proof for the two examples above about continuity of noncommutative tori and convergence of matrix algebras to integral coadjoint orbits, and show in general that a parameterized family of compact quantum metric spaces induced by ergodic finite actions of G is continuous with respect to $\mathrm{dist_{oq}}$ if and only if the multiplicities of the actions are locally constant:

Theorem 1.3. *Let $\{\alpha_t\}$ be a continuous field of strongly continuous finite ergodic actions of G on a continuous field of order-unit spaces $(\{\overline{A_t}\}, \Gamma)$ over a locally compact Hausdorff space T. Then the induced field $(\{(A_t, L_t)\}, \Gamma)$ (for a fixed l) is a continuous field of compact quantum metric spaces. For any $t_0 \in T$ the following are equivalent*:

(i) $\lim_{t\to t_0} \mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma) = \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma)$ *for all* $\gamma \in \hat{G}$;
(ii) $\limsup_{t\to t_0} \mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma) \leqslant \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma)$ *for all* $\gamma \in \hat{G}$;
(iii) $\mathrm{dist}_{\mathrm{oq}}(A_t, A_{t_0}) \to 0$ *as* $t \to t_0$.

In [13] Connes and Landi introduced a one-parameter deformation S^4_θ of the 4-sphere with the property that the Hochschild dimension of S^4_θ equals that of S^4. They also considered general θ-deformations, which was studied further by Connes and Dubois-Violette in [12] (see also [44]). In general, the θ-deformation M_θ of a manifold M equipped with a smooth action of the n-torus T^n is determined by defining the algebra of smooth functions $C^\infty(M_\theta)$ as the invariant subalgebra (under the diagonal action of T^n) of the algebra $C^\infty(M \times T_\theta) := C^\infty(M)\hat{\otimes}C^\infty(T_\theta)$ of smooth functions on $M \times T_\theta$; here θ is a real skew-symmetric $n \times n$ matrix and T_θ is the corresponding noncommutative n-torus. When M is a compact spin manifold, Connes and Landi showed that the canonical Dirac operator $(D, \mathcal{H})$ on M and a deformed anti-unitary operator J_θ together give a spectral triple for $C^\infty(M_\theta)$, fitting it into Connes' noncommutative Riemannian geometry framework [10,11].

Intuitively, the θ-deformations are continuous in the parameter θ. Quantum distances provide a concrete way for us to express the continuity. In [27] we showed that when M is connected, $(C^\infty(M_\theta))_{\mathrm{sa}}$ equipped with the seminorm L_θ determined by the Dirac operator D is a compact quantum metric space. Denote by Θ the space of all $n \times n$ real skew-symmetric matrices. In Section 11 we shall see that there is a natural continuous field of C^*-algebras over Θ with fibers $C(M_\theta)$. Denote by Γ^M the space of continuous sections of this field. As another application of our criteria for quantum distance convergence, we show that θ-deformations are continuous with respect to $\mathrm{dist}_{\mathrm{oq}}$:

Theorem 1.4. *Let M be a connected compact spin manifold with a smooth action of $\mathbb{T}^n$. Then the field $(\{((C^\infty(M_\theta))_{\mathrm{sa}}, L_\theta)\}, (\Gamma^M)_{\mathrm{sa}})$ is a continuous field of compact quantum metric spaces over Θ. And $\mathrm{dist}_{\mathrm{oq}}((C^\infty(M_\theta))_{\mathrm{sa}}, (C^\infty(M_{\theta_0}))_{\mathrm{sa}}) \to 0$ as $\theta \to \theta_0$.*

This paper is organized as follows. In Section 2, we review briefly the Gromov–Hausdorff distance for compact metric spaces and Rieffel's quantum distance for compact quantum metric spaces. Via a characterization of state spaces of compact quantum metric spaces, a formula for Rieffel's distance $\mathrm{dist}_{\mathrm{q}}$ is given in Section 3.

In Section 4, we define the order-unit Gromov–Hausdorff distance $\mathrm{dist}_{\mathrm{oq}}$ and prove Theorem 1.1. One important aspect of the theory of (quantum) Gromov–Hausdorff distance is the (quantum) compactness theorem. In Section 5, we give a reformulation of Rieffel's quantum compactness theorem in terms of the balls we use to define the order-unit distance. The notion of continuous fields of compact quantum metric spaces is introduced in Section 6. In Section 7, we prove our criteria for quantum distance convergence.

The Sections 8–10 are devoted to an extensive study of compact quantum metric spaces induced by ergodic compact group actions, where we show how multiplicities of the actions dominate the metric aspect of such spaces. In Section 8, we show that an ergodic action induces a compact quantum metric space only when the action is finite. In Section 9, we investigate when a family of compact quantum metric spaces induced from ergodic actions of a fixed compact group is totally bounded. Theorem 1.3 is proved in Section 10.

Finally, we prove Theorem 1.4 in Section 11.

2. Preliminaries

In this section we review briefly the Gromov–Hausdorff distance for compact metric spaces [18,42,8] and Rieffel's quantum distance for compact quantum metric spaces [35,36,38–40].

Let (X, ρ) be a metric space, i.e. ρ is a metric on the space X. For any subset $Y \subseteq X$ and $r > 0$, let

$$\mathcal{B}_r(Y) = \{x \in X : \rho(x, y) < r \text{ for some } y \in Y\}$$

be the set of points with distance less than r from Y. When $Y = \{x\}$, we also write it as $\mathcal{B}_r(x)$ and call it the *open ball* of radius r centered at x.

For nonempty subsets $Y, Z \subseteq X$, we can measure the distance between Y and Z inside of X by the *Hausdorff distance* $\mathrm{dist}^{\rho}_{\mathrm{H}}(Y, Z)$ defined by

$$\mathrm{dist}^{\rho}_{\mathrm{H}}(Y, Z) := \inf\{r > 0 : Y \subseteq \mathcal{B}_r(Z),\ Z \subseteq \mathcal{B}_r(Y)\}.$$

We will also use the notation $\mathrm{dist}^{X}_{\mathrm{H}}(Y, Z)$ when there is no confusion about the metric on X.

For any compact metric spaces X and Y, Gromov [17] introduced the *Gromov–Hausdorff distance*, $\mathrm{dist}_{\mathrm{GH}}(X, Y)$, which is defined as

$$\mathrm{dist}_{\mathrm{GH}}(X, Y) := \inf\{\mathrm{dist}^{Z}_{\mathrm{H}}(h_X(X), h_Y(Y)) \,|\, h_X : X \to Z,\ h_Y : Y \to Z \text{ are isometric embeddings into some metric space } Z\}.$$

It is possible to reduce the space Z in above to be the disjoint union $X \coprod Y$. A distance ρ on $X \coprod Y$ is said to be *admissible* if the inclusions $X, Y \hookrightarrow X \coprod Y$ are isometric embeddings. Then it is not difficult to check that

$$\mathrm{dist}_{\mathrm{GH}}(X, Y) = \inf\left\{\mathrm{dist}^{\rho}_{\mathrm{H}}(X, Y) : \rho \text{ is an admissible distance on } X \coprod Y\right\}.$$

For a compact metric space (X, ρ), we shall denote by $\mathrm{diam}(X) := \max\{\rho(x, y) \mid x, y \in X\}$ the *diameter* of X. Also let $r_X = \frac{\mathrm{diam}(X)}{2}$ be the *radius* of X. For any $\varepsilon > 0$, the *covering number* $\mathrm{Cov}_\rho(X, \varepsilon)$ is defined as the smallest number of open balls of radius ε whose union covers X.

Denote by CM the set of isometry classes of compact metric spaces. One important property of Gromov–Hausdorff distance is the completeness and compactness theorems by Gromov [17]:

Theorem 2.1 (*Gromov's completeness and compactness theorems*). *The space* (CM, $\mathrm{dist_{GH}}$) *is a complete metric space. A subset* $\mathcal{S} \subseteq$ CM *is totally bounded* (*i.e. has compact closure*) *if and only if*

(1) *there is a constant D such that* $\mathrm{diam}(X, \rho) \leqslant D$ *for all* $(X, \rho) \in \mathcal{S}$;
(2) *for any* $\varepsilon > 0$, *there exists a constant* $K_\varepsilon > 0$ *such that* $\mathrm{Cov}_\rho(X, \varepsilon) \leqslant K_\varepsilon$ *for all* $(X, \rho) \in \mathcal{S}$.

Next we recall Rieffel's quantum Gromov–Hausdorff distance $\mathrm{dist_q}$ for compact quantum metric spaces.

Rieffel has found that the right framework for compact quantum metric spaces is that of order-unit spaces. There is an abstract characterization of order-unit spaces due to Kadison [20,1]. An *order-unit space* is a real partially ordered vector space, A, with a distinguished element e (the order unit) satisfying:

(1) (Order unit property): For each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leqslant re$;
(2) (Archimedean property): For $a \in A$, if $a \leqslant re$ for all $r \in \mathbb{R}$ with $r > 0$, then $a \leqslant 0$.

On an order-unit space (A, e), we can define a norm as

$$\|a\| = \inf\{r \in \mathbb{R} : -re \leqslant a \leqslant re\}.$$

Then A becomes a normed vector space and we can consider its dual, A', consisting of the bounded linear functionals, equipped with the dual norm $\|\cdot\|'$.

By a *state* of an order-unit space (A, e), we mean a $\mu \in A'$ such that $\mu(e) = \|\mu\|' = 1$. States are automatically positive. Denote the set of all states of A by $S(A)$. It is a compact convex subset of A' under the weak-$*$ topology. Kadison's basic representation theorem [1] says that the natural pairing between A and $S(A)$ induces an isometric order isomorphism of A onto a dense subspace of the space $\mathrm{Af}_{\mathbb{R}}(S(A))$ of all affine $\mathbb{R}$-valued continuous functions on $S(A)$, equipped with the supremum norm and the usual order on functions.

For an order-unit space (A, e) and a seminorm L on A, we can define an ordinary metric, ρ_L, on $S(A)$ (which may take value $+\infty$) by (2). We say that L is a *Lipschitz seminorm* on A if it satisfies:

(1) For $a \in A$, we have $L(a) = 0$ if and only if $a \in \mathbb{R}e$.

We call L a *Lip-norm*, and call the pair (A, L) a *compact quantum metric space* [38, Definitions 2.1, 2.2] if L satisfies further:

(2) The topology on $S(A)$ induced by the metric ρ_L is the weak-$*$ topology.

The *diameter* $\mathrm{diam}(A)$, the *radius* r_A, and the *covering number* $\mathrm{Cov}(A, \varepsilon)$ of (A, L) are defined to be those of $(S(A), \rho_L)$.

Let (A, e) be an order-unit space with a Lipschitz seminorm L. Then L and $\|\cdot\|$ induce norms $\tilde{L}$ and $\|\cdot\|^{\sim}$ respectively on the quotient space $\tilde{A} = A/\mathbb{R}e$. The dual of $(\tilde{A}, \|\cdot\|^{\sim})$ is exactly $A'^0 = \{\lambda \in A' : \lambda(e) = 0\}$. Now $\tilde{L}$ induces a dual seminorm L' on A'^0, which may take value $+\infty$. The metric on $S(A)$ induced by (2) is related to L' by:

$$\rho_L(\mu, \nu) = L'(\mu - \nu) \tag{3}$$

for all $\mu, \nu \in S(A)$.

Notation 2.2. For any $r \geqslant 0$, let

$$\mathcal{D}_r(A) := \{a \in A : L(a) \leqslant 1, \|a\| \leqslant r\}.$$

When L is a Lip-norm on A, set

$$\mathcal{D}(A) := \mathcal{D}_{r_A}(A).$$

Proposition 2.3 (*Rieffel [35, Proposition 1.6, Theorem 1.9]*). *Let (A, e) be an order-unit space with a Lipschitz seminorm L. Then L is a Lip-norm if and only if*

(1) *there is a constant $K \geqslant 0$ such that $L' \leqslant K\|\cdot\|'$ on A'^0;*
or (1′) *there is a constant $K \geqslant 0$ such that $\|\cdot\|^{\sim} \leqslant K\tilde{L}$ on $\tilde{A}$;*
and (2) *for any $r \geqslant 0$, the ball $\mathcal{D}_r(A)$ is totally bounded in A for $\|\cdot\|$;*
or (2′) *for some $r > 0$, the ball $\mathcal{D}_r(A)$ is totally bounded in A for $\|\cdot\|$.*
In this event, the minimal K is exactly r_A.

Let A be an order-unit space. By a *quotient* (π, B) of A, we mean an order-unit space B and a surjective linear positive map $\pi : A \to B$ preserving the order-unit. Via the dual map $\pi' : B' \to A'$, one may identify $S(B)$ with a closed convex subset of $S(A)$. This gives a bijection between isomorphism classes of quotients of A and closed convex subsets of $S(A)$ [38, Proposition 3.6]. If L is a Lip-norm on A, then the quotient seminorm L_B on B, defined by

$$L_B(b) := \inf\{L(a) : \pi(a) = b\}$$

is a Lip-norm on B, and $\pi'|_{S(B)} : S(B) \to S(A)$ is an isometry for the corresponding metrics ρ_L and ρ_{L_B} [38, Proposition 3.1].

Let (A, L_A) and (B, L_B) be compact quantum metric spaces. The direct sum $A \oplus B$, of vector spaces, with (e_A, e_B) as order-unit, and with the natural order structure is also an order-unit space. We call a Lip-norm L on $A \oplus B$ *admissible* if it induces

L_A and L_B under the natural quotient maps $A \oplus B \to A$ and $A \oplus B \to B$. Rieffel's *quantum Gromov–Hausdorff distance* $\mathrm{dist}_{\mathrm{q}}(A, B)$ [38, Definition 4.2] is defined by

$$\mathrm{dist}_{\mathrm{q}}(A, B) = \inf\{\mathrm{dist}_{\mathrm{H}}^{\rho_L}(S(A), S(B)) : L \text{ is an admissible Lip-norm on } A \oplus B\}.$$

Let (A, L) be a compact quantum metric space. Let $\bar{A}$ be the completion of A for $\|\cdot\|$. Define a seminorm, $\bar{L}$, on $\bar{A}$ (which may take value $+\infty$) by

$$\bar{L}(b) := \inf\left\{\liminf_{n\to\infty} L(a_n) : a_n \in A, \lim_{n\to\infty} a_n = b\right\}.$$

The *closure* of L, denoted by L_A^{c}, is defined as the restriction of $\bar{L}$ to the subspace

$$A^{\mathrm{c}} := \{b \in \bar{A} : \bar{L}(b) < \infty\}.$$

Then L^{c} is a Lip-norm on A^{c}, and $\rho_L = \rho_{L^{\mathrm{c}}}$ on $S(A) = S(A^{\mathrm{c}})$ [36, Theorem 4.2, Proposition 4.4]. Identify $\bar{A}$ with $\mathrm{Af}_{\mathbb{R}}(S(A))$. Then A^{c} is exactly the space of Lipschitz functions in $\mathrm{Af}_{\mathbb{R}}(S(A))$, and L^{c} is just the Lipschitz seminorm defined by (1) [38, Proposition 6.1]. We say that L is *closed* if L equals its closure.

Let (A, L_A) and (B, L_B) be compact quantum metric spaces. By an *isometry* from (A, L_A) to (B, L_B) we mean an order isomorphism φ from A^{c} onto B^{c} such that $L_A^{\mathrm{c}} = L_B^{\mathrm{c}} \circ \varphi$. The isometries from (A, L_A) to (B, L_B) are in natural bijective correspondence with the affine isometries from $(S(B), \rho_{L_B})$ onto $(S(A), \rho_{L_A})$ through $\varphi \mapsto \varphi'|_{S(B)}$ [38, Corollary 6.4].

Denote by CQM the set of isometry classes of compact quantum metric spaces. Rieffel also proved a quantum version of Gromov's completeness and compactness theorems [38, Theorems 12.11 and 13.5]:

Theorem 2.4 (*Rieffel's quantum completeness and compactness theorems*). *The space* $(\mathrm{CQM}, \mathrm{dist}_{\mathrm{q}})$ *is a complete metric space. A subset* $\mathcal{S} \subseteq \mathrm{CQM}$ *is totally bounded if and only if*

(1) *there is a constant D such that* $\mathrm{diam}(A, L) \leqslant D$ *for all* $(A, L) \in \mathcal{S}$;
(2) *for any* $\varepsilon > 0$, *there exists a constant* $K_\varepsilon > 0$ *such that* $\mathrm{Cov}(A, \varepsilon) \leqslant K_\varepsilon$ *for all* $(A, L) \in \mathcal{S}$.

3. A characterization of state spaces of compact quantum metric spaces

In this section we give a characterization of state spaces of compact quantum metric spaces in Proposition 3.1, and use it to give a formula for Rieffel's $\mathrm{dist}_{\mathrm{q}}$ in Proposition 3.2.

Proposition 5.7 and Corollary 6.4 in [38] tell us that for compact quantum metric spaces (B_i, L_i), $i = 1, 2$, if their state spaces are affinely isometrically embedded into the state space $S(A)$ of some other compact quantum metric space (A, L), then

$$\mathrm{dist}_{\mathrm{q}}(B_1, B_2) \leqslant \mathrm{dist}_{\mathrm{H}}^{S(A)}(S(B_1), S(B_2)).$$

This provides a powerful way of getting upper bounds for $\mathrm{dist}_{\mathrm{q}}(B_1, B_2)$. In practice, it is quite easy to embed the state space of a quantum metric space into some other compact metric space. So we need to find out what kind of compact metric spaces can be the state space of a compact quantum metric space.

Throughout the rest of this section, locally convex topological vector spaces (LCTVS) will all be Hausdorff. Let $\mathfrak{X}$ be a compact convex subset of a LCTVS V over $\mathbb{R}$. Then $(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}), 1_{\mathfrak{X}})$ is an order-unit space. For each $\mu \in \mathfrak{X}$, the evaluation at μ induces a linear function $\sigma(\mu)$ on $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$. Clearly

$$(\sigma(\mu))(1_{\mathfrak{X}}) = 1 = \|\sigma(\mu)\|.$$

So $\sigma(\mu)$ is a state of $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$. This defines an affine map $\sigma : \mathfrak{X} \to S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))$. Let ρ be a metric on $\mathfrak{X}$. We say that ρ is *midpoint-balanced* [36, Definition 9.3] if for any $\mu, \nu, \mu', \nu' \in \mathfrak{X}$ with $\frac{\mu+\nu'}{2} = \frac{\mu'+\nu}{2}$, we have $\rho(\mu, \nu) = \rho(\mu', \nu')$. We say that ρ is *convex* if for any $\mu, \nu, \mu', \nu' \in \mathfrak{X}$ and $0 \leqslant t \leqslant 1$, we have

$$\rho(t\mu + (1-t)\mu', t\nu + (1-t)\nu') \leqslant t\rho(\mu, \nu) + (1-t)\rho(\mu', \nu').$$

Proposition 3.1. *Let $\mathfrak{X}$ be a compact convex subset of a* LCTVS V, *and let ρ be a metric on $\mathfrak{X}$ compatible with the topology. Then $(\mathfrak{X}, \rho)$ is affinely isometric to $(S(A), \rho_L)$ for some compact quantum metric space (A, L) if and only if the metric ρ is convex and midpoint-balanced. In this event, the closed compact quantum metric space is $(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})_L, L_\rho)$, unique up to isometry, where $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})_L$ is the space of Lipschitz functions in $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$ and L_ρ is the Lipschitz seminorm defined by* (1).

Proof. Assume that $(\mathfrak{X}, \rho)$ is affinely isometric to $(S(A), \rho_L)$ for some compact quantum metric space (A, L). It is easy to check directly from (2) that the metric ρ_L and hence ρ are convex and midpoint-balanced.

Conversely, assume that the metric ρ is convex and balanced. Elements in the dual V' separate the points in V by the Hahn–Banach theorem. Since the restrictions of elements in V' to $\mathfrak{X}$ are all in $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$, we see that functions in $\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})$ separate the points of $\mathfrak{X}$. Theorem II.2.1 in [1] tells us that σ is a homeomorphic embedding of $\mathfrak{X}$ into $S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))$, and that $\sigma(\mathfrak{X})$ contains the set of extreme points of $S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))$. Since $\sigma(\mathfrak{X})$ is convex and closed, we see that σ is surjective. Hence we may identify $\mathfrak{X}$ and $S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))$. By [36, Lemma 2.1] we have $(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))'^0 = \mathbb{R}(S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})) - S(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))) = \mathbb{R}(\mathfrak{X} - \mathfrak{X})$ (see the discussion preceding Notation 2.2). By [36, Theorem 9.7] there is a norm M on $(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X}))'^0 = \mathbb{R}(\mathfrak{X} - \mathfrak{X})$ such that $\rho(\mu, \nu) = M(\mu - \nu)$ for all $\mu, \nu \in \mathfrak{X}$. Then [36, Theorem 9.8] (see also the discussion right after the proof of Proposition 1.1 in [37]) asserts that $(\mathrm{Af}_{\mathbb{R}}(\mathfrak{X})_L, L_\rho)$ is a closed compact quantum metric space and

$(\mathfrak{X}, \rho)$ is its state space. The uniqueness of such a closed compact quantum metric space follows from [38, Corollary 6.4]. $\square$

Consequently we have the following description of the quantum distance dist_q:

Proposition 3.2. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. Then we have*

$$\mathrm{dist}_\mathrm{q}(A, B) = \inf\{\mathrm{dist}_\mathrm{H}^V(h_A(S(A)), h_B(S(B))) : h_A \text{ and } h_B \text{ are affine isometric embeddings of } S(A) \text{ and } S(B) \text{ into some real normed space } V\}.$$

Proof. Denote the right-hand side of the above identity by $\mathrm{dist}'_\mathrm{q}(A, B)$. For any admissible Lip-norm L on $A \oplus B$ let $V = (A \oplus B)'^0$ equipped with the norm L' (see the discussion preceding Notation 2.2). Pick an element p in $S(A \oplus B)$, and let $\varphi : S(A \oplus B) \to V$ be the translation $x \mapsto x - p$. Then φ is an affine isometric embedding from $(S(A \oplus B), \rho_L)$ to V according to (3). Hence $\mathrm{dist}'_\mathrm{q}(A, B) \leqslant \mathrm{dist}_\mathrm{H}^V(\varphi(S(A)), \varphi(S(B))) = \mathrm{dist}_\mathrm{H}^{\rho_L}(S(A), S(B))$. Thus $\mathrm{dist}'_\mathrm{q}(A, B) \leqslant \mathrm{dist}_\mathrm{q}(A, B)$.

Now let V, h_A and h_B be as in Proposition 3.2. Let $\mathfrak{X}$ be the convex hull of $h_A(S(A)) \cup h_B(S(B))$. Clearly $\mathfrak{X}$ equipped with the distance induced from the norm in V is compact, and hence is the state space of some compact quantum metric space (C, L_C) by Proposition 3.1. Therefore $\mathrm{dist}_\mathrm{q}(A, B) \leqslant \mathrm{dist}_\mathrm{H}^V(h_A(S(A)), h_B(S(B)))$ by [38, Proposition 5.7, Corollary 6.4]. Consequently $\mathrm{dist}_\mathrm{q}(A, B) \leqslant \mathrm{dist}'_\mathrm{q}(A, B)$. $\square$

4. Definition of the order-unit quantum Gromov–Hausdorff distance

In this section we define the order-unit Gromov–Hausdorff distance and prove Theorem 1.1.

Rieffel's definition of quantum Gromov–Hausdorff distance is a modified ordinary Gromov–Hausdorff distance for the state spaces. In the view of noncommutative geometry, whose principle is the duality between ordinary spaces and appropriate vector spaces of functions over the spaces, it may be more natural to do everything on the vector spaces of functions directly, avoiding referring back to the state spaces. So it may be more natural to measure the ordinary Gromov–Hausdorff distance for the vector spaces of functions directly. But the order-unit spaces of functions are not compact, so we cannot apply the ordinary Gromov–Hausdorff distance to them. One way to get around this difficulty is to consider some core of the vector spaces of functions which captures all the information of the order-unit spaces. One natural choice is the unit ball. But, unless the order-unit space is finite dimensional, the unit ball is not compact either. It also does not remember the Lip-norm. Now comes the candidate, $\mathcal{D}(A)$ (see Notation 2.2) for closed (A, L_A). When $r_A > 0$, $\mathcal{D}(A)$ is absorbing, i.e. for every $a \in A$ there is some $\varepsilon > 0$ such that $\lambda a \in \mathcal{D}(A)$ for all $0 \leqslant \lambda < \varepsilon$. Thus

$\mathcal{D}(A)$ equipped with the metric induced by the norm of A encodes the normed space structure of A. It also captures the Lip-norm:

Lemma 4.1. *Let (A, L) be a closed compact quantum metric space. Then for any $R \geqslant r_A$ we have*

$$\{a \in A : L(a) \leqslant 1\} = \mathbb{R}e_A + \mathcal{D}_R(A). \tag{4}$$

Conversely, let (B, e_B) be an order-unit space, and let X be a balanced (i.e. $\lambda x \in X$ for all $x \in X$ and $\lambda \in \mathbb{R}$ with $|\lambda| \leqslant 1$), absorbing (i.e. $\{\lambda x : \lambda \in \mathbb{R}_+, x \in X\} = B$), compact convex subset of B (under the order-unit norm topology). Let R be the radius of X. If $X = \{b \in (X + \mathbb{R}e_B) : \|b\| \leqslant R\}$, then there is a unique closed Lip-norm L on B such that $X = \mathcal{D}_R(B)$. In this case L is also characterized as the unique seminorm on B satisfying $X + \mathbb{R}e_B = \{b \in B : L(b) \leqslant 1\}$.

Proof. Eq. (4) follows directly from Proposition 2.3. Now let X be as in Lemma 4.1. Then clearly $X + \mathbb{R}e_B$ is also a balanced absorbing convex set. Since X is compact, $X + \mathbb{R}e_B$ is closed. Let L be the Minkowski functional [2, Theorem 37.4] corresponding to $X + \mathbb{R}e_B$, i.e. the unique seminorm on B satisfying that $X + \mathbb{R}e_B = \{b \in B : L(b) \leqslant 1\}$. Clearly $L(e_B) = 0$. Suppose that $L(b) = 0$. Then for any $n \in \mathbb{N}$ we have $nb \in X + \mathbb{R}e_B$. Thus there exist $x_n \in X$ and $\lambda_n \in \mathbb{R}$ such that $nb = x_n + \lambda_n e_B$. Since $\|x_n\| \leqslant R$, we have $\|\tilde{b}\|^\sim = \|\frac{1}{n}x_n\|^\sim \leqslant \frac{1}{n}R$ in $\tilde{B} = B/\mathbb{R}e_B$. Thus $\|\tilde{b}\|^\sim = 0$, and hence $b \in \mathbb{R}e_B$. Therefore L is a Lipschitz seminorm on B. Clearly condition (1′) in Proposition 2.3 is satisfied with $K = R$. The assumption $X = \{b \in (X + \mathbb{R}e_B) : \|b\| \leqslant R\}$ means that $X = \mathcal{D}_R(B)$. Note that $R > 0$ since X is absorbing. Thus condition (2′) in Proposition 2.3 is also satisfied with $r = R$. By Proposition 2.3 L is a Lip-norm on B, and $r_B \leqslant R$. Since $X + \mathbb{R}e_B$ is closed, L is closed. The uniqueness of such a closed Lip-norm follows from (4). □

Most importantly, $\mathcal{D}(A)$ is compact with the distance induced from the norm on A by Proposition 2.3. So we can use it to redefine the quantum Gromov–Hausdorff distance. There is one subtle point: we do not know whether $\mathcal{D}(A)$ remembers the order-unit e_A or not (see Remark 4.13). We shall come back to this point later.

Now the question is what kind of modified Gromov–Hausdorff distance we should put on $\mathcal{D}(A)$. Certainly this modified Gromov–Hausdorff distance should reflect the convex structure on $\mathcal{D}(A)$. If we look at the definition of $\mathrm{dist}_{\mathrm{GH}}$ in Section 2, one immediate choice for the modified distance is $\inf\{\mathrm{dist}_{\mathrm{H}}^V(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B)))\}$, where the infimum runs over affine isometric embeddings h_A and h_B of $\mathcal{D}(A)$ and $\mathcal{D}(B)$ into some real normed space V. On the other hand, notice that $\mathcal{D}(A)$ is the state space of some compact quantum metric space $(A, L_A)'$ according to Proposition 3.1. So we may try to use Rieffel's quantum distance for $(A, L_A)'$ and $(B, L_B)'$. Proposition 3.2 tells us that these two possible definitions agree. Notice that when $r_A > 0$ we can extend h_A uniquely to an affine isometric embedding of A into V. When $r_A = 0$, the space A is one dimensional, so we can also extend h_A to A (by enlarging V if $V = \{0\}$). Therefore the infimum actually runs over affine isometric embeddings h_A and h_B of A and B into real normed spaces V. These embeddings may not be linear since $h_A(0_A)$ and $h_B(0_B)$ need not be 0_V. But we can always assume that h_A is linear by composing both h_A and h_B with the translation $x \mapsto x - h_A(0_A)$ in V. To makes things easier, we choose to require both h_A and h_B to be linear. Since we do not know whether $\mathcal{D}(A)$ remembers the order-unit e_A or not (see Remark 4.13), we need to consider also $\|h_A(r_A e_A) - h_B(r_B e_B)\|$. Now we get to:

Definition 4.2. Let (A, L_A) and (B, L_B) be compact quantum metric spaces. We define the *order-unit quantum Gromov–Hausdorff distance* between them, denoted by $\mathrm{dist}_{\mathrm{oq}}(A, B)$, by

$$\mathrm{dist}_{\mathrm{oq}}(A, B) := \inf\{\max(\mathrm{dist}_{\mathrm{H}}^{V}(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B))), \|h_A(r_A e_A) - h_B(r_B e_B)\|)\},$$

and, for $R \geqslant 0$, the *R-order-unit quantum Gromov–Hausdorff distance* between them, denoted by $\mathrm{dist}_{\mathrm{oq}}^{R}(A, B)$, by

$$\mathrm{dist}_{\mathrm{oq}}^{R}(A, B) := \inf\{\max(\mathrm{dist}_{\mathrm{H}}^{V}(h_A(\mathcal{D}_R(A)), h_B(\mathcal{D}_R(B))), \|h_A(Re_A) - h_B(Re_B)\|)\},$$

where the infima are taken over all triples (V, h_A, h_B) consisting of a real normed space V and linear isometric embeddings $h_A : A \to V$ and $h_B : B \to V$.

Remark 4.3. (1) To simply the notation, usually we shall identify A and B with their images $h_A(A)$ and $h_B(B)$ respectively, and just say that V is a normed space containing both A and B;

(2) See the discussion preceding Theorem 7.1 for the motivation of introducing $\mathrm{dist}_{\mathrm{oq}}^{R}$;

(3) We choose to use the terms $\|h_A(r_A e_A) - h_B(r_B e_B)\|$ and $\|h_A(Re_A) - h_B(Re_B)\|$ to take care of the order-units. As another choice, one may also omit these terms and require $h_A(e_A) = h_B(e_B)$ in Definition 4.2. Denote the resulting distances by $\mathrm{dist}_{\mathrm{oq}}^{*}$ and $\mathrm{dist}_{\mathrm{oq}}^{R*}$. It is easy to see that $\mathrm{dist}_{\mathrm{oq}} \leqslant \mathrm{dist}_{\mathrm{oq}}^{*}$ and $\mathrm{dist}_{\mathrm{oq}}^{R} \leqslant \mathrm{dist}_{\mathrm{oq}}^{R*}$. One may also check that the proofs of Propositions 4.8, 4.10, and Theorem 1.1 hold with $\mathrm{dist}_{\mathrm{oq}}$ and $\mathrm{dist}_{\mathrm{oq}}^{R}$ replaced by $\mathrm{dist}_{\mathrm{oq}}^{*}$ and $\mathrm{dist}_{\mathrm{oq}}^{R*}$;

(4) For any ordinary compact metric space (X, ρ), let A_X be the space of Lipschitz $\mathbb{R}$-valued functions on X and let L_ρ be the Lipschitz seminorm defined by (1). Then (A_X, L_ρ) is a closed compact quantum metric space, called the *associated compact quantum metric space* of (X, ρ). For any compact metric spaces (X, ρ_X) and (Y, ρ_Y), by [38, Proposition 4.7] and Theorem 1.1 we have $\mathrm{dist}_{\mathrm{oq}}(A_X, A_Y) \leqslant 3\mathrm{dist}_{\mathrm{q}}(A_X, A_Y) \leqslant 3\mathrm{dist}_{\mathrm{GH}}(X, Y)$. Using [38, Theorem 13.16] and Theorems 2.1, 2.4, and 1.1, one can see that the distance $(X, Y) \mapsto \mathrm{dist}_{\mathrm{oq}}(A_X, A_Y)$ determines the same topology on CM as does $\mathrm{dist}_{\mathrm{GH}}$.

As in the discussion for Gromov–Hausdorff distance in Section 2, it suffices to have V to be $A \oplus B$ (equipped with certain norms) in Definition 4.2. To this end, for any normed spaces V and W we call a norm $\|\cdot\|_{V\oplus W}$ on $V\oplus W$ *admissible* if it extends the norms on V and W.

Proposition 4.4. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces. Then*

$$\mathrm{dist}_{\mathrm{oq}}(A, B) = \inf\{\max(\mathrm{dist}_{\mathrm{H}}^{A\oplus B}(\mathcal{D}(A), \mathcal{D}(B)), \|r_A e_A - r_B e_B\|_{A\oplus B})\},$$

and, for any $R \geqslant 0$,

$$\mathrm{dist}^R_{\mathrm{oq}}(A, B) = \inf\{\max(\mathrm{dist}_{\mathrm{H}}^{A\oplus B}(\mathcal{D}_R(A), \mathcal{D}_R(B)), \|Re_A - Re_B\|_{A\oplus B})\},$$

where the infima are taken over all admissible norms $\|\cdot\|_{A\oplus B}$ on $A\oplus B$.

Proof. We prove the case of $\mathrm{dist}_{\mathrm{oq}}(A, B)$. That of $\mathrm{dist}^R_{\mathrm{oq}}$ is similar. The proof here could be thought of as a dual of Example 5.6 and Proposition 5.7 in [38]. Let (V, h_A, h_B) be as in Definition 4.2. Let $1 > \varepsilon > 0$ be given. We will construct an admissible norm on $V\oplus V$ such that the two copies of V are ε-close to each other, i.e. $\|(v, -v)\|_{V\oplus V} \leqslant \varepsilon\|v\|$. Clearly $\|(u, v)\|_{V\oplus V} := \max(\|u + v\|, \varepsilon\|u\|, \varepsilon\|v\|)$ satisfies the requirement. Now we identify $A\oplus B$ with the subspace $h_A(A)\oplus h_B(B)$ of $V\oplus V$. Then the induced norm on $A\oplus B$ is admissible. And

$$\begin{aligned}\mathrm{dist}_{\mathrm{H}}^{A\oplus B}(\mathcal{D}(A), \mathcal{D}(B)) \leqslant\ & \mathrm{dist}_{\mathrm{H}}^{V}(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B))) \\ & + \mathrm{dist}_{\mathrm{H}}^{V\oplus V}((h_B(\mathcal{D}(B)), 0), (0, h_B(\mathcal{D}(B)))) \\ \leqslant\ & \mathrm{dist}_{\mathrm{H}}^{V}(h_A(\mathcal{D}(A)), h_B(\mathcal{D}(B))) + \varepsilon r_B.\end{aligned}$$

Similarly, $\|r_A e_A - r_B e_B\|_{A\oplus B} \leqslant \|h_A(r_A e_A) - h_B(r_B e_B)\|_V + \varepsilon r_B$. This gives the desired result. $\square$

We start to prove Theorem 1.1. We prove the triangle inequality first. For this we need the amalgamation of normed spaces:

Lemma 4.5. *Let $\varphi_j : A \hookrightarrow B_j$ be linear isometric embeddings of normed spaces (over $\mathbb{R}$ or $\mathbb{C}$) for $j \in J$, where J is an index set. Then there is a normed space C and linear isometric embeddings $\psi_j : B_j \hookrightarrow C$ such that $\psi_j \circ \varphi_j = \psi_k \circ \varphi_k$ for all $j, k \in J$.*

Proof. Let $\|\cdot\|_1$ be the L_1-norm on $\oplus_{j\in J} B_j$, i.e. $\|(u_j)\|_1 = \sum_{j\in J}\|u_j\|$. Let

$$W = \left\{(u_j) : u_j \in \varphi_j(A) \text{ for all } j \in J, \text{ and } \sum_{j\in J}(\varphi_j)^{-1}(u_j) = 0\right\},$$

which is a linear subspace of $\oplus_{j\in J} B_j$. Let $q : \oplus_{j\in J} B_j \to (\oplus_{j\in J} B_j)/W$ be the quotient map, and let $\psi_j : B_j \to (\oplus_{j\in J} B_j)/W$ be the composition of $B_j \to \oplus_{j\in J} B_j$ and q.

Then clearly $\psi_j \circ \varphi_j = \psi_k \circ \varphi_k$ for all $j, k \in J$, and ψ_j is contractive. For any $u \in B_k$ and $(\varphi_j(v_j)) \in W$ we have

$$\begin{aligned}\|u + (\varphi_j(v_j))\|_1 &= \|u + \varphi_k(v_k)\| + \sum_{j \in J, j \neq k} \|v_j\| \\ &= \left\| u - \varphi_k\left(\sum_{j \in J,\, j \neq k} v_j\right)\right\| + \sum_{j \in J,\, j \neq k} \|v_j\| \geqslant \|u\|.\end{aligned}$$

Therefore ψ_k is isometric. $\square$

Using Lemma 4.5 one gets immediately the triangle inequality:

Lemma 4.6. *For any compact quantum metric spaces (A, L_A), (B, L_B), and (C, L_C) we have*

$$\mathrm{dist}_{\mathrm{oq}}(A, C) \leqslant \mathrm{dist}_{\mathrm{oq}}(A, B) + \mathrm{dist}_{\mathrm{oq}}(B, C).$$

For $R \geqslant 0$ we also have

$$\mathrm{dist}^R_{\mathrm{oq}}(A, C) \leqslant \mathrm{dist}^R_{\mathrm{oq}}(A, B) + \mathrm{dist}^R_{\mathrm{oq}}(B, C).$$

Next we compare $\mathrm{dist}_{\mathrm{oq}}$ (and $\mathrm{dist}^R_{\mathrm{oq}}$) with $\mathrm{dist}_{\mathrm{q}}$. For this purpose we express first $\mathrm{dist}_{\mathrm{q}}$ in a form similar to that of $\mathrm{dist}_{\mathrm{oq}}$. For any compact quantum metric space (A, L_A) denote by $\mathcal{E}(A)$ the unit ball of A under L_A.

Proposition 4.7. *For any compact quantum metric spaces (A, L_A) and (B, L_B) we have*

$$\mathrm{dist}_{\mathrm{q}}(A, B) = \inf\{\mathrm{dist}^V_{\mathrm{H}}(\mathcal{E}(A), \mathcal{E}(B))\},$$

where the infimum is taken over all order-unit spaces V containing both A and B as order-unit subspaces. The equation also holds if the infimum is taken over all normed spaces V containing both A and B such that $e_A = e_B$.

Proof. Denote the right-hand side of the above identity by $\mathrm{dist}'_{\mathrm{q}}(A, B)$. Also denote by $\mathrm{dist}''_{\mathrm{q}}(A, B)$ the corresponding term for the infimum being taken over all normed spaces V containing both A and B such that $e_A = e_B$. Clearly $\mathrm{dist}'_{\mathrm{q}}(A, B) \geqslant \mathrm{dist}''_{\mathrm{q}}(A, B)$.

Let L be an admissible Lip-norm on $A \oplus B$, and set $d = \mathrm{dist}^{\rho_L}_{\mathrm{H}}(S(A), S(B))$. Denote by Z the subset of $S(A) \times S(B)$ consisting of pairs (p, q) with $\rho_L(p, q) \leqslant d$. Since $S(A)$ and $S(B)$ are compact, the projections $Z \to S(A)$ and $Z \to S(B)$ are surjective. Think of A and B as subspaces of $C(S(A))$ and $C(S(B))$, respectively. Then the induced

$\mathbb{R}$-linear maps $A \to C(Z)$ and $B \to C(Z)$ are unital isometric embeddings. Notice that for any $a \in A$ and $b \in B$ we have

$$\|a - b\| = \sup\{|p(a) - q(b)| : (p, q) \in Z\} \leqslant L(a, b)d.$$

Let $a \in \mathcal{E}(A)$. For any $\varepsilon > 0$ pick $b \in B$ with $L(a, b) < 1 + \varepsilon$. Then $\|a - b\| \leqslant L(a, b)d \leqslant (1 + \varepsilon)d$, and hence

$$\|b\| \leqslant \|b - a\| + \|a\| \leqslant (1 + \varepsilon)d + \|a\|.$$

Also $L_B(b) \leqslant L(a, b) < 1 + \varepsilon$. Let $b' = b/(1 + \varepsilon)$. Then $b' \in \mathcal{E}(B)$, and

$$\|a - b'\| \leqslant \|a - b\| + \|b - b'\| \leqslant (1 + \varepsilon)d + \frac{\varepsilon}{1 + \varepsilon}\|b\| \leqslant (1 + 2\varepsilon)d + \frac{\varepsilon}{1 + \varepsilon}\|a\|.$$

Similarly, for any $b \in \mathcal{E}(B)$ and $\varepsilon > 0$ we can find $a' \in \mathcal{E}(A)$ such that $\|b - a'\| \leqslant (1+2\varepsilon)d + \frac{\varepsilon}{1+\varepsilon}\|b\|$. Letting $\varepsilon \to 0$ we get $\mathrm{dist}'_{\mathrm{q}}(A, B) \leqslant d$. Consequently, $\mathrm{dist}'_{\mathrm{q}}(A, B) \leqslant \mathrm{dist}_{\mathrm{q}}(A, B)$.

Let V be a normed space V containing both A and B such that $e_A = e_B$, and set $d = \mathrm{dist}^V_{\mathrm{H}}(\mathcal{E}(A), \mathcal{E}(B))$. Let $\varepsilon > 0$ be given. Define a seminorm L on $A \oplus B$ via $L(a, b) = \max(L_A(a), L_B(b), \|a - b\|/(d + \varepsilon))$. It follows easily from Proposition 2.3 that L is an admissible Lip-norm on $A \oplus B$. For any $p \in S(A)$, by the Hahn–Banach theorem extend p to a linear functional φ on V with $\|\varphi\| = 1$ and set q to be the restriction of φ on B. Since $e_A = e_B$ we have $q(e_B) = 1$ and hence $q \in S(B)$. For any $(a, b) \in \mathcal{E}(A \oplus B)$ we have $|p(a) - q(b)| = |\varphi(a - b)| \leqslant \|a - b\| \leqslant d + \varepsilon$. Therefore $\rho_L(p, q) \leqslant d + \varepsilon$. Similarly, for any $q' \in S(B)$ we can find $p' \in S(A)$ with $\rho_L(p', q') \leqslant d + \varepsilon$. Thus $\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant \mathrm{dist}^{\rho_L}_{\mathrm{H}}(S(A), S(B)) \leqslant d + \varepsilon$. Letting $\varepsilon \to 0$ we get $\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant d$. Consequently, $\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant \mathrm{dist}''_{\mathrm{q}}(A, B)$. This finishes the proof of Proposition 4.7. $\square$

We remark that though $\mathrm{dist}_{\mathrm{q}}$ has a form similar to those of $\mathrm{dist}_{\mathrm{oq}}$ and $\mathrm{dist}^R_{\mathrm{oq}}$, to prove the criteria Theorems 1.2 and 7.1 we have to use $\mathrm{dist}_{\mathrm{oq}}$ and $\mathrm{dist}^R_{\mathrm{oq}}$ in an essential way.

Proposition 4.8. *For any compact quantum metric spaces* (A, L_A) *and* (B, L_B) *we have*

$$|r_A - r_B| \leqslant \mathrm{dist}_{\mathrm{GH}}(\mathcal{D}(A), \mathcal{D}(B)) \leqslant \mathrm{dist}_{\mathrm{oq}}(A, B) \leqslant r_A + r_B, \tag{5}$$

$$|\mathrm{dist}_{\mathrm{oq}}(A, B) - \mathrm{dist}^{r_B}_{\mathrm{oq}}(A, B)| \leqslant |r_A - r_B|, \tag{6}$$

$$\mathrm{dist}_{\mathrm{oq}}(A, B) \leqslant 3\mathrm{dist}_{\mathrm{q}}(A, B). \tag{7}$$

For $R \geqslant 0$ we also have

$$\mathrm{dist}^R_{\mathrm{oq}}(A, B) \leqslant 2\mathrm{dist}_{\mathrm{q}}(A, B). \tag{8}$$

Proof. For any compact metric spaces X and Y, one has $|r_X - r_Y| \leqslant \mathrm{dist}_{\mathrm{GH}}(X, Y)$ [8, Exercise 7.3.14]. Thus (5) is trivial once we notice that $\mathcal{D}(A)$ has radius r_A. To show (6) it suffices to show that $\mathrm{dist}^A_{\mathrm{H}}(\mathcal{D}(A), \mathcal{D}_{r_B}(A)) \leqslant |r_A - r_B|$. In fact we have:

Lemma 4.9. *For any compact quantum metric space (A, L_A) and any $R > r \geqslant 0$ we have*

$$\mathrm{dist}^A_{\mathrm{H}}(\mathcal{D}_R(A), \mathcal{D}_r(A)) \leqslant R - r.$$

Proof. Notice that $\mathcal{D}_r(A)$ is a subset of $\mathcal{D}_R(A)$. For each $a \in \mathcal{D}_R(A)$ let $a' = \frac{r}{R}a$. Then $a' \in \mathcal{D}_r(A)$ and

$$\|a - a'\| = \frac{R - r}{R}\|a\| \leqslant R - r.$$

Hence $\mathrm{dist}^A_{\mathrm{H}}(\mathcal{D}_R(A), \mathcal{D}_r(A)) \leqslant R - r$. □

Back to the proof of Proposition 4.8. Inequality (7) follows from (6), (8), and the fact that $|r_A - r_B| \leqslant \mathrm{dist}_{\mathrm{GH}}(S(A), S(B)) \leqslant \mathrm{dist}_{\mathrm{q}}(A, B)$. So we are left to prove (8). Let V be a normed space containing both A and B such that $e_A = e_B$, and set $d = \mathrm{dist}^V_{\mathrm{H}}(\mathcal{E}(A), \mathcal{E}(B))$. For any $a \in \mathcal{D}_R(A)$ and $\varepsilon > 0$ pick $b \in \mathcal{E}(B)$ such that $\|a - b\| \leqslant d + \varepsilon$. Then $\|b\| \leqslant \|b - a\| + \|a\| \leqslant d + \varepsilon + R$. By Lemma 4.9 we can find $b' \in \mathcal{D}_R(B)$ with $\|b - b'\| \leqslant d + \varepsilon$. Then $\|a - b'\| \leqslant 2(d + \varepsilon)$. Similarly, for any $b \in \mathcal{D}_R(B)$ we can find $a' \in \mathcal{D}_R(A)$ with $\|a' - b\| \leqslant 2(d + \varepsilon)$. It follows that $\mathrm{dist}^R_{\mathrm{oq}}(A, B) \leqslant 2d$. Then (8) follows from Proposition 4.7. □

Proposition 4.10. *Let (A, L_A) and (B, L_B) be compact quantum metric spaces with $R \geqslant r_A, r_B$. Then we have*

$$\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant \tfrac{5}{2}\mathrm{dist}^R_{\mathrm{oq}}(A, B), \tag{9}$$

$$\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant 5\mathrm{dist}_{\mathrm{oq}}(A, B). \tag{10}$$

Proof. Note that (10) follows immediately from (9), (6), and (5). We prove (9). We may assume that both (A, L_A) and (B, L_B) are closed. The case $R = 0$ is trivial, so we assume that $R > 0$. Let V be a normed space containing A and B, and let $d = \max(\mathrm{dist}^V_{\mathrm{H}}(\mathcal{D}_R(A), \mathcal{D}_R(B)), \|Re_A - Re_B\|)$. If $d = 0$ then it is easy to see from Lemma 4.1 that (A, L_A) and (B, L_B) are isometric. So we assume that $d > 0$. Rieffel used *bridges* in [38] to get upper bounds for $\mathrm{dist}_{\mathrm{q}}(A, B)$. Recall that a bridge between (A, L_A) and (B, L_B) [38, Definition 5.1] is a seminorm, N, on $A \oplus B$ such that N is continuous for the order-unit norm on $A \oplus B$, $N(e_A, e_B) = 0$ but $N(e_A, 0) \neq 0$, and for any $a \in A$ and $\delta > 0$ there is a $b \in B$ such that $\max(L_B(b), N(a, b)) \leqslant L_A(a) + \delta$, and similarly for A and B interchanged. The importance of bridges is that the seminorm L on $A \oplus B$ defined by $L(a, b) = \max(L_A(a), L_B(b), N(a, b))$ is an admissible Lip-norm [38, Theorem 5.2]. In our situation one natural choice of N is (the seminorm induced from the quotient map $A \oplus B \to (A \oplus B)/\mathbb{R}(e_A, e_B)$ and) the quotient norm on $(A \oplus B)/\mathbb{R}(e_A, e_B)$ induced by the norm $\|(a, b)\|_* = \max(\|a\|, \|b\|, \|a - b\|)$. Let $a \in A$ with $L_A(a) = 1$. We can write a as $a' + \lambda e_A$ with $a' \in \mathcal{D}_R(A)$ and $\lambda \in \mathbb{R}$ by Lemma 4.1. Since $\mathcal{D}_R(B)$ is compact we can find $b' \in \mathcal{D}_R(B)$ with $\|a' - b'\| \leqslant d$. If we let $b = b' + \lambda e_B$, then we have $N(a, b) = N(a', b')$, and we just need $N(a', b') \leqslant 1$. So we need to replace the norm $\|\cdot\|_*$ by $\|(a, b)\|_1 = \max(\|a\|/R, \|b\|/R, \|a - b\|/d)$. Then define N as $N(a, b) = \inf\{\|(a, b) + \lambda(e_A, e_B)\|_1 : \lambda \in \mathbb{R}\}$. The above discussion shows that N is a bridge. Then we have the admissible Lip-norm L associated to N.

Now let $p \in S(A)$. We need to find $q \in S(B)$ such that $\rho_L(p, q) \leqslant \frac{5}{2}d$. Let $(a, b) \in A \oplus B$ with $L(a, b) \leqslant 1$. Adding a scalar multiple of (e_A, e_B), we may assume that $L_A(a)$, $L_B(b)$, $\|(a, b)\|_1 \leqslant 1$. Then $\|a - b\| \leqslant d$, $\|a\| \leqslant R$, and $\|b\| \leqslant R$. Hence $a \in \mathcal{D}_R(A)$ and $b \in \mathcal{D}_R(B)$. So we are looking for $q \in S(B)$ such that $|p(a) - q(b)| \leqslant \frac{5}{2}d$ for all $a \in \mathcal{D}_R(A), b \in \mathcal{D}_R(B)$ with $\|a - b\| \leqslant d$. Denote the set of such pairs (a, b) by X. By the Hahn–Banach theorem we can extend $p \in A'$ to a $P \in V'$ with $\|P\|_{V'} = \|p\|_{A'} = 1$. Let $g = P|_B$. Then $|p(a) - g(b)| = |P(a - b)| \leqslant \|a - b\| \leqslant d$ for all $(a, b) \in X$, and $\|g\|_{B'} \leqslant \|P\|_{V'} = 1$. Also $|1 - g(e_B)| = |P(e_A - e_B)| \leqslant \|e_A - e_B\| \leqslant d/R$. Now we need:

Lemma 4.11. *Let $g \in B'$ and $\delta \geqslant 0$ with $1 \geqslant \|g\| \geqslant g(e_B) \geqslant 1 - \delta > 0$. Then there is a $q \in S(B)$ such that $\|q - g\| \leqslant \frac{3}{2}\delta$.*

Proof. We use the idea in Lemma 2.1 of [36]. Think of B as a subspace of $C_{\mathbb{R}}(S(B))$, the space of $\mathbb{R}$-valued continuous functions on $S(B)$. Then by the Hahn–Banach theorem g extends to a continuous linear functional on $C_{\mathbb{R}}(S(B))$ with the same norm. Using the Jordan decomposition we can write g as $\mu - \nu$ with $\|g\| = \|\mu\| + \|\nu\|$, where μ and ν are disjoint nonnegative measures on $S(B)$. Then $1 \geqslant \|\mu\| + \|\nu\|$ and $\|\mu\| - \|\nu\| = \mu(e_B) - \nu(e_B) = g(e_B) \geqslant 1 - \delta$. Consequently $\|\mu\| \geqslant 1 - \delta > 0$ and $\|\nu\| \leqslant \delta/2$. Note that $\|\mu\| = \mu(e_B) = \|\mu\|_{B'}$. Let $q = \mu/\|\mu\|$. Then $q \in S(B)$ and $\|g - q\| \leqslant \|\nu\| + \|\mu - q\| \leqslant \delta/2 + \|q(1 - \|\mu\|)\| \leqslant \frac{3}{2}\delta$. $\square$

Proof of Proposition 4.10 (*Conclusion*). Pick q for g and $\delta = d/R$ as in Lemma 4.11. Then $|p(a) - q(b)| \leqslant |p(a) - g(b)| + |q(b) - g(b)| \leqslant d + \frac{3}{2}(d/R)R = \frac{5}{2}d$ for all $(a, b) \in X$. Consequently $\mathrm{dist}_{\mathrm{q}}(A, B) \leqslant \mathrm{dist}_{\mathrm{H}}^{\rho_L}(S(A), S(B)) \leqslant \frac{5}{2}d$. Letting V run over all normed spaces containing A and B, we get (9). $\square$

Now Theorem 1.1 follows from Lemma 4.6 and Propositions 4.8 and 4.10. We do not know whether the constants in Theorem 1.1 are the best ones or not.

Remark 4.12. Notice that the terms $\|h_A(r_A e_A) - h_B(r_B e_B)\|$ and $\|h_A(R e_A) - h_B(R e_B)\|$ in Definition 4.2 are used only in the proof of Proposition 4.10 (and hence

Theorem 1.1). Denote by $\mathrm{dist}'_{\mathrm{oq}}(A, B)$ the distance omitting the term $\|h_A(r_A e_A) - h_B(r_B e_B)\|$. If one can show that any compact quantum metric spaces (A, L_A) and (B, L_B) with $\mathrm{dist}'_{\mathrm{oq}}(A, B) = 0$ are isometric, i.e. $\mathrm{dist}'_{\mathrm{oq}}$ is a metric on CQM, then it is not hard to use (5), (7), Lemma 5.4, and Theorems 2.1 and 2.4 to show that $\mathrm{dist}_{\mathrm{q}}$ and $\mathrm{dist}'_{\mathrm{oq}}$ define the same topology on CQM. When (A, L_A) and (B, L_B) are closed, it is easy to see that $\mathrm{dist}'_{\mathrm{oq}}(A, B) = 0$ if and only if there is an affine isometry from $\mathcal{D}(A)$ onto $\mathcal{D}(B)$ (which has to map 0_A to 0_B). Clearly such isometry extends to a linear isometry from A onto B. Thus the question is:

Question 4.13. Let (A, L_A) and (B, L_B) be compact quantum metric spaces. If there is a linear isometry (for the norms) φ from A onto B mapping $\mathcal{D}(A)$ onto $\mathcal{D}(B)$, then are (A, L_A) and (B, L_B) isometric as quantum metric spaces?

Notice that if $\varphi(e_A) = e_B$ then φ is an isometry as quantum metric spaces. A related question is:

Question 4.14. Let (A, e_A) and (B, e_B) be order-unit spaces. If they are isometric as normed spaces, then must they be isomorphic as order-unit spaces?

5. Quantum compactness theorem

In this section we prove Theorem 5.5, which describes Rieffel's quantum compactness theorem in terms of the balls $\mathcal{D}(A)$.

The main fact we need is Corollary 5.3, which can be proved directly. Since Proposition 5.2 will be useful at other places, we include Proposition 5.2 here, and deduce Corollary 5.3 from it. We shall need the following well-known fact several times. We omit the proof.

Lemma 5.1. *Let (X, ρ) be a metric space and Y a subset of X. Then for any $\varepsilon > 0$ we have $\mathrm{Cov}_\rho(Y, 2\varepsilon) \leqslant \mathrm{Cov}_\rho(X, \varepsilon)$, where $\mathrm{Cov}_\rho(X, \varepsilon)$ is the smallest number of open balls of radius ε whose union covers X.*

Proposition 5.2 is the dual version of the fact that if a subset $\mathcal{S} \subseteq \mathrm{CM}$ satisfies the two conditions in Theorem 2.1, then there is a compact metric space (Z, ρ) such that each $X \in \mathcal{S}$ can be isometrically embedded into Z [17, p. 65]. The proof here is a modification of that for this fact given in [17].

Proposition 5.2. *Let $R \geqslant 0$. For any compact metric space (X, ρ) let $C(X)_R := \{f \in C(X) : L_\rho(X) \leqslant 1, \|f\| \leqslant R\}$, equipped with the metric induced from the supremum norm in the algebra $C(X)$ of $\mathbb{C}$-valued continuous functions on X, where L_ρ is the Lipschitz seminorm as defined by* (1). *If a subset $\mathcal{S} \subseteq \mathrm{CM}$ satisfies the condition* (2) *in Theorem* 2.1, *then there exist a complex Banach space V and a compact convex subset $Z \subseteq V$ such that for every $(X, \rho) \in \mathcal{S}$ there is a linear isometric embedding $h_X : C(X) \hookrightarrow V$ with $h_X(C(X)_R) \subseteq Z$.*

Proof. For any $(X, \rho) \in \mathcal{S}$ if we pick a dense sequence in X, then the linear map $C(X) \to \ell^\infty$ given by the evaluations at these points is an isometric embedding. What we shall do is to choose this dense sequence carefully such that the image of $C(X)_R$ is contained in some compact $Z \subseteq \ell^\infty$ which does not depend on (X, ρ).

Let $\varepsilon_j = 2^{-j}$ for all $j \in \mathbb{N}$. Also let $K_1 = \sup\{\mathrm{Cov}(X, \varepsilon_1) : (X, \rho) \in \mathcal{S}\}$ and $K_j = \sup\{\mathrm{Cov}(X, \frac{\varepsilon_j}{2}) : (X, \rho) \in \mathcal{S}\}$ for all $j > 1$. Denote by D_j the set of all finite sequences of the form $(n_1, n_2, \ldots, n_j)$, $1 \leqslant n_1 \leqslant K_1, 1 \leqslant n_2 \leqslant K_2, \ldots, 1 \leqslant n_j \leqslant K_j$, and denote by $p_j : D_{j+1} \to D_j$ the natural projection.

We claim that for each $(X, \rho) \in \mathcal{S}$ there are maps $I_X^j : D_j \to X$ with the following properties:

(a) the image of I_X^j forms an ε_j-net in X, i.e. the open ε_j-balls centered at the points of this image cover X;

(b) for each $\omega \in D_{j+1}$, $j = 1, 2, \ldots,$ the point $I_X^{j+1}(\omega)$ is contained in the open ε_j-ball centered at $I_X^j(p_j(\omega))$.

These maps are constructed as follows. Notice that $K_1 \geqslant \mathrm{Cov}(X, \varepsilon_1)$. So we can cover X by K_1 open balls of radius ε_1, and we take any bijective map from D_1 onto the set of centers of these balls. This is our map I_X^1. For any ε_1-ball $\mathcal{B}$, by Lemma 5.1, we have $K_2 \geqslant \mathrm{Cov}(X, \frac{\varepsilon_2}{2}) \geqslant \mathrm{Cov}(\mathcal{B}, \varepsilon_2)$. So we can cover each open ε_1-ball by K_2 balls of radius ε_2 and map D_2 onto the set of centers of these ε_2-balls so that (n_1, n_2) goes to the center of a ball which we used to cover the ε_1-ball with center at $I_X^1((n_1))$. This is our I_X^2. Then we cover each ε_2-ball by K_3 open balls of radius ε_3 and map D_3 onto the set of centers of these ε_3-balls so that $(n_1, n_2, n_3) \in D_3$ goes to the center of a ball which was used in covering the ε_2-ball with center at $I_X^2((n_1, n_2))$, and so on.

Denote by D the union $\cup_{j=1}^\infty D_j$, and let V be the space of all bounded $\mathbb{C}$-valued functions on D. Then V is a Banach space under the supremum norm $\|\cdot\|$. Denote by $Z \subseteq V$ the set which consists of all functions f satisfying the following inequalities:

if $\omega \in D_1 \subseteq D$, then $|f(\omega)| \leqslant R$,

if $\omega \in D_j$ and $j > 1$, then $|f(\omega) - f(p_{i-1}(\omega))| \leqslant \varepsilon_{j-1}$.

Clearly Z is a closed convex subset of V. We show that Z is totally bounded. For any $\varepsilon > 0$, pick k such that $\varepsilon_k < \varepsilon$. Let P_k be the map restricting functions on D to $\cup_{j=1}^k D_j$. From the inequalities above we see that $|f(\omega)| \leqslant R + \sum_{j=1}^{i-1} \varepsilon_j$ for each $f \in Z$ and $\omega \in D_i$. So $P_k(Z)$ is contained in $F_k := \{g \in C(\cup_{j=1}^k D_j) : \|g\| \leqslant R + \sum_{j=1}^{k-1} \varepsilon_j\}$. Hence $P_k(Z)$ is totally bounded. Pick $f_1, \ldots, f_m$ in Z such that the open ε-balls around $P_k(f_1), \ldots, P_k(f_m)$ cover $P_k(Z)$. Then for any $f \in Z$, there is some $1 \leqslant l \leqslant m$ so that $\|P_k(f) - P_k(f_l)\| < \varepsilon$. This means that $|f(\omega) - f_l(\omega)| < \varepsilon$ for all $\omega \in \cup_{j=1}^k D_j$. In particular, $|f(\omega) - f_l(\omega)| < \varepsilon$ for all $\omega \in D_k$. From the second inequality above we see that $|f(\omega) - f_l(\omega)| < \varepsilon + \sum_{j=k}^\infty \varepsilon_j = \varepsilon + 2\varepsilon_k < 3\varepsilon$ for all $\omega \in D \setminus \cup_{j=k+1}^\infty D_j$. So $\|f - f_l\| < 3\varepsilon$. Therefore $f_1, \ldots, f_m$ is a 3ε-net of Z, and hence Z is totally bounded. So Z is compact.

Denote by $I_X : D \to X$ the map corresponding to all I_X^j, $j = 1, 2, \ldots$. Then we can define $h_X : C(X) \to V$ as the pull back of I_X:

$$(h_X(f))(\omega) = f(I_X(\omega)), \quad f \in C(X), \ \omega \in D.$$

Clearly h_X is linear. The property (a) implies that $I_X(D)$ is dense in X. Thus the map h_X is isometric. For each $f \in C(X)_R$ and $\omega \in D_1$, we have $|(h_X(f))(\omega)| \leqslant \|h_X(f)\| = \|f\| \leqslant R$. If $\omega \in D_j$, and $j > 1$, then by property (b), we have

$$|(h_X(f))(\omega) - (h_X(f))(p_{j-1}(\omega))| = |f(I_X(\omega)) - f(I_X(p_{j-1}(\omega)))|$$
$$\leqslant L_\rho(f)\rho(I_X(\omega), I_X(p_{j-1}(\omega))) \leqslant \varepsilon_{j-1}.$$

So $h_X(f) \in Z$. Therefore $h_X(C(X)_R)$ is contained in Z. $\square$

Corollary 5.3. *Let the notation and hypothesis be as in Proposition* 5.2. *Then the set* $\{C(X)_R : (X, \rho) \in \mathcal{S}\}$ *satisfies condition* (2) *in Theorem* 2.1.

Proof. This is a direct consequence of Proposition 5.2 and Lemma 5.1. $\square$

Lemma 5.4. *Let* $\mathcal{S}$ *be a subset of* CQM. *Pick a closed representative for each element in* $\mathcal{S}$. *Then the set* $\{S(A) : (A, L) \in \mathcal{S}\}$ *satisfies conditions* (1) *and* (2) *in Theorem* 2.1 *if and only if the set* $\{\mathcal{D}(A) : (A, L) \in \mathcal{S}\}$ *does. If* $R \geqslant \sup\{r_A : (A, L) \in \mathcal{S}\}$, *then the set* $\{S(A) : (A, L) \in \mathcal{S}\}$ *satisfies condition* (2) *in Theorem* 2.1 *if and only if the set* $\{\mathcal{D}_R(A) : (A, L) \in \mathcal{S}\}$ *does.*

Proof. We prove the first equivalence. The proof for the second one is similar. Notice that the radius of $\mathcal{D}(A)$ is exactly r_A. Thus $\{S(A) : (A, L) \in \mathcal{S}\}$ satisfies condition (1) in Theorem 2.1 if and only if $\{\mathcal{D}(A) : (A, L) \in \mathcal{S}\}$ does.

Assume that condition (1) is satisfied now. Let $R \geqslant \sup\{r_A : (A, L) \in \mathcal{S}\}$. Notice that the natural inclusion $\mathcal{D}(A) \to C(S(A))$ is isometric, and has image in $C(S(A))_R$, where $C(S(A))_R$ is defined as in Proposition 5.2. Then the "only if" part follows from Corollary 5.3 and Lemma 5.1.

Notice that the natural pairing between A and A' gives a map $\psi : S(A) \to C(\mathcal{D}(A))$. Clearly ψ maps $S(A)$ into $C(\mathcal{D}(A))_R$. From Lemma 4.1 it is easy to see that ψ is isometric. Then the "if" part also follows from Corollary 5.3 and Lemma 5.1. $\square$

Combining Lemma 5.4 and Theorem 2.4 together we get:

Theorem 5.5. *A subset* $\mathcal{S} \subseteq$ CQM *is totally bounded if and only if*

(1′) *there is a constant* D' *such that* $\mathrm{diam}(\mathcal{D}(A)) \leqslant D'$ *for all* $(A, L) \in \mathcal{S}$;
(2′) *for any* $\varepsilon > 0$, *there exists a constant* $K'_\varepsilon > 0$ *such that* $\mathrm{Cov}(\mathcal{D}(A), \varepsilon) \leqslant K'_\varepsilon$ *for all* $(A, L) \in \mathcal{S}$.

One can also give a direct proof of Theorem 5.5 (see [26, Remark 4.10]).

In Section 9 we shall use Theorem 5.5 to prove Theorem 9.2, which tells us when a family of compact quantum metric spaces induced from ergodic actions of a fixed compact group is totally bounded.

6. Continuous fields of compact quantum metric spaces

In this section we define continuous fields of compact quantum metric spaces, a framework we shall use in Section 7 to discuss the continuity of families of compact quantum metric spaces with respect to $\mathrm{dist}_{\mathrm{oq}}$. The main results of this section are Theorem 6.12 and Proposition 6.16. We refer the reader to [15, Sections 10.1, 10.2] for basic definitions and facts about continuous fields of Banach spaces.

We first define continuous fields of order-unit spaces. To reflect the continuity of the order structures, clearly we should require that the order-unit section is continuous.

Definition 6.1. Let T be a locally compact Hausdorff space. A *continuous field of order-unit spaces* over T is a continuous field $(\{A_t\}, \Gamma)$ of Banach spaces over T, each A_t being a complete order-unit space with its order-unit norm, and the unit section e given by $e_t = e_{A_t}, t \in T$ being in the space Γ of continuous sections.

Remark 6.2. Not every continuous field of Banach spaces consisting of order-unit spaces is continuous as a field of order-unit spaces. For a trivial example, let $T = [0, 1]$, and let $(\{A_t\}, \Gamma)$ be the trivial field over T with fibers $(A_t, e_{A_t}) = (\mathbb{R}, 1)$. For each $f \in \Gamma$ define a section f^* as $f^*_t = f_t$ for $0 \leqslant t < 1$ and $f^*_1 = -f_1$. Then $\Gamma^* = \{f^* : f \in \Gamma\}$ defines a continuous field of Banach spaces over T with the same fibers, but with the section $t \mapsto 1$ no longer being continuous.

Before we define continuous fields of compact quantum metric spaces, let us take a look at one example:

Example 6.3 (*Quotient field of a compact quantum metric space*). Let (B, L_B) be a closed compact quantum metric space. Let T be the set of all nonempty convex closed subsets of $S(B)$. Notice that for any compact metric space (X, ρ), the space $\mathrm{SUB}(X)$ of closed nonempty subsets of X is compact equipped with the metric $\mathrm{dist}^X_{\mathrm{H}}$ [8, Proposition 7.3.7]. It is easy to see that $(T, \mathrm{dist}^{S(B)}_{\mathrm{H}})$ is a closed subspace of $(\mathrm{SUB}(S(B)), \mathrm{dist}^{S(B)}_{\mathrm{H}})$, and hence is a compact metric space. Now each $t \in T$ is a closed convex subset of $S(B)$. Let (A_t, L_t) be the corresponding quotient of (B, L_B) (see the discussion right after Proposition 2.3). Then $\overline{A_t} = \mathrm{Af}_{\mathbb{R}}(t)$. Let $\pi_t : \mathrm{Af}_{\mathbb{R}}(S(B)) \to \mathrm{Af}_{\mathbb{R}}(t)$ be the restriction map. Since each $w \in \mathrm{Af}_{\mathbb{R}}(S(B))$ is uniformly continuous over $S(B)$, clearly the function $t \mapsto \|\pi_t(w)\|$ is continuous over T. Hence the sections $\pi(w) = \{\pi_t(w)\}$ for all $w \in \mathrm{Af}_{\mathbb{R}}(S(B))$ generate a continuous field of Banach spaces over T with fibers $\mathrm{Af}_{\mathbb{R}}(t) = \overline{A_t}$. Notice that the unit section is just $\pi(e_B)$. So this is a continuous field of order-unit spaces. We shall call it the *quotient field* of (B, L_B). According to [38, Proposition 5.7] we have that $\mathrm{dist}_{\mathrm{q}}(A_t, A_{t_0}) \to 0$ as $t \to t_0$ for any $t_0 \in T$.

Certainly the above example deserves to be called a continuous field of compact quantum metric spaces. In general, we start with a continuous field of order-unit spaces $(\{A_t\}, \Gamma)$ over some T and a Lip-norm L_t on (a dense subspace of) each A_t. These L_t's should satisfy certain continuity conditions for the field to be called a continuous field of compact quantum metric spaces. If we look back at the definition of continuous fields of Banach spaces, we see that the main ingredient is that there are enough continuous sections. Thus one may want to require that there are enough sections f with the functions $t \mapsto L_t(f_t)$ being continuous. But in the above example of the quotient field, clearly $t \mapsto L_t(\pi_t(w))$ is always lower semi-continuous, and there are no obvious w's except the scalars for which the functions $t \mapsto L_t(\pi_t(w))$ are continuous. Thus this requirement is too strong. Now there are two weaker ways to explain "enough continuous sections". The first one is that the structure (which is L_{t_0} in our case) at A_{t_0} should be determined by the sections "continuous at t_0". Let $\Gamma^L_{t_0}$ be the set of sections f in Γ such that $t \mapsto L_t(f_t)$ is continuous at t_0. Then when L_{t_0} is closed, one wants every $a \in A_{t_0}$ to have a lifting in $\Gamma^L_{t_0}$. When L_{t_0} is not closed (which happens in a lot of natural examples), recalling how the closure $L^c_{t_0}$ is defined, one wants $L^c_{t_0}$ to be determined by these f_{t_0} when f runs over $\Gamma^L_{t_0}$. The second way to think of "enough continuous sections" is that there should be enough continuous sections to connect the fibers. Then one wants that for every $a \in A_{t_0}$ there is some $f \in \Gamma$ such that $f_{t_0} = a$ and $L_t(f_t) \leqslant L_{t_0}(a)$ for all $t \in T$. This implies that for every $a \in A_{t_0}$ there is some $f \in \Gamma$ with $f_{t_0} = a$ and $t \mapsto L_t(f_t)$ being upper semi-continuous at t_0. This is weaker than what we get above in the first way. However, it turns out that this condition is strong enough for us to prove some properties of continuous fields of compact quantum metric spaces (see Theorem 6.12 and Proposition 6.16), especially the criteria for continuity under the order-unit quantum distance (Theorems 1.2 and 7.1).

Definition 6.4. Let T be a locally compact Hausdorff space, and let (A_t, L_t) be a compact quantum metric space for each $t \in T$, with completion $\overline{A_t}$. Let Γ be the set of continuous sections of a continuous field of order-unit spaces over T with fibers $\overline{A_t}$. For each $t_0 \in T$ set

$$\Gamma^L_{t_0} = \{f \in \Gamma : \text{ the function } t \mapsto L_t(f_t) \text{ is upper semi-continuous at } t_0\},$$

where we use the convention that $L_t = +\infty$ on $\overline{A_t} \setminus A_t$. We call $(\{(A_t, L_t)\}, \Gamma)$ a *continuous field of compact quantum metric spaces* over T if for any $t \in T$ the restriction of L_t to $\{f_t : f \in \Gamma^L_t\}$ determines the closure of L_t, i.e. for any $a \in A_t$ and $\varepsilon > 0$ there exists $f \in \Gamma^L_t$ such that $\|f_t - a\| < \varepsilon$ and $L_t(f_t) < L_t(a) + \varepsilon$. Sections in Γ^L_t are called *Lipschitz sections* at t. If every L_t is closed, we say that $(\{(A_t, L_t)\}, \Gamma)$ is *closed*.

Remark 6.5. At first sight, for the restriction of L_t to $\{f_t : f \in \Gamma^L_t\}$ to determine the closure of L_t, we should require that for any $a \in \overline{A_t}$ and $\varepsilon > 0$ there exists $f \in \Gamma^L_t$ such that $\|f_t - a\| < \varepsilon$ and $L_t(f_t) < \overline{L_t}(a) + \varepsilon$. This seems stronger than the condition we put in Definition 6.4. In fact they are equivalent. By the definition of $\overline{L_t}$ we can find $a' \in A_t$ with $\|a' - a\| < \frac{1}{2}\varepsilon$ and $L_t(a') < \overline{L_t}(a) + \frac{1}{2}\varepsilon$. Assume that the condition in Definition 6.4 holds. Then there exists $f \in \Gamma^L_t$ such that $\|f_t - a'\| < \frac{1}{2}\varepsilon$ and $L_t(f_t) < L_t(a') + \frac{1}{2}\varepsilon$. Consequently, $\|f_t - a\| < \varepsilon$ and $L_t(f_t) < \overline{L_t}(a) + \varepsilon$.

As one would expect, the fiberwise closure of a continuous field of compact quantum metric spaces is still such a field:

Proposition 6.6. *If* $(\{(A_t, L_t)\}, \Gamma)$ *is a continuous field of compact quantum metric spaces over a locally compact Hausdorff space T, then so is* $(\{(A^c_t, L^c_t)\}, \Gamma)$. *If* $(\{(A_t, L_t)\}, \Gamma)$ *is closed, then* $\overline{A_t} = \{f_t : f \in \Gamma^L_t\}$ *for every* $t \in T$.

Proof. Let $t_0 \in T$, and let $a \in A^c_{t_0}$. We need to find $g \in \Gamma$ such that $g_{t_0} = a$ and $L^c_{t_0}(a) \geqslant \limsup_{t \to t_0} L^c_t(g_t)$. Take a section $f \in \Gamma$ with $f_{t_0} = a$. For each $n \in \mathbb{N}$ by Remark 6.5 we can find an $f_n \in \Gamma^L_{t_0}$ such that $\|(f_n)_{t_0} - a\| < \frac{1}{n}$ and $L_{t_0}((f_n)_{t_0}) < L^c_{t_0}(a) + \frac{1}{n}$. There is an open neighborhood $\mathcal{U}_n$ of t_0 with compact closure such that $\|(f_n)_t - f_t\| < \frac{1}{n}$ and $L_t((f_n)_t) < L^c_{t_0}(a) + \frac{1}{n}$ for all $t \in \mathcal{U}_n$. By shrinking these neighborhoods we may assume that $\overline{\mathcal{U}_{n+1}} \subseteq \mathcal{U}_n$ for all n. By Urysohn's lemma [21, p. 115] we can find a continuous function w_n on T with $0 \leqslant w_n \leqslant 1$, $w_n|_{T \setminus \mathcal{U}_n} = 0$, and $w_n|_{\mathcal{U}_{n+1}} = 1$. Define a section g by $g_t = (f_1)_t$ for $t \in T \setminus \mathcal{U}_1$, $g_t = w_n(t)(f_{n+1})_t + (1 - w_n(t))(f_n)_t$ for $t \in \mathcal{U}_n \setminus \mathcal{U}_{n+1}$, and $g_t = f_t$ for $t \in \cap_{n=1}^{\infty} \mathcal{U}_n$. Clearly $g \in \Gamma$, $g_{t_0} = a$, and $L^c_{t_0}(a) \geqslant \limsup_{t \to t_0} L^c_t(g_t)$. $\square$

Example 6.7 (*Pull back*). Let $(\{(A_t, L_t)\}, \Gamma)$ be a continuous field of compact quantum metric spaces over a locally compact Hausdorff space T. Let T' be another locally compact Hausdorff space, and let $\Phi : T' \to T$ be a continuous map. Set $(A_{t'}, L_{t'}) = (A_{\Phi(t')}, L_{\Phi(t')})$ for each $t' \in T'$. For each $f \in \Gamma$, define a section $\Phi^*(f)$ over T' by $(\Phi^*(f))_{t'} = f_{\Phi(t')}$. Then the set $\Phi^*(\Gamma)$ of all these sections generates a continuous field of Banach spaces over T' with fibers $\overline{A_{t'}} = \overline{A_{\Phi(t')}}$. This is called the *pull back* of the continuous field $(\{\overline{A_t}\}, \Gamma)$. Let $\overline{\Phi^*(\Gamma)}$ be the set of all continuous sections of this field. Notice that the pull back of the unit section is exactly the unit section on T'. So the pull back is a continuous field of order-unit spaces. Clearly for each $t'_0 \in T'$ and $f \in \Gamma^L_{\Phi(t'_0)}$ the function $t' \mapsto L_{t'}((\Phi^*(f))_{t'}) = L_{\Phi(t')}(f_{\Phi(t')})$ is upper semi-continuous at t'_0. Hence $\Phi^*(\Gamma^L_{\Phi(t'_0)}) \subseteq \overline{\Phi^*(\Gamma)}^L_{t'_0}$. Then it is easy to see that $(\{(A_{t'}, L_{t'})\}, \overline{\Phi^*(\Gamma)})$ is a continuous field of compact quantum metric spaces over T'. We shall call it the *pull back* of $(\{(A_t, L_t)\}, \Gamma)$.

Example 6.8 (*Quotient field continued*). Let the notation be as in Example 6.3. For each $t_0 \in T$ and $a \in A_{t_0}$, the proof of [38, Proposition 3.3] shows that we can find $b \in B$ with $\pi_{t_0}(b) = a$ and $L_B(b) = L_{t_0}(a)$. Then obviously $\pi(b)$ is in $\Gamma^L_{t_0}$. Therefore $(\{(A_t, L_t)\}, \Gamma)$ is a closed continuous field of compact quantum metric spaces.

In fact, we can say more about the Lip-norms in the quotient field:

Proposition 6.9. *Let* (B, L_B) *be a closed compact quantum metric space. Let* $(\{(A_t, L_t)\}, \Gamma)$ *be the corresponding quotient field of compact quantum metric spaces. Then* *for any* $t_0 \in T$ *and* $a \in \overline{A_{t_0}}$ *we have that*

$$L_{t_0}(a) = \inf\left\{\limsup_{t\to t_0} L_t(f_t) : f \in \Gamma,\ f_{t_0} = a\right\}$$
$$= \inf\left\{\liminf_{t\to t_0} L_t(f_t) : f \in \Gamma,\ f_{t_0} = a\right\}.$$

Proof. By Proposition 6.6 we have $L_{t_0}(a) \geqslant \inf\{\limsup_{t\to t_0} L_t(f_t) : f \in \Gamma, f_{t_0} = a\}$. So we just need to show that for any $a \in \overline{A_{t_0}}$ and $f \in \Gamma$ with $f_{t_0} = a$ we have $L_{t_0}(a) \leqslant \liminf_{t\to t_0} L_t(f_t)$. If $L_{t_0}(a) = 0$, this is trivial. So we assume that $L_{t_0}(a) > 0$. We prove the case $L_{t_0}(a) < +\infty$. The proof for the case $L_{t_0}(a) = +\infty$ is similar.

Let $\rho = \rho_{L_B}$. By [38, Proposition 3.3] we can find $b \in B$ with $\pi_{t_0}(b) = a$. For any $\varepsilon > 0$, since L_{t_0} coincides with the Lip-norm induced by $\rho|_{t_0}$, we can pick distinct points p_1, p_2 in t_0 with $\frac{|a(p_1)-a(p_2)|}{\rho(p_1,p_2)} \geqslant L_{t_0}(a) - \varepsilon$. For any $\delta > 0$ and $t \in T$ with $\mathrm{dist}_{\mathrm{H}}^{S(B)}(t, t_0) < \delta$, we can find $q_1, q_2 \in t$ with $\rho(p_j, q_j) < \delta$. Since b is uniformly continuous on $S(B)$, when δ is small enough, we have

$$\left|\frac{|(\pi(b))_t(q_1)-(\pi(b))_t(q_2)|}{\rho(q_1,q_2)} - \frac{|a(p_1)-a(p_2)|}{\rho(p_1,p_2)}\right| = \left|\frac{|b(q_1)-b(q_2)|}{\rho(q_1,q_2)} - \frac{|b(p_1)-b(p_2)|}{\rho(p_1,p_2)}\right|$$
$$< \varepsilon.$$

Then $\frac{|(\pi(b))_t(q_1)-(\pi(b))_t(q_2)|}{\rho(q_1,q_2)} \geqslant L_{t_0}(a) - 2\varepsilon$. Now $f, \pi(b) \in \Gamma$ and $f_{t_0} = (\pi(b))_{t_0} = a$. This implies that $\|f_t - (\pi(b))_t\| \to 0$ as $t \to t_0$. For $\delta < \frac{1}{3}\rho(p_1,p_2)$, we have $\rho(q_1,q_2) \geqslant \frac{\rho(p_1,p_2)}{3}$. Hence when t is close enough to t_0 we have that

$$\left|\frac{|f_t(q_1)-f_t(q_2)|}{\rho(q_1,q_2)} - \frac{|(\pi(b))_t(q_1)-(\pi(b))_t(q_2)|}{\rho(q_1,q_2)}\right| < \varepsilon.$$

Therefore

$$L_t(f_t) \geqslant \frac{|f_t(q_1)-f_t(q_2)|}{\rho(q_1,q_2)} \geqslant \frac{|(\pi(b))_t(q_1)-(\pi(b))_t(q_2)|}{\rho(q_1,q_2)} - \varepsilon \geqslant L_{t_0}(a) - 3\varepsilon.$$

Thus $L_{t_0}(a) \leqslant \liminf_{t\to t_0} L_t(f_t)$. □

Example 6.10. Let (B, L_B) be a closed compact quantum metric space, and let $(\{(A_t, L_t)\}, \Gamma)$ be the corresponding quotient field of compact quantum metric spaces. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of closed convex subsets of $S(B)$ converging to some closed convex subset t_0 under $\mathrm{dist}_{\mathrm{H}}^{S(B)}$. Set $T' = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ and define $\Phi : T' \to T$ by $\Phi(\frac{1}{n}) = t_n$ and $\Phi(0) = t_0$. Then Φ is continuous. By Example 6.7 we have the pull back continuous field $(\{(A_{t'}, L_{t'})\}, \overline{\Phi^*(\Gamma)})$ of compact quantum metric spaces over T'.

We will call it *the continuous field corresponding to* $t_n \to t_0$. Clearly the fibers are $A_{\frac{1}{n}} = A_{t_n}$ and $A_0 = A_{t_0}$. The field of Banach spaces $(\{\overline{A_{t'}}\}, \overline{\Phi^*(\Gamma)})$ is generated by the restrictions of functions in $\mathrm{Af}_{\mathbb{R}}(S(B))$.

In the same way as for Proposition 6.9, one can show:

Proposition 6.11. *Let (B, L_B) be a closed compact quantum metric space. Let $(\{(A_t, L_t)\}, \Gamma)$ be the corresponding quotient field of compact quantum metric spaces. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence of closed convex subsets of $S(B)$ converging to some closed convex subset t_0 under $\mathrm{dist}_{\mathrm{H}}^{S(B)}$, and let $(\{(A_{t'}, L_{t'})\}, \overline{\Phi^*(\Gamma)})$ be the pull back field as in Example* 6.10. *Then for any $a \in \overline{A_{t_0}}$ we have that*

$$L_{t_0}(a) = \inf\left\{\limsup_{n\to\infty} L_{t_n}\left(f_{\frac{1}{n}}\right) : f \in \overline{\Phi^*(\Gamma)},\ f_0 = a\right\}$$

$$= \inf\left\{\liminf_{n\to\infty} L_{t_n}(f_{\frac{1}{n}}) : f \in \overline{\Phi^*(\Gamma)},\ f_0 = a\right\}.$$

Theorem 6.12. *Let $(\{(A_t, L_t)\}, \Gamma)$ be a continuous field of compact quantum metric spaces over a locally compact Hausdorff space T. Then the radius function $t \mapsto r_{A_t}$ is lower semi-continuous over T.*

Lemma 6.13. *Let $(\{A_t\}, \Gamma)$ be a continuous field of real Banach spaces over a locally compact Hausdorff space T, and let f be a nowhere-vanishing section in Γ. Let $\|\cdot\|_t^{\sim}$ be the quotient norm in $A_t/\mathbb{R}f_t$. Then for any $g \in \Gamma$ the function $t \mapsto \|\tilde{g}_t\|_t^{\sim}$ is continuous over T.*

Proof. Replacing f by $t \mapsto \frac{f_t}{\|f_t\|}$, we may assume that $\|f_t\| = 1$ for all $t \in T$. For every $t \in T$ pick $c_t \in \mathbb{R}$ with $\|g_t - c_t f_t\| = \|\tilde{g}_t\|_t^{\sim}$. Let $t_0 \in T$ and $\varepsilon > 0$ be given. Since $g - c_{t_0} f \in \Gamma$, the function $t \mapsto \|g_t - c_{t_0} f_t\|$ is continuous over T. So there is a neighborhood $\mathcal{U}$ of t_0 such that for any $t \in \mathcal{U}$, we have $\|g_t - c_{t_0} f_t\| < \|\tilde{g}_{t_0}\|_{t_0}^{\sim} + \varepsilon$, and hence $\|\tilde{g}_t\|_t^{\sim} < \|\tilde{g}_{t_0}\|_{t_0}^{\sim} + \varepsilon$. This shows that the function $t \mapsto \|\tilde{g}_t\|_t^{\sim}$ is upper semi-continuous over T.

We proceed to show that the function $t \mapsto \|\tilde{g}_t\|_t^{\sim}$ is lower semi-continuous over T. We may assume that the neighborhood $\mathcal{U}$ in the above is compact. Let $M := \sup\{\|g_t\| : t \in \mathcal{U}\} < \infty$. Then for every $t \in \mathcal{U}$ we have that

$$|c_t| = \|c_t f_t\| \leqslant \|g_t\| + \|g_t - c_t f_t\| \leqslant M + \|\tilde{g}_t\|_t^{\sim} < M + \|\tilde{g}_{t_0}\|_{t_0}^{\sim} + \varepsilon.$$

Let $I = [-(M + \|\tilde{g}_{t_0}\|_{t_0}^{\sim} + \varepsilon),\ M + \|\tilde{g}_{t_0}\|_{t_0}^{\sim} + \varepsilon]$. Clearly the function $(c, t) \mapsto \|g_t - cf_t\|$ is continuous over $I \times \mathcal{U}$. Since I is compact, we can find a neighborhood $\mathcal{U}_1 \subseteq \mathcal{U}$ of t_0 so that $|\|g_t - cf_t\| - \|g_{t_0} - cf_{t_0}\|| < \varepsilon$ for all $(c, t) \in I \times \mathcal{U}_1$. Then for any $t \in \mathcal{U}_1$

we have that

$$\|\tilde{g}_{t_0}\|^{\sim}_{t_0} \leqslant \|g_{t_0} - c_t f_{t_0}\| < \|g_t - c_t f_t\| + \varepsilon = \|\tilde{g}_t\|^{\sim}_t + \varepsilon$$

So the function $t \mapsto \|\tilde{g}_t\|^{\sim}_t$ is lower semi-continuous, and hence continuous, over T. □

Taking f in Lemma 6.13 to be the unit section, we get immediately:

Lemma 6.14. *Let $(\{A_t\}, \Gamma)$ be a continuous field of order-unit spaces over a locally compact Hausdorff space T. Let $\|\cdot\|^{\sim}_t$ be the quotient norm in $\tilde{A}_t = A_t/\mathbb{R}e_{A_t}$. Then for any $f \in \Gamma$, the function $t \mapsto \|\tilde{f}_t\|^{\sim}_t$ is continuous over T.*

We are ready to prove Theorem 6.12.

Proof of Theorem 6.12. By Proposition 6.6 we may assume that $(\{(A_t, L_t)\}, \Gamma)$ is closed. Let $t_0 \in T$ and $\varepsilon > 0$ be given. If $A_{t_0} = \mathbb{R}e_{A_{t_0}}$, then $r_{A_{t_0}} = 0$ and the radius function is obviously lower semi-continuous at t_0. So we may assume that $A_{t_0} \neq \mathbb{R}e_{A_{t_0}}$. Then $r_{A_{t_0}} = \sup\{\|\tilde{a}\|^{\sim}_{t_0} : \tilde{a} \in \tilde{A}_{t_0} \text{ with } \tilde{L}_{t_0}(\tilde{a}) = 1\}$ by Proposition 2.3. Pick $a \in A_{t_0}$ with $L_{t_0}(a) = \tilde{L}_{t_0}(\tilde{a}) = 1$ and $\|\tilde{a}\|^{\sim}_{t_0} > r_{A_{t_0}} - \varepsilon$. By Proposition 6.6 we can find $f \in \Gamma^L_{t_0}$ with $f_{t_0} = a$. Then the function $t \mapsto L_t(f_t)$ is upper semi-continuous at t_0. By Lemma 6.14 the function $t \mapsto \|\widetilde{f_t}\|^{\sim}_t$ is continuous over T. So there is some neighborhood $\mathcal{U}$ of t_0 in T such that $\|\widetilde{f_t}\|^{\sim}_t / L_t(f_t) > \|\widetilde{f_{t_0}}\|^{\sim}_{t_0} - \varepsilon > r_{A_{t_0}} - 2\varepsilon$ for all $t \in \mathcal{U}$. Then $r_{A_t} > r_{A_{t_0}} - 2\varepsilon$ for all $t \in \mathcal{U}$ by Proposition 2.3. So the radius function is lower semi-continuous at t_0. □

Next we show that there are enough Lipschitz sections to connect the fibers. The next lemma is probably known, but we cannot find a reference, so we include a proof.

Lemma 6.15. *Let T be a locally compact Hausdorff space, and let w be a nonnegative function on T. If w is lower semi-continuous at some point $t_0 \in T$, then there is a continuous nonnegative function w' over T with $w'(t_0) = w(t_0)$ and $w' \leqslant w$ on T.*

Proof. If $w(t_0) = 0$, we may take $w' = 0$. So assume $w(t_0) > 0$. Replacing w by $\frac{w}{w(t_0)}$, we may assume $w(t_0) = 1$.

Since w is lower semi-continuous at t_0, for each $n \in \mathbb{N}$ we can find an open neighborhood $\mathcal{U}_n$ of t_0 such that $w \geqslant 1 - 2^{-n}$ on $\mathcal{U}_n$. By shrinking $\mathcal{U}_n$ we may assume that the closure of $\mathcal{U}_n$ is compact and contained in $\mathcal{U}_{n-1}$ for each $n \in \mathbb{N}$, where $\mathcal{U}_0 = T$. By Urysohn's lemma [21, p. 115], we can find a continuous function w'_n over T with $0 \leqslant w'_n \leqslant 2^{-n}$, $w'_n|_{T \setminus \mathcal{U}_n} = 0$, and $w'_n|_{\overline{\mathcal{U}_{n+1}}} = 2^{-n}$. Then $w' = \sum_{n=1}^{\infty} w'_n$ is continuous over T.

If $t \in T \setminus \mathcal{U}_1$, then $w'_j(0) = 0$ for all j and hence $w'(0) = 0 \leqslant w(t)$. If $t \in \mathcal{U}_n \setminus \mathcal{U}_{n+1}$ for some $n \in \mathbb{N}$, then $w'_j(t) = 2^{-j}$ for all $1 \leqslant j < n$, $0 \leqslant w'_n(t) \leqslant \frac{1}{2^n}$, and $w'_j(t) = 0$ for all

$j > n$. So $w'(t) \leqslant \sum_{j=1}^{n} 2^{-j} = 1 - 2^{-n} \leqslant w(t)$. If $t \in \cap_{n=1}^{\infty} \mathcal{U}_n$, then $w(t) \geqslant 1$ according to the construction of $\mathcal{U}_n$. In this case, $w'_j(t) = 2^{-j}$ for all j. So $w'(t) = 1 \leqslant w(t)$. In particular, we see that $w'(t_0) = 1 = w(t_0)$. So w' satisfies our requirement. $\square$

Proposition 6.16. *Let* $(\{(A_t, L_t)\}, \Gamma)$ *be a continuous field of compact quantum metric spaces over a locally compact Hausdorff space* T. *Then for any* $t_0 \in T$ *and* $a \in \{f_{t_0} : f \in \Gamma^L_{t_0}\} \cap \mathcal{D}(A_{t_0})$, *there exists* $f \in \Gamma^L_{t_0}$ *with* $f_t \in \mathcal{D}(A_t)$ *for all* $t \in T$ *and* $f_{t_0} = a$. *In particular, when* $(\{(A_t, L_t)\}, \Gamma)$ *is closed, such* f *exists for every* $a \in \mathcal{D}(A_{t_0})$.

Proof. If $a = 0$, we can pick $f = 0$. So suppose that $a \neq 0$. Take $g \in \Gamma^L_{t_0}$ with $g_{t_0} = a$. Then $0 < \|g_{t_0}\| = \|a\| \leqslant r_{A_{t_0}}$. Since $t \mapsto \|g_t\|$ is continuous on T, there is some neighborhood $\mathcal{U}$ of t_0 such that $\|g_t\| > 0$ for all $t \in \mathcal{U}$. Define a nonnegative function w on T by $w(t_0) = 1$, $w(t) = \frac{r_{A_t}}{\|g_t\|}$ for $t \in \mathcal{U} \setminus \{t_0\}$, and $w(t) = 0$ for $t \in T \setminus \mathcal{U}$. By Theorem 6.12 the radius function $t \mapsto r_{A_t}$ is lower semi-continuous over T. Then it is easy to see that the function w is lower semi-continuous at t_0. According to Lemma 6.15 we can find a continuous nonnegative function w' on T such that $w'(t_0) = 1$ and $w'(t) \leqslant w(t)$ for all $t \in T$. Then $w'(t) \leqslant \frac{r_{A_t}}{\|g_t\|}$ for all $t \in \mathcal{U}$, and $w'(t) = 0$ for $t \in T \setminus \mathcal{U}$. Set $h_t = w'(t) g_t$. Then $h \in \Gamma^L_{t_0}$. Also, $h_{t_0} = a$ and $\|h_t\| \leqslant r_{A_t}$ for all $t \in T$.

If $L_{t_0}(a) < 1$, then $L_{t_0}(h_{t_0}) < 1$. Since $h \in \Gamma^L_{t_0}$, there is an open neighborhood $\mathcal{U}_1$ of t_0 with compact closure such that $L_t(h_t) < 1$ for all $t \in \mathcal{U}_1$. Take a continuous function w'' on T with $0 \leqslant w'' \leqslant 1$, $w''(t_0) = 1$, and $w''|_{T \setminus \mathcal{U}_1} = 0$. Define a section f by $f_t = w''(t) h_t$. Then f is in $\Gamma^L_{t_0}$, and satisfies $L_t(f_t) \leqslant 1$, $\|f_t\| \leqslant \|h_t\| \leqslant r_{A_t}$ for all $t \in T$. So $f_t \in \mathcal{D}(A_t)$ for all $t \in T$. Also $f_{t_0} = w''(t_0) h_{t_0} = a$. Hence f satisfies our requirement.

Now suppose that $L_{t_0}(a) = 1$. Then $L_{t_0}(h_{t_0}) = 1$. Define a nonnegative function w_1 on T as $w_1(t) = \min(\frac{1}{L_t(h_t)}, 1)$ for all $t \in T$, where $\frac{1}{L_t(h_t)} = \infty$ if $L_t(h_t) = 0$. Then $w_1(t_0) = 1$. Since $h \in \Gamma^L_{t_0}$, it is easy to see that w_1 is lower semi-continuous at t_0. According to Lemma 6.15 we can find a continuous nonnegative function w'_1 on T such that $w'_1(t_0) = 1$ and $w'_1(t) \leqslant w_1(t)$ for all $t \in T$. Then $w'_1 \leqslant w_1 \leqslant 1$ on T. Define a section f by $f_t = w'_1(t) h_t$ for all $t \in T$. Then $f \in \Gamma^L_{t_0}$ and $f_{t_0} = w'_1(t_0) h_{t_0} = a$. Clearly $L_t(f_t) = w'_1(t) L_t(h_t) \leqslant w_1(t) L_t(h_t) \leqslant 1$ and $\|f_t\| \leqslant \|h_t\| \leqslant r_{A_t}$ for all $t \in T$. So $f_t \in \mathcal{D}(A_t)$ for all $t \in T$. Hence f satisfies our requirement.

The assertion about closed $(\{(A_t, L_t)\}, \Gamma)$ follows from Proposition 6.6. $\square$

7. Criteria for metric convergence

In this section we prove Theorems 1.2 and 7.1.

When applying Theorem 1.2, usually we need to show that the radius function $t \mapsto r_{A_t}$ is continuous at t_0. This is often quite difficult. However, sometimes we can show easily that the radii are bounded (for example, compact quantum metric spaces induced by ergodic actions of compact groups on complete order-unit spaces of finite multiplicity, see Theorem 8.2). In these cases, the next criterion is more useful. This is also the reason we introduced $\mathrm{dist}^R_{\mathrm{oq}}$.

Theorem 7.1. *Let $(\{(A_t, L_t)\}, \Gamma)$ be a continuous field of compact quantum metric spaces over a locally compact Hausdorff space T. Let $R \geqslant 0$. Let $t_0 \in T$, and let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in Γ such that $(f_n)_{t_0} \in \mathcal{D}_R(A_{t_0})$ for each $n \in \mathbb{N}$ and the set $\{(f_n)_{t_0} : n \in \mathbb{N}\}$ is dense in $\mathcal{D}_R(A_{t_0})$. Then the following are equivalent*:

(i) $\mathrm{dist}^R_{\mathrm{oq}}(A_t, A_{t_0}) \to 0$ *as* $t \to t_0$;
(ii) $\mathrm{dist}_{\mathrm{GH}}(\mathcal{D}_R(A_t), \mathcal{D}_R(A_{t_0})) \to 0$ *as* $t \to t_0$;
(iii) *for any* $\varepsilon > 0$, *there is an* N *such that the open* ε*-balls in* A_t *centered at* $(f_1)_t, \ldots, (f_N)_t$ *cover* $\mathcal{D}_R(A_t)$ *for all* t *in some neighborhood* $\mathcal{U}$ of t_0.

The proof of Theorem 7.1 is similar to that of Theorem 1.2. So we shall prove only Theorem 1.2. We need some preparation. The next lemma generalizes Lemma 4.5 to deal with "almost amalgamation":

Lemma 7.2. *Let A and B be normed spaces* (over $\mathbb{R}$ *or* $\mathbb{C}$). *Let X be a linear subspace of A, and let $\varepsilon \geqslant 0$. Let $\varphi : X \to B$ be a linear map with $(1-\varepsilon)\|x\| \leqslant \|\varphi(x)\| \leqslant (1+\varepsilon)\|x\|$ for all $x \in X$. Then there are a normed space V and linear isometric embeddings $h_A : A \hookrightarrow V$ and $h_B : B \hookrightarrow V$ such that $\|h_A(x) - (h_B \circ \varphi)(x)\| \leqslant \varepsilon\|x\|$ for all $x \in X$.*

Proof. We define a seminorm, $\|\cdot\|_*$, on $A \oplus B$ by

$$\|(a, b)\|_* := \inf\{\|a - x\| + \|b + \varphi(x)\| + \varepsilon\|x\| : x \in X\}.$$

We claim that $\|\cdot\|_*$ extends the norm of A. Let $a \in A$. Taking $x = 0$ we get $\|(a, 0)\|_* \leqslant \|a\|$. For any $x \in X$ we have $\|a - x\| + \|0 + \varphi(x)\| + \varepsilon\|x\| \geqslant \|a - x\| + (1 - \varepsilon)\|x\| + \varepsilon\|x\| \geqslant \|a\|$. So $\|(a, 0)\|_* \geqslant \|a\|$, and hence $\|(a, 0)\|_* = \|a\|$. Similarly, $\|\cdot\|_*$ extends the norm of B. For any $x \in X$ we have $\|(x, -\varphi(x))\|_* \leqslant \|x - x\| + \|-\varphi(x) + \varphi(x)\| + \varepsilon\|x\| = \varepsilon\|x\|$. Let N be the null space of $\|\cdot\|_*$, and let V be $(A \oplus B)/N$. Then $\|\cdot\|_*$ induces a norm on V, and the natural maps $A, B \to V$ satisfy the requirement. □

A *subtrivialization* [5, p. 133] of a continuous field of C^*-algebras $(\{\mathcal{A}_t\}, \Gamma)$ over a locally compact Hausdorff space T is a faithful $*$-homomorphism h_t of each $\mathcal{A}_t$ into a common C^*-algebra $\mathcal{A}$ such that for every $f \in \Gamma$ the $\mathcal{A}$-valued function $t \mapsto h_t(f_t)$ is continuous over T. Not every continuous field of C^*-algebras can be subtrivialized [22, Remark 5.1]. We can talk about the subtrivialization of continuous fields of Banach spaces similarly by requiring h_t's to be linear isometric embeddings into some common Banach space. One natural question is:

Question 7.3. Can every continuous field of Banach spaces over a locally compact Hausdorff space be subtrivialized?

Blanchard and Kirchberg gave affirmative answer for separable continuous fields of complex Banach spaces over compact metric spaces [6, Corollary 2.8]. However, to prove Theorem 1.2 we have to deal with continuous fields of real separable Banach spaces over general locally compact Hausdorff spaces. For us the following weaker answer to Question 7.3 is sufficient:

Proposition 7.4. *Let* $(\{A_t\}, \Gamma)$ *be a continuous field of Banach spaces (over* $\mathbb{R}$ *or* $\mathbb{C}$*) over a locally compact Hausdorff space* T*. Let* $t_0 \in T$ *with* A_{t_0} *separable. Then there are a normed space* V *and linear isometric embeddings* $h_t : A_t \hookrightarrow V$ *such that for every* $f \in \Gamma$ *the* V*-valued map* $t \mapsto h_t(f_t)$ *is continuous at* t_0*.*

Proof. We prove the case where A_{t_0} is infinite-dimensional. The case where A_{t_0} is finite-dimensional is similar and easier. Since A_{t_0} is separable we can find a linearly independent sequence $x_1, x_2, \ldots$ in A_{t_0} such that the linear span of $\{x_k\}_{k\in\mathbb{N}}$ is dense in A_{t_0}. For each k pick a section $f_k \in \Gamma$ with $(f_k)_{t_0} = x_k$. Then for each t the map $\varphi_t : x_k \mapsto (f_k)_t$, $k = 1, 2, \ldots$, extends uniquely to a linear map from $\mathrm{span}\{x_k : k \in \mathbb{N}\}$ to A_t, which we still denote by φ_t. Let $X_n = \mathrm{span}\{x_1, \ldots, x_n\}$, and let $\varphi_{n,t}$ be the restriction of φ_t on X_n. Notice that for each $x \in X_n$ the section $t \mapsto \varphi_{n,t}(x)$ is in Γ. Then using a standard compactness argument we can find a neighborhood $\mathcal{U}_n$ of t_0 such that $1 - \frac{1}{n} \leqslant \|\varphi_{n,t}(x)\| \leqslant 1 + \frac{1}{n}$ for all x in the unit sphere of X_n and $t \in \mathcal{U}_n$. We may assume that $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \cdots$. We shall find a normed space V_t containing both A_t and A_{t_0} for each $t \in \Gamma$ such that A_t and A_{t_0} are kind of close to each other inside of V_t. If $t \notin \mathcal{U}_1$ we let V_t simply be $A_t \oplus A_{t_0}$ equipped with any admissible norm. If $t \in \mathcal{U}_n \setminus \mathcal{U}_{n+1}$, then by Lemma 7.2 we can find a normed space V_t containing both A_t and A_{t_0} such that $\|x - \varphi_{n,t}(x)\| \leqslant \frac{1}{n}\|x\|$ for all $x \in X_n$. If $t \in \bigcap_{n=1}^{\infty} \mathcal{U}_n$, then $\varphi_{n,t}$ is an isometric embedding for all n, and hence φ_t extends to a linear isometric embedding from A_{t_0} into A_t. So for $t \in \bigcap_{n=1}^{\infty} \mathcal{U}_n$ we can identify A_{t_0} with $\varphi_t(A_{t_0})$, and let $V_t = A_t$. Now by Lemma 4.5 we can find a normed space V containing all these V_t's such that the copies of A_{t_0} are identified. Let h_t be the composition $A_t \hookrightarrow V_t \hookrightarrow V$. Then for each $x \in \mathrm{span}\{x_k : k \in \mathbb{N}\}$ clearly the map $t \mapsto h_t(\varphi_t(x))$ is continuous at t_0. Now it is easy to see that for every section $f \in \Gamma$ the map $t \mapsto h_t(f_t)$ is continuous at t_0. $\square$

Remark 7.5. The C^*-algebraic analogue of Proposition 7.4 is not true, i.e. for a continuous field $(\{\mathcal{A}_t\}, \Gamma)$ of C^*-algebras, in general we cannot find a C^*-algebra $\mathcal{B}$ and faithful $*$-homomorphisms $h_t : \mathcal{A}_t \hookrightarrow \mathcal{B}$ such that for every $f \in \Gamma$ the map $t \mapsto h_t(f_t)$ is continuous at t_0. The reason is that such $\mathcal{B}$ and h_t's will imply that for any C^*-algebra $\mathcal{C}$ and any $\sum_j f_j \otimes c_j$ in the algebraic tensor product $\Gamma \otimes_{\mathrm{alg}} \mathcal{C}$ the function $t \mapsto \|\sum_j (f_j)_t \otimes c_j\|_{\mathcal{A}_t \otimes \mathcal{C}}$ is continuous at t_0, where $\mathcal{A}_t \otimes \mathcal{C}$ is the minimal tensor product. But there are examples [22, Proposition 4.3] where $t \mapsto \|\sum_j (f_j)_t \otimes c_j\|_{\mathcal{A}_t \otimes \mathcal{C}}$ is not continuous, even when T is simply the one-point compactification of $\mathbb{N}$. Notice that in the proof of Proposition 7.4 we used only Lemmas 4.5 and 7.2. The C^*-algebraic analogue of Lemma 4.5 has been proved by Blackadar [4, Theorem 3.1]. Thus the C^*-algebraic analogue of Lemma 7.2 (with φ still being a linear map, but h_A and h_B being faithful $*$-homomorphisms) is not true.

Recall that for a metric space X and $\varepsilon > 0$ the *packing number* $P(X, \varepsilon)$ is the maximal cardinality of an ε-separated (i.e. $\rho_X(x, x') > \varepsilon$ if $x \neq x'$) subset in X. When X is compact, $P(X, \varepsilon)$ is finite. In fact clearly $P(X, \varepsilon) \leqslant \mathrm{Cov}(X, \frac{1}{2}\varepsilon)$.

Lemma 7.6. *Let X be a compact metric space, and let $\varepsilon > 0$. For any closed subset Y of X if* $\mathrm{dist}_{\mathrm{GH}}(X, Y) < \frac{1}{4P(X,\varepsilon/2)}\varepsilon$, *then the open ε-balls centered at points of Y cover X.*

Proof. Let $N = P(X, \varepsilon/2)$. Let $h_X : X \to Z$ and $h_Y : Y \to Z$ be isometric embeddings into some metric space Z such that $\frac{1}{4N}\varepsilon > \mathrm{dist}^Z_{\mathrm{H}}(h_X(X), h_Y(Y))$. For each $x \in X$ pick $\varphi(x) \in Y$ with $\frac{1}{4N}\varepsilon > \rho_Z(h_X(x), h_Y(\varphi(x)))$. Let $x \in X$. Define x_n inductively by $x_0 = x$ and $x_n = \varphi(x_{n-1})$. Then for any $m > n \geqslant 1$ we have that

$$
\begin{aligned}
&\rho_X(x_n, x_m)\\
&\quad = \rho_Y(x_n, x_m)\\
&\quad \geqslant \rho_X(x_{n-1}, x_{m-1}) - \rho_Z(h_X(x_{n-1}), h_Y(x_n)) - \rho_Z(h_X(x_{m-1}), h_Y(x_m))\\
&\quad \geqslant \rho_X(x_{n-1}, x_{m-1}) - \frac{1}{2N}\varepsilon.
\end{aligned}
$$

Consequently $\rho_X(x_n, x_m) \geqslant \rho_X(x_0, x_{m-n}) - \frac{n}{2N}\varepsilon \geqslant \rho_X(x, Y) - \frac{n}{2N}\varepsilon$ for all $m > n \geqslant 0$. Therefore $x_0, x_1, \ldots, x_N$ are $(\rho_X(x, Y) - \frac{1}{2}\varepsilon)$-separated. Thus $\rho_X(x, Y) - \frac{1}{2}\varepsilon < \frac{1}{2}\varepsilon$. Then $\rho_X(x, Y) < \varepsilon$ follows. $\square$

Lemma 7.7. *Let X and Y be compact metric spaces, and let $\varepsilon > 0$. If* $\mathrm{dist}_{\mathrm{GH}}(X, Y) < \frac{1}{4}\varepsilon$ *then* $P(X, \varepsilon) \leqslant P(Y, \frac{1}{2}\varepsilon)$.

Proof. Let ρ be an admissible metric on $X \coprod Y$ with $\mathrm{dist}^{\rho}_{\mathrm{H}}(X, Y) < \frac{1}{4}\varepsilon$ (see the discussion preceding Theorem 2.1). Let $\{x_1, \ldots, x_n\}$ be an ε-separated set in X. For each k pick $y_k \in Y$ such that $\rho(x_k, y_k) < \frac{1}{4}\varepsilon$. Then clearly $\{y_1, \ldots, y_n\}$ is $\frac{1}{2}\varepsilon$-separated in Y. Therefore $n \leqslant P(Y, \frac{1}{2}\varepsilon)$. $\square$

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. We claim first that (iii) does not depend on the choice of the sequence $\{f_n\}_{n\in\mathbb{N}}$. Suppose that $\{f'_n\}_{n\in\mathbb{N}}$ is another sequence in Γ satisfying the conditions in the theorem. If (iii) holds for $\{f_n\}_{n\in\mathbb{N}}$, then for any $\varepsilon > 0$, we can find N and a neighborhood $\mathcal{U}$ as in (iii). Since $\{(f'_n)_{t_0} : n \in \mathbb{N}\}$ is dense in $\mathcal{D}(A_{t_0})$, there is some $N' \in \mathbb{N}$ so that for each $1 \leqslant n \leqslant N$, there is some $1 \leqslant \sigma(n) \leqslant N'$ with $\|(f_n)_{t_0} - (f'_{\sigma(n)})_{t_0}\|_{t_0} < \varepsilon$. Then we can find a neighborhood $\mathcal{U}' \subseteq \mathcal{U}$ of t_0 such that $\|(f_n)_t - (f'_{\sigma(n)})_t\|_{t_0} < 2\varepsilon$ for all $1 \leqslant n \leqslant N$ and all $t \in \mathcal{U}'$. It is clear that the open 3ε-balls in A_t centered at $(f'_1)_t, \ldots, (f'_{N'})_t$ cover $\mathcal{D}(A_t)$ for all $t \in \mathcal{U}'$. So (iii) is also satisfied for $\{f'_n\}_{n\in\mathbb{N}}$, and hence it does not depend on the choice of the sequence f_n.

Since $\mathcal{D}(A_t)$ is dense in $\mathcal{D}(A_t^c)$ for every t, (iii) does not depend on whether we take $(\{(A_t, L_t)\}, \Gamma)$ or its closure. Clearly neither does (i) nor (ii). So we may assume that $(\{(A_t, L_t)\}, \Gamma)$ is closed. Take a dense sequence $\{a_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(A_{t_0})$. According to Proposition 6.16 we can find $f_n \in \Gamma$ with $(f_n)_{t_0} = a_n$ and $(f_n)_t \in \mathcal{D}(A_t)$ for all $t \in T$. Then $\{f_n\}_{n\in\mathbb{N}}$ satisfies the condition in the theorem. In the rest of the proof we will use this sequence $\{f_n\}_{n\in\mathbb{N}}$.

Since $S(A_{t_0})$ is a compact metric space, $A_{t_0} \subseteq C(S(A_{t_0}))$ is separable. So by Proposition 7.4 we can find a normed space V containing all V_t's such that for every $f \in \Gamma$ the map $t \mapsto f_t$ from T to V is continuous at t_0. For any $n \in \mathbb{N}$ and $t \in T$ let $Y_{n,t} = \{(f_1)_t, \ldots, (f_n)_t\}$. Also let $X_t = \mathcal{D}(A_t)$ for all $t \in T$.

We show first that (iii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be given. Pick N and a neighborhood $\mathcal{U}$ of t_0 for ε as in (iii). Then $\mathrm{dist}_{\mathrm{H}}^V(Y_{N,t}, X_t) \leqslant \varepsilon$ throughout $\mathcal{U}$. By shrinking $\mathcal{U}$ we may assume that $\|e_{A_t} - e_{t_0}\| \leqslant \varepsilon$ and $\|(f_k)_t - (f_k)_{t_0}\| \leqslant \varepsilon$ for all $t \in \mathcal{U}$ and $1 \leqslant k \leqslant N$. Let $t \in \mathcal{U}$. Then $\mathrm{dist}_{\mathrm{H}}^V(Y_{N,t}, Y_{N,t_0}) \leqslant \varepsilon$. Hence $\mathrm{dist}_{\mathrm{H}}^V(X_t, X_{t_0}) \leqslant \mathrm{dist}_{\mathrm{H}}^V(X_t, Y_{N,t}) + \mathrm{dist}_{\mathrm{H}}^V(Y_{N,t}, Y_{N,t_0}) + \mathrm{dist}_{\mathrm{H}}^V(Y_{N,t_0}, X_{t_0}) \leqslant 3\varepsilon$. Therefore $\mathrm{dist}_{\mathrm{oq}}(A_t, A_{t_0}) \leqslant 3\varepsilon$. This proves (iii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii) follows from (5). So we are left to show that (ii) $\Rightarrow$ (iii). Let $\delta = \min((12P(X_{t_0}, \frac{1}{4}\varepsilon))^{-1}, \frac{1}{8})\varepsilon$. Take N so that the set Y_{N,t_0} is δ-dense in X_{t_0}. Then we have $\mathrm{dist}_{\mathrm{H}}^V(Y_{N,t_0}, X_{t_0}) \leqslant \delta$. Similarly as above there is some neighborhood $\mathcal{U}$ of t_0 such that $\mathrm{dist}_{\mathrm{H}}^V(Y_{N,t}, Y_{N,t_0}) \leqslant \delta$ for all $t \in \mathcal{U}$. By shrinking $\mathcal{U}$ we may assume that $\mathrm{dist}_{\mathrm{GH}}(X_t, X_{t_0}) < \delta \leqslant \frac{1}{8}\varepsilon$ for all $t \in \mathcal{U}$. Let $t \in \mathcal{U}$. Then

$$
\begin{aligned}
\mathrm{dist}_{\mathrm{GH}}(Y_{N,t}, X_t) &\leqslant \mathrm{dist}_{\mathrm{GH}}(Y_{N,t}, Y_{N,t_0}) + \mathrm{dist}_{\mathrm{GH}}(Y_{N,t_0}, X_{t_0}) + \mathrm{dist}_{\mathrm{GH}}(X_{t_0}, X_t) \\
&< 3\delta.
\end{aligned}
$$

Also $P(X_t, \frac{1}{2}\varepsilon) \leqslant P(X_{t_0}, \frac{1}{4}\varepsilon)$ by Lemma 7.7. So $\mathrm{dist}_{\mathrm{GH}}(Y_{N,t}, X_t) < \varepsilon/(4P(X_t, \frac{1}{2}\varepsilon))$. Then Lemma 7.6 tells us that the open ε-balls centered at points of $Y_{N,t}$ cover X_t. $\square$

We give one example to illustrate how to apply Theorem 7.1. Later in Sections 10 and 11 we shall use Theorem 7.1 to study the continuity of compact quantum metric spaces induced by ergodic actions (Theorem 1.3) and the continuity of θ-deformations (Theorem 1.4).

Example 7.8. Let V be a finite-dimensional real vector space equipped with a distinguished element e. Let T be a locally compact Hausdorff space. Suppose that for each $t \in T$ there is an order-unit space structure on V with unit e. Denote the order-unit space for t by (V_t, e) and the norm by $\|\cdot\|_t$. If the function $t \mapsto \|v\|_t$ is continuous on T for each $v \in V$, this is called a *continuous field of finite-dimensional order-unit spaces* by Rieffel [38, Section 10]. Clearly this fits into our Definition 6.1. If there is also a Lip-norm L_t for each t such that $t \mapsto L_t(v)$ is continuous on T for each $v \in V$, then $\{L_t\}$ is called a *continuous field of Lip-norms* [38, Section 11]. Again, this fits into our Definition 6.4. Rieffel proved that for a continuous field of Lip-norms, $\mathrm{dist}_{\mathrm{q}}(V_t, V_{t_0}) \to 0$ as $t \to t_0$ for each $t_0 \in T$ [38, Theorem 11.2]. By Theorem 1.1 this is equivalent to saying that $\mathrm{dist}_{\mathrm{oq}}(V_t, V_{t_0}) \to 0$ as $t \to t_0$. We use Theorem 7.1 to give the latter a new proof.

For later use we consider a more general case. We want to allow V to be infinite-dimensional. To still get the continuity under $\mathrm{dist}_{\mathrm{oq}}$ we need stronger conditions.

Definition 7.9. Let V be a real vector space equipped with a distinguished element e. Let T be a locally compact Hausdorff space. Suppose that for each $t \in T$ there is an order-unit space structure on V with unit e, for which we denote the order-unit norm by $\|\cdot\|_t$. We call $(V, e, \{\|\cdot\|_t\})$ *a uniformly continuous field of order-unit spaces* if for any $t_0 \in T$ and $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of t_0 such that $(1-\varepsilon)\|\cdot\|_{t_0} \leqslant \|\cdot\|_t \leqslant (1+\varepsilon)\|\cdot\|_{t_0}$ throughout $\mathcal{U}$. Let L_t be a Lip-norm on $(V, e, \|\cdot\|_t)$ for each $t \in T$. We call $\{L_t\}$ *a uniformly continuous field of Lip-norms* if for any $v \in V$ the function $t \mapsto L_t(v)$ is continuous, and if for any $t_0 \in T$ and $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of t_0 such that $(1-\varepsilon)L_{t_0} \leqslant L_t$ throughout $\mathcal{U}$.

Notice that we do not need $L_t \leqslant (1+\varepsilon)L_{t_0}$. For a continuous field of finite-dimensional order-unit spaces (resp. finite-dimensional Lip-norms), by a standard compactness argument we can find a neighborhood $\mathcal{U}$ of t_0 such that $1-\varepsilon \leqslant \|v\|_t \leqslant 1+\varepsilon$ (resp. $1-\varepsilon \leqslant L_t^{\sim}(\tilde{v})$) for all $t \in \mathcal{U}$ and v (resp. $\tilde{v}$) in the unit sphere of $(V, \|\cdot\|_{t_0})$ (resp. $(\tilde{V}, L_{t_0}^{\sim})$). Therefore continuous fields of finite-dimensional order-unit spaces and Lip-norms are uniformly continuous. The assertion that $\mathrm{dist}_{\mathrm{oq}}(V_t, V_{t_0}) \to 0$ as $t \to t_0$ follows directly from Theorem 7.1 and the next lemma:

Lemma 7.10. *Let $(V, e, \{\|\cdot\|_t\}, \{L_t\})$ be a uniformly continuous field of order-unit spaces and Lip-norms over T. Denote the order-unit space for t by (V_t, e). Then the radius function $t \mapsto r_{V_t}$ is upper semi-continuous over T. Let $t_0 \in T$, and let $R > 0$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a sequence dense in $\mathcal{D}_R(V_{t_0})$. Then for any $\varepsilon > 0$, there is an N such that the open ε-balls in V_t centered at $v_1, \ldots, v_N$ cover $\mathcal{D}_R(V_t)$ throughout some neighborhood of t_0.*

Proof. Let $1 > \varepsilon > 0$ be given. Let $\mathcal{U}$ be a neighborhood of t_0 such that $(1-\varepsilon)\|\cdot\|_{t_0} \leqslant \|\cdot\|_t \leqslant (1+\varepsilon)\|\cdot\|_{t_0}$ and $(1-\varepsilon)L_{t_0} \leqslant L_t$ throughout $\mathcal{U}$. Let $t \in \mathcal{U}$. Then $(1-\varepsilon)\|\cdot\|_{t_0}^{\sim} \leqslant \|\cdot\|_t^{\sim} \leqslant (1+\varepsilon)\|\cdot\|_{t_0}^{\sim}$ and $(1-\varepsilon)L_{t_0}^{\sim} \leqslant L_t^{\sim}$. By Proposition 2.3 $\|\cdot\|_{t_0}^{\sim} \leqslant r_{V_{t_0}} L_{t_0}^{\sim}$. Thus $\|\cdot\|_t^{\sim} \leqslant (1+\varepsilon)\|\cdot\|_{t_0}^{\sim} \leqslant (1+\varepsilon) r_{V_{t_0}} L_{t_0}^{\sim} \leqslant \frac{1+\varepsilon}{1-\varepsilon} r_{V_{t_0}} L_t^{\sim}$. Applying Proposition 2.3 again we see that $r_{V_t} \leqslant \frac{1+\varepsilon}{1-\varepsilon} r_{V_{t_0}}$. This shows that the radius function $t \mapsto r_{V_t}$ is upper semi-continuous. Pick N such that the open ε-balls in V_{t_0} centered at $v_1, \ldots, v_N$ cover $\mathcal{D}_R(V_{t_0})$. Let $v \in \mathcal{D}_R(V_t)$. Then $\|v\|_{t_0} \leqslant \frac{1}{1-\varepsilon}\|v\|_t \leqslant \frac{1}{1-\varepsilon}R$ and $L_{t_0}(v) \leqslant \frac{1}{1-\varepsilon} L_t(v) \leqslant \frac{1}{1-\varepsilon}$. Thus $(1-\varepsilon)v \in \mathcal{D}_R(V_{t_0})$. Then $\|(1-\varepsilon)v - v_n\|_{t_0} < \varepsilon$ for some $1 \leqslant n \leqslant N$. Consequently

$$
\begin{aligned}
\|v - v_n\|_t &\leqslant \|v - (1-\varepsilon)v\|_t + \|(1-\varepsilon)v - v_n\|_t \\
&\leqslant \varepsilon\|v\|_t + (1+\varepsilon)\|(1-\varepsilon)v - v_n\|_{t_0} \leqslant \varepsilon(R + 1 + \varepsilon).
\end{aligned}
$$

Thus the open $(R+2)\varepsilon$-balls in V_t centered at $v_1, \ldots, v_N$ cover $\mathcal{D}_R(V_t)$. $\square$

8. Lip-norm and finite multiplicity

In this section we prove Theorem 8.3 to determine when an ergodic action of a compact group induces a Lip-norm.

Throughout the rest of this paper G will be a nontrivial compact group with identity e_G, endowed with the normalized Haar measure. Denote by $\hat{G}$ the dual of G, and by γ_0 the class of trivial representations. For any $\gamma \in \hat{G}$ let χ_γ be the corresponding character on G, and let $\bar{\gamma}$ be the contragradient representation. For any $\gamma \in \hat{G}$ and any representation of G on some complex vector space V, we denote by V_γ the γ-isotypic component of V. If $\mathcal{J}$ is a finite subset of $\hat{G}$, we also let $V_{\mathcal{J}} = \sum_{\gamma \in \mathcal{J}} V_\gamma$, and let $\bar{\mathcal{J}} = \{\bar{\gamma} : \gamma \in \mathcal{J}\}$. For a strongly continuous action α of G on a complete order-unit space $(\bar{A}, e)$ as automorphisms, we endow $\bar{A}^{\mathbb{C}} = \bar{A} \otimes_{\mathbb{R}} \mathbb{C} = \bar{A} + i\bar{A}$ with the diagonal action $\alpha^{\mathbb{C}} := \alpha \otimes I$. We say that α is of *finite multiplicity* if $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) < \infty$ for all $\gamma \in \hat{G}$, and that Γ is *ergodic* if the only α-invariant elements are the scalar multiples of e.

We also fix a length function on G, i.e. a continuous real-valued function, l, on G such that

$$\begin{aligned} l(xy) &\leqslant l(x) + l(y) \text{ for all } x, y \in G, \\ l(x^{-1}) &= l(x) \text{ for all } x \in G, \\ l(x) &= 0 \text{ if and only if } x = e_G. \end{aligned}$$

Remark 8.1. One can verify easily that a length function l on G is equivalent to a left invariant metric ρ on G under the correspondence $\rho(x, y) = l(x^{-1}y)$. Since every metric on a compact group could be integrated to be a left invariant one, we see that a compact group G has a length function if and only if it is metrizable.

Let $\mathcal{A}$ be a unital C^*-algebra, and let α be a strongly continuous ergodic action of G on $\mathcal{A}$ by automorphisms. In [35] Rieffel defined a (possibly $+\infty$-valued) semi-norm L on $\mathcal{A}$ by

$$L(a) = \sup\left\{ \frac{\|\alpha_x(a) - a\|}{l(x)} : x \in G, x \neq e_G \right\}. \tag{11}$$

He showed that the set $A = \{a \in \mathcal{A}_{\mathrm{sa}} : L(a) < \infty\}$ is a dense subspace of $\mathcal{A}_{\mathrm{sa}}$ containing the identity e, and that $(A, L|A)$ is a closed compact quantum metric space [35, Theorem 2.3]. In fact, the proof there shows more:

Theorem 8.2 (*Rieffel [35, Theorem 2.3]*). *Let α be a strongly continuous isometric action of G on a (real or complex) Banach space $\bar{V}$. Define a (possibly $+\infty$-valued) seminorm L on $\bar{V}$ by* (11). *Then $V := \{v \in \bar{V} : L(v) < \infty\}$ is always a dense subspace of $\bar{V}$. If $\bar{V} = (\bar{A}, e)$ is a complete order-unit space and G acts as automorphisms of $\bar{A}$,* *then $A = \{a \in \bar{A} : L(a) < \infty\}$ also contains e, and hence we can identify $S(A)$ with $S(\bar{A})$. If furthermore α is ergodic and of finite multiplicity, then A with the restriction of L is a closed compact quantum metric space, and $r_A \leqslant \int_G l(x)\,dx$.*

The aim of this section is to show that the converse of Theorem 8.2 is also true:

Theorem 8.3. *Let α be an ergodic strongly continuous action of G on a complete unit-order space $(\bar{A}, e)$. Define L and A as in Theorem* 8.2. *If the restriction of L on A is a Lip-norm, then α is of finite multiplicity.*

The intuition is that covering numbers of $\mathcal{D}_r(A)$ increase (fast) as the multiplicities $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \Gamma)$ increase. Thus the compactness of $\mathcal{D}_r(A)$ in Proposition 2.3 forces $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \Gamma)$ to be finite.

Lemma 8.4. *For any finite subset $\mathcal{J}$ of $\hat{G}$ and any map $\omega : \mathcal{J} \to \mathbb{N} \cup \{0\}$, there is a constant $M_{\mathcal{J},\omega} > 0$ such that for any strongly continuous isometric action α of G on a finite-dimensional complex Banach space V with* $\mathrm{mul}(V, \gamma) \leqslant \omega(\gamma)$ *for $\gamma \in \mathcal{J}$ and* $\mathrm{mul}(V, \gamma) = 0$ *for $\gamma \in \hat{G} \setminus \mathcal{J}$, we have $L \leqslant M_{\mathcal{J},\omega} \| \cdot \|$ on V.*

Proof. Let X be the set of all functions $\omega_1 : \mathcal{J} \to \mathbb{N} \cup \{0\}$ with $\omega_1 \leqslant \omega$. Set $\omega_1(\gamma) = \mathrm{mul}(V, \gamma)$ for all $\gamma \in \mathcal{J}$. Let $\| \cdot \|_V$ be the norm on V, and let $N = \dim V$. Let W be a Hilbert space with dimension N. It is a theorem of John [46, Proposition 9.12] that there is a linear isomorphism $\phi : V \to W$ such that $\|\phi\|, \|\phi^{-1}\| \leqslant \sqrt[4]{N}$. Define an inner product, $\langle , \rangle_*$, on V by $\langle u, v\rangle_* = \int_G \langle \phi(\alpha_x(u)), \phi(\alpha_x(v))\rangle\,dx$. Then $\langle , \rangle_*$ is G-invariant. Let $\| \cdot \|_*$ be the corresponding norm. Since α is isometric with respect to $\| \cdot \|_V$, for any $v \in V$ we have $\|v\|_*^2 = \int_G \|\phi(\alpha_x(v))\|^2\,dx \leqslant \|\phi\|^2 \int_G \|\alpha_x(v)\|_V^2\,dx = \|\phi\|^2 \|v\|_V^2 \leqslant \sqrt{N}\|v\|_V^2$. Thus $\| \cdot \|_* \leqslant \sqrt[4]{N} \| \cdot \|_V$. Similarly, $\| \cdot \|_* \geqslant \frac{1}{\sqrt[4]{N}} \| \cdot \|_V$.

For each $\gamma \in \hat{G}$ fix a Hilbert space H_γ with an irreducible unitary action of type γ. Let H be the Hilbert space direct sum $\oplus_{\gamma\in\mathcal{J}} \oplus_{j=1}^{\omega_1(\gamma)} H_\gamma$ equipped with the natural action β of G. By Theorem 8.2 L_{H} is finite on a dense subspace of H. Since H is finite-dimensional, L_{H} is finite on the whole H. Clearly there is a constant M such that $L_{\mathrm{H}} \leqslant M \| \cdot \|_{\mathrm{H}}$ on H.

Since $(V, \langle , \rangle_*)$ has the same multiplicities as does H, there is a G-equivariant unitary map $\varphi : (V, \langle , \rangle_*) \to H$. Then for any $v \in V$ and $x \in G$ we have that

$$\begin{aligned}
\|v - \alpha_x(v)\| &\leqslant \sqrt[4]{N}\|v - \alpha_x(v)\|_* = \sqrt[4]{N}\|\varphi(v) - \varphi(\alpha_x(v))\|_{\mathrm{H}} \\
&= \sqrt[4]{N}\|\varphi(v) - \beta_x(\varphi(v))\|_{\mathrm{H}} \leqslant M \cdot \sqrt[4]{N} \cdot l(x)\|\varphi(v)\|_{\mathrm{H}} \\
&= M \cdot \sqrt[4]{N} \cdot l(x)\|v\|_* \leqslant M \cdot \sqrt{N} \cdot l(x)\|v\|_V.
\end{aligned}$$

Let $M'_{\mathcal{J},\omega_1} = M \cdot \sqrt{N}$. Then $L_V \leqslant M_{\mathcal{J},\omega} \| \cdot \|_V$ on V. Thus $M_{\mathcal{J},\omega} := 1 + \max\{M'_{\mathcal{J},\omega_1} : \omega_1 \in X\}$ satisfies the requirement. $\square$

In particular, let M_γ be the constant $M_{\mathcal{J},\omega}$ for $\mathcal{J} = \{\gamma\}$ and $\omega(\gamma) = 1$.

We shall need a well-known fact (cf. the discussion at the end of page 217 of [41], noticing that δ can be 1 in Lemma 2 when E_1 is finite-dimensional):

Lemma 8.5. *For any (real or complex) normed space V there is a sequence $p_1, p_2, \ldots$ in the unit sphere of V, with length $\dim(V)$ when V is finite-dimensional or length ∞ otherwise, satisfying that $\|p_m - q\| \geqslant 1$ for all m and all $q \in \mathrm{span}\{p_1, \ldots, p_{m-1}\}$.*

Lemma 8.6. *Let α be a strongly continuous isometric action of G on a complex Banach space V, and let $\gamma \in \hat{G}$. Then there is a subset $X \subseteq \mathcal{D}_{1/M_\gamma}(V) = \{v \in V : L(v) \leqslant 1, \|v\| \leqslant 1/M_\gamma\}$, with $\mathrm{mul}(V, \gamma)$ many elements when $\mathrm{mul}(V, \gamma) < \infty$ or infinitely many elements otherwise, such that any two distinct points in X have distance no less than $1/M_\gamma$.*

Proof. Fix a Banach space H_γ with an irreducible action of type γ. We have $V_\gamma = \oplus_{j=1}^{\mathrm{mul}(V,\gamma)} V_j$ with G-equivariant isomorphisms $\varphi_j : H_\gamma \to V_j$. Take a nonzero u in H_γ, and let $W = \mathrm{span}\{\varphi_j(u)\}_j$. Then for any nonzero v in V', v is "purely" of type γ, i.e. the action of G on $\mathrm{span}\{\alpha_x(v)\}_{x \in G}$ is an irreducible one of type γ. By Lemma 8.5 we can find a subset Y in the unit sphere of W, with $\dim(W)$ many elements when W is finite-dimensional or infinitely many elements otherwise, such that any two distinct points in Y have distance no less than 1. According to Lemma 8.4 any element in the unit sphere of W has L no bigger than M_γ. Therefore $Y/M_\gamma \subseteq \mathcal{D}_{1/M_\gamma}(V)$. $\square$

For a set S and a subset X of S we say that X is an n-subset if X consists of n elements. For $q_1, q_2 \geqslant 2$ let $N(q_1, q_2; 2)$ be the Ramsey number [28], i.e. the minimal number n such that for any set S with at least n elements, if the set of all 2-subsets of S is divided into 2 disjoint families Y_1 and Y_2 ("colors"), then there are a j and some q_j-subset of S for which every 2-subset is in Y_j. Consequently, for any set S with at least n elements, if the set of all 2-subsets of S is the union of 2 (not necessarily disjoint) families T_1 and T_2, then there are a j and some q_j-subset of S for which every 2-subset is in T_j.

Lemma 8.7. *Let α be a strongly continuous action of G on a complete order-unit space $(\bar{A}, e)$ as automorphisms. Suppose that for some $\gamma \in \hat{G} \setminus \{\gamma_0\}$ and $q > 0$, $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) \geqslant N(q, q; 2)$. Then for any $0 < \varepsilon < 1/(4M_\gamma)$ we have that $\mathrm{Cov}(\mathcal{D}_{1/M_\gamma}(A), \varepsilon) \geqslant q$.*

Proof. Since $\bar{A} = \mathrm{Af}_{\mathbb{R}}(S(\bar{A}))$, we can identify $\bar{A}^{\mathbb{C}}$ with $\mathrm{Af}_{\mathbb{C}}(S(\bar{A}))$, the space of $\mathbb{C}$-valued continuous affine functions on $S(\bar{A})$ equipped with the supremum norm, and hence $\bar{A}^{\mathbb{C}}$ becomes a complex Banach space whose norm extends that of $\bar{A}$. Notice that the action α corresponds to an action α' of G on $S(\bar{A})$. Then clearly $\alpha^{\mathbb{C}}$ is isometric and strongly continuous with respect to this norm.

According to Lemma 8.6 we can find a subset $X \subseteq \mathcal{D}_{1/M_\gamma}(\bar{A}^{\mathbb{C}})$, with $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma)$ many elements when $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) < \infty$ or infinitely many elements otherwise, such that any two distinct points in X have distance at least $1/M_\gamma$. Then $|X| \geqslant N(q, q; 2)$. For any distinct $a_1 + ia_2,\ b_1 + ib_2 \in X$ with $a_j, b_j \in \bar{A}$, we have that $\|(a_1 + ia_2) - (b_1 +$

$ib_2)\| \geqslant 1/M_\gamma$, and hence $|a_1 - b_1| \geqslant 1/(2M_\gamma)$ or $|a_2 - b_2| \geqslant 1/(2M_\gamma)$. Denote by $X^{(2)}$ the set of all 2-subsets of X. Let $T_j = \{\{a_1 + ia_2, b_1 + ib_2\} \in X^{(2)} : |a_j - b_j| \geqslant 1/(2M_\gamma)\}$. Then $T_1 \cup T_2 = X^{(2)}$. So there are a j and some q-subset X' of X for which every 2-subset is in T_j. Let $Y = \{a_j : a_1 + ia_2 \in X'\}$. Clearly Y is contained in $\mathcal{D}_{1/M_\gamma}(A)$ and any two distinct points in Y have distance at least $1/(2M_\gamma)$.

Let $0 < \varepsilon < 1/(4M_\gamma)$. Suppose that $p_1, \ldots, p_k \in \mathcal{D}_{1/M_\gamma}(A)$ and the open ε-balls centered at $p_1, \ldots, p_k$ cover $\mathcal{D}_{1/M_\gamma}(A)$. Then each such open ball could contain at most one point in Y. So $k \geqslant |Y| = q$, and hence $\mathrm{Cov}(\mathcal{D}_{1/M_\gamma}(A), \varepsilon) \geqslant q$. □

Proof of Theorem 8.3. Suppose that $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) = \infty$ for some $\gamma \in \hat{G}$. For any $0 < \varepsilon < 1/(4M_\gamma)$ by Lemma 8.7 we have $\mathrm{Cov}(\mathcal{D}_{1/M_\gamma}(A), \varepsilon) = \infty$. Therefore $\mathcal{D}_{1/M_\gamma}(A)$ is not totally bounded. By Proposition 2.3 L is not a Lip-norm on A. □

It should be pointed out that there do exist examples of ergodic strongly continuous action of G on a complete unit-order space $(\bar{A}, e)$, for which $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) = \infty$ for some $\gamma \in \hat{G}$. We shall give such an example in Section 9.

9. Compactness and bounded multiplicity

In this section we investigate when a family of compact quantum metric spaces induced from ergodic actions of G is totally bounded.

In the discussion after Theorem 13.5 of [38] Rieffel observed that the set of all isometry classes of compact quantum metric spaces (for given l) induced from ergodic actions of G on unital C^*-algebras is totally bounded under $\mathrm{dist}_{\mathrm{q}}$. In fact, the argument there works for general ergodic actions of G on complete order-unit spaces:

Theorem 9.1 (*Rieffel [38, Section 13]*). *Let $\mathcal{S}$ be a set of compact quantum metric spaces (A, L) induced by ergodic actions α of G on $\bar{A}$ for a fixed l. Let* $\mathrm{mul}(\mathcal{S}, \gamma) = \sup\{\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma) : (A, L) \in \mathcal{S}\}$ *for each $\gamma \in \hat{G}$. If* $\mathrm{mul}(\mathcal{S}, \gamma) < \infty$ *for all $\gamma \in \hat{G}$, then $\mathcal{S}$ is totally bounded under* $\mathrm{dist}_{\mathrm{q}}$.

We show that the converse is also true:

Theorem 9.2. *Let $\mathcal{S}$ be a set of compact quantum metric spaces (A, L) induced by ergodic actions α of G on $\bar{A}$ for a fixed l. If $\mathcal{S}$ is totally bounded under* $\mathrm{dist}_{\mathrm{q}}$, *then* $\mathrm{mul}(\mathcal{S}, \gamma) < \infty$ *for all $\gamma \in \hat{G}$.*

Proof. Suppose that $\mathcal{S}$ is totally bounded. For any $R > \int_G l(x)\,dx$, by Theorems 5.5, 8.2, and Lemma 5.4 we have that $\sup\{\mathrm{Cov}(\mathcal{D}_R(A), \varepsilon) : (A, L) \in \mathcal{S}\} < \infty$ for all $\varepsilon > 0$. Taking $R > \max(\int_G l(x)\,dx, 1/M_\gamma)$, by Lemma 5.1 we get $\sup\{\mathrm{Cov}(\mathcal{D}_{1/M_\gamma}(A), \varepsilon) : (A, L) \in \mathcal{S}\} < \infty$ for all $\varepsilon > 0$. Let

$$M = \sup\{\mathrm{Cov}(\mathcal{D}_{1/M_\gamma}(A), 1/(5M_\gamma)) : (A, L) \in \mathcal{S}\}.$$

Then Lemma 8.7 tells us that $\mathrm{mul}(\mathcal{S}, \gamma) < N(M + 1, M + 1; 2)$. □

It should be pointed out that there do exist ergodic actions of G on complete order-unit spaces with big multiplicity:

Example 9.3. Let $\{(A_j, L_j)\}_{j\in I}$ be a family of compact quantum metric spaces induced by ergodic actions α_j of G on $(\bar{A}_j, e_j)$. Also let $\eta_j(a) = \int_G (\alpha_j)_x(a)\, dx$ be the unique G-invariant state on $\bar{A}_j$ and $V_j = \ker(\eta_j)$. Then $\bar{A}_j = \mathbb{R}e_j \oplus V_j$ as vector spaces. As in Section 12 of [38], consider $\prod^b \bar{A}_j$ the subspace of the full product which consists of sequences $\{a_j\}$ for which $\|a_j\|$ is bounded. This is a complete order-unit space with unit $e = \{e_j\}$. Consider the reduced product $\prod^{rb} \bar{A}_j = \{\{a_j\} \in \prod^b \bar{A}_j : \eta_j(a_j) = \eta_k(a_k)$ for all $j, k \in I\}$. Then $\prod^{rb} \bar{A}_j$ is a closed subspace in $\prod^b \bar{A}_j$, and is also a complete order-unit space. Clearly $\prod^{rb} A_j = \mathbb{R}e \oplus \prod^b V_j$ as vector spaces. The actions α_j of G on the components $\bar{A}_j$ give an isometric action on $\prod^b \bar{A}_j$, which we denote by α. Although α is not ergodic on $\prod^b \bar{A}_j$, it is on $\prod^{rb} \bar{A}_j$ because of the above decomposition as a direct sum. By the natural G-equivariant embedding $\bar{A}_j \hookrightarrow \prod^b \bar{A}_j$, we see that $\mathrm{mul}((\prod^{rb} \bar{A}_j)^{\mathbb{C}}, \gamma) \geqslant \sum_{j\in I} \mathrm{mul}(\bar{A}_j^{\mathbb{C}}, \gamma)$ for every $\gamma \in \hat{G} \setminus \{\gamma_0\}$.

In general, α may not be strongly continuous on $\prod^{rb} \bar{A}_j$. But there are two special cases in which it is strongly continuous:

(1). When $(\bar{A}_j, e_j)$ and α_j are all the same and finite-dimensional, say $(\bar{A}_j, e_j) = (\bar{A}, e)$. Then $\bar{A} = A$ and there is some constant $M > 0$ such that $L \leqslant M\|\cdot\|$ on A. Therefore $\|a - (\alpha_j)_x(a)\| \leqslant L(a)l(x) \leqslant M\|a\|l(x)$ for all $a \in A$ and $x \in G$. Then it is easy to see that α is strongly continuous on $\prod^{rb} A_j$. It is standard [45] that in the left regular representation of G on $C(G)$ the multiplicity of each $\gamma \in \hat{G}$ equals $\dim(\gamma)$. For a finite subset $\mathcal{J} \subseteq \hat{G}$ and any $a + ia' \in C(G)$, clearly $a + ia' \in (C(G))_{\mathcal{J}}$ if and only if $a - ia' \in (C(G))_{\bar{\mathcal{J}}}$. Therefore if $\mathcal{J} = \bar{\mathcal{J}}$ and $\gamma_0 \in \mathcal{J}$, then $(C(G))_{\mathcal{J}}$ is closed under the involution and contains the constant functions. Hence $((C(G))_{\mathcal{J}})_{\mathrm{sa}}$ is a complete finite-dimensional order-unit space and $(C(G))_{\mathcal{J}} = (((C(G))_{\mathcal{J}})_{\mathrm{sa}})^{\mathbb{C}}$. Taking $I = \mathbb{N}$ and $A = ((C(G))_{\mathcal{J}})_{\mathrm{sa}}$, we get $\mathrm{mul}\left(\left(\prod^{rb} A_j\right)^{\mathbb{C}}, \gamma\right) = \infty$ for every $\gamma \in \mathcal{J} \setminus \{\gamma_0\}$ as promised at the end of Section 8.

(2). When I is finite, α is always strongly continuous on $\prod^{rb} \bar{A}_j$. In particular, take $\bar{A} = (C(G))_{\mathrm{sa}}$, and $\bar{A}_j = \bar{A}$ for all $j \in I$. Since $\mathrm{mul}(\bar{A}^{\mathbb{C}}, \gamma)$ equals $\dim(\gamma)$ for all $\gamma \in \hat{G}$, we see that $\mathrm{mul}\left(\left(\prod^{rb} \bar{A}_j\right)^{\mathbb{C}}, \gamma\right)$ could be as big as we want for any $\gamma \neq \gamma_0$.

10. Continuous fields of compact quantum metric spaces induced by ergodic compact group actions

In this section we study continuous fields of compact quantum metric spaces induced by ergodic actions, and prove Theorem 1.3. In Examples 10.11 and 10.12 we use Theorem 1.3 to give a unified treatment of the continuity of noncommutative tori and integral coadjoint orbits, which were studied by Rieffel before.

Rieffel has defined continuous fields of actions of a locally compact group on C^*-algebras [33, Definition 3.1]. We adapt it to actions on order-unit spaces:

Definition 10.1. Let $(\{\overline{A_t}\}, \Gamma)$ be a continuous field of order-unit spaces over a locally compact Hausdorff space T, and let α_t be a strongly continuous action of G on $\overline{A_t}$ for each $t \in T$. We say that $\{\alpha_t\}$ is a *continuous field of strongly continuous actions* of G on $(\{\overline{A_t}\}, \Gamma)$ if the action of G on Γ_∞ is strongly continuous, where $\Gamma_\infty := \{f \in \Gamma :$ the function $t \mapsto \|f_t\|$ vanishes at $\infty\}$ is the space of continuous sections vanishing at ∞. If each α_t is ergodic, we say that this is a *field of ergodic actions*. If each α_t is of finite multiplicity, we say that this is a *field of finite actions*.

Remark 10.2. For a continuous field $(\{\overline{A_t}\}, \Gamma)$ of order-unit spaces over a compact Hausdorff space T, it is easy to see that Γ is a complete order-unit space with the unit e and the order defined as $f \geqslant g$ if and only if $f_t \geqslant g_t$ for all $t \in T$. Then the natural projections $\Gamma \to \overline{A_t}$ become order-unit space quotient maps. According to the discussion right after Proposition 2.3 we may identify $S(\overline{A_t})$ with a closed convex subset of $S(\Gamma)$. From the definition of the order in Γ it is easy to see that the convex hull of the union of all the $S(\overline{A_t})$'s is dense in $S(\Gamma)$. If we identify $\overline{A_t}^{\mathbb{C}}$ and $\Gamma^{\mathbb{C}}$ with $\mathrm{Af}_{\mathbb{C}}(S(\overline{A_t}))$ and $\mathrm{Af}_{\mathbb{C}}(S(\Gamma))$ respectively, then they are endowed with complex vector space norms and $\|f + ig\| = \sup\{\|f_t + ig_t\| : t \in T\}$ for all $f + ig \in \Gamma^{\mathbb{C}}$. When we talk about $\overline{A_t}^{\mathbb{C}}$ and $\Gamma^{\mathbb{C}}$ as complex Banach spaces, we always mean these norms. If (α_t) is a continuous field of strongly continuous actions of G on $(\{\overline{A_t}\}, \Gamma)$, then the action of G on $\Gamma^{\mathbb{C}}$ is easily seen to be also strongly continuous.

For a continuous field of strongly continuous finite ergodic actions of G on order-unit spaces, obviously we get a field of compact quantum metric spaces. Theorem 1.3 indicates that this is indeed a continuous field, as one may expect. However, as Theorems 1.2 and 7.1 indicate, as $t \to t_0$ the corresponding compact quantum metric spaces do not necessarily converge to that at t_0. We give a trivial example here:

Example 10.3. Take a complete order-unit space $\bar{A}$ with a nontrivial ergodic action of G with finite multiplicity (for example, $(C(G))_{\mathrm{sa}}$ with the left regular representation of G). Let $T = [0, 1]$. Then we have the trivial field $(\{\overline{A_t^1}\}, \Gamma^1)$ with $\overline{A_t^1} = \bar{A}$ for all $t \in T$. The action of G on Γ^1 is clearly strongly continuous. Now we take the subfield $((\overline{A_t}), \Gamma)$ with $\overline{A_t} = \bar{A}$ for all $0 < t \leqslant 1$ and $\overline{A_0} = \mathbb{R}e_A$. The action of G restricted on Γ is still strongly continuous. But $r_{A_0} = 0$ and $r_{A_t} = r_A > 0$ for all $0 < t \leqslant 1$. So $\mathrm{dist}_{\mathrm{oq}}(A_t, A_0) = \mathrm{dist}_{\mathrm{oq}}(A, \mathbb{R}e_A)$ does not converge to 0 as $t \to 0$.

Notice that in the above example the multiplicities degenerate at $t_0 = 0$. Theorem 1.3 tells us that this is exactly why we do not get $\mathrm{dist}_{\mathrm{oq}}(A_t, A_0) \to 0$.

We start to prove Theorem 1.3. We show first that the multiplicity function $t \mapsto \mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma)$ is lower semi-continuous. This shows (ii) $\Longrightarrow$ (i) in Theorem 1.3.

Lemma 10.4. *Let* $(\{V_t\}, \Gamma)$ *be a continuous field of* (*real or complex*) *Banach spaces over a locally compact Hausdorff space T. For any* $f_1, \ldots, f_m \in \Gamma$ *the set* $\mathrm{Ind}(f_1, \ldots, f_m) = \{t \in T : ((f_1)_t, \ldots, (f_m)_t)$ *are linearly independent*$\}$ *is open.*

Proof. Let $t_0 \in \mathrm{Ind}(f_1, \ldots, f_m)$. Then for any $t \in T$ the map $(f_j)_{t_0} \mapsto (f_j)_t$, $j = 1, \ldots, m$, extends uniquely to a linear map φ_t from $W := \mathrm{span}\{(f_1)_{t_0}, \ldots, (f_m)_{t_0}\}$ to V_t. Notice that for any $v \in W$ the section $t \mapsto \varphi_t(v)$ is in Γ. A standard compactness argument shows that there is a neighborhood $\mathcal{U}$ of t_0 such that $\frac{1}{2} < \|\varphi_t(v)\|$ for all $t \in \mathcal{U}$ and v in the unit sphere of W. In particular, φ_t is injective throughout $\mathcal{U}$. Thus $(f_1)_t, \ldots, (f_m)_t$ are linear independent throughout $\mathcal{U}$. $\square$

We shall need the following well-known fact several times. We omit the proof.

Lemma 10.5. *Let G be a compact group. Let* α *be a continuous action of G on a complex Banach space V. For a continuous* $\mathbb{C}$*-valued function* φ *on G let*

$$\alpha_\varphi(v) = \int_G \varphi(x)\alpha_x(v)\,dx$$

for $v \in V$. *Then* $\alpha_\varphi : V \to V$ *is a continuous linear map. If* $\mathcal{J}$ *is a finite subset of* $\hat{G}$ *and if* φ *is a linear combination of the characters of* $\gamma \in \bar{\mathcal{J}}$, *then* $\alpha_\varphi(V) \subseteq V_{\mathcal{J}}$. *Let*

$$\alpha_{\mathcal{J}} = \alpha_{\sum_{\gamma \in \mathcal{J}} \dim(\gamma)\overline{\chi_\gamma}}.$$

(*When* $\mathcal{J}$ *is a one-element set* $\{\gamma\}$, *we will simply write* α_γ *for* $\alpha_{\{\gamma\}}$.) *Then* $\alpha_{\mathcal{J}}(v) = v$ *for all* $v \in V_{\mathcal{J}}$, *and* $\alpha_{\mathcal{J}}(v) = 0$ *for all* $v \in V_\gamma$ *with* $\gamma \in \hat{G} \setminus \mathcal{J}$.

Lemma 10.6. *Let* $\{\alpha_t\}$ *be a continuous field of strongly continuous actions of G on a continuous field of order-unit spaces* $(\{\overline{A_t}\}, \Gamma)$ *over a compact Hausdorff space T. Then for any* $\gamma \in \hat{G}$ *the multiplicity function* (*possibly* $+\infty$*-valued*) $t \mapsto \mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma)$ *is lower semi-continuous over T. For any finite subset* $\mathcal{J}$ of $\hat{G}$ *and* $v \in (\overline{A_t}^{\mathbb{C}})_{\mathcal{J}}$ *we can lift* v *to* f *in* $\Gamma^{\mathbb{C}}_{\mathcal{J}}$. *If furthermore* $\mathcal{J} = \overline{\mathcal{J}}$ *and* v *is in* A_t, *we may take* f *to be in* Γ.

Proof. Let α be the action of G on Γ. Suppose that $v_1, \ldots, v_m$ are linearly independent vectors in $(\overline{A_t}^{\mathbb{C}})_{\mathcal{J}}$. Let $f_1, \ldots, f_m$ be lifts of $v_1, \ldots, v_m$ in $\Gamma^{\mathbb{C}}$. Let $\alpha^{\mathbb{C}}_{\mathcal{J}}$ and $(\alpha^{\mathbb{C}}_t)_{\mathcal{J}}$ be the maps for $\alpha^{\mathbb{C}}$ and $\alpha^{\mathbb{C}}_t$ as defined in Lemma 10.5. Let $\tilde{f}_j = \alpha^{\mathbb{C}}_{\mathcal{J}}(f_j)$. Then $\tilde{f}_1, \ldots, \tilde{f}_m$ are in $\Gamma^{\mathbb{C}}_{\mathcal{J}}$. Since the projection $\Gamma^{\mathbb{C}} \to \overline{A_t}^{\mathbb{C}}$ is G-equivariant, by [27, Lemma 3.3] we have $(\alpha^{\mathbb{C}}_{\mathcal{J}}(f_j))_t = (\alpha^{\mathbb{C}}_t)_{\mathcal{J}}((f_j)_t)$. Thus $(\tilde{f}_j)_t = (\alpha^{\mathbb{C}}_{\mathcal{J}}(f_j))_t = (\alpha^{\mathbb{C}}_t)_{\mathcal{J}}((f_j)_t) = (\alpha^{\mathbb{C}}_t)_{\mathcal{J}}(v_j) = v_j$.

By Lemma 10.4 $(\tilde{f}_1)_{t'}, \ldots, (\tilde{f}_m)_{t'}$ are linearly independent in some open neighborhood $\mathcal{U}$ of t. Since $\tilde{f}_i \in \Gamma^{\mathbb{C}}_{\mathcal{J}}$, $(\tilde{f}_i)_{t'} \in (\overline{A_{t'}}^{\mathbb{C}})_{\mathcal{J}}$ for all $t' \in T$. Taking $\mathcal{J} = \{\gamma\}$, we get the lower semi-continuity of the multiplicity function.

If furthermore $\mathcal{J} = \bar{\mathcal{J}}$ and $v_1, \ldots, v_m$ are all in A_t, we may take $f_1, \ldots, f_m$ to be all in Γ. Notice that $\alpha^{\mathbb{C}}_{\mathcal{J}} = \alpha^{\mathbb{C}}_{\varphi}$, where $\varphi = \sum_{\gamma \in \mathcal{J}} \dim(\gamma)\overline{\chi_\gamma}$. Since the function $x \mapsto (\sum_{\gamma \in \mathcal{J}} \dim(\gamma)\overline{\chi_\gamma})(x)$ is real-valued in this case, we see that $\tilde{f}_1, \ldots, \tilde{f}_m$ are also in Γ. $\square$

Next we show that there are enough Lipschitz sections. Recall that a vector $f \in \Gamma^{\mathbb{C}}$ is called *G-finite* if the linear span of its orbit under α is finite dimensional. We will show that the Lip-norm function $t \mapsto L_t(f_t)$ is continuous for G-finite f. The case of this for quantum tori is proved by Rieffel in Lemma 9.3 of [38]. Our proof for the general case follows the way given there.

Lemma 10.7. *Let (α_t) be a continuous field of strongly continuous actions of G on a continuous field of order-unit spaces $(\{\overline{A_t}\}, \Gamma)$ over a compact Hausdorff space T. Then for any G-finite $f \in \Gamma^{\mathbb{C}}$ the function $t \to L_t(f_t)$ takes finite values and is continuous on T.*

Proof. Let L be the seminorm on $\Gamma^{\mathbb{C}}$ defined by (11). Let $f \in \Gamma^{\mathbb{C}}$ be G-finite. By Theorem 8.2 L takes finite value on a dense subspace of $\mathrm{span}\{\alpha_x(f) : x \in G\}$. Since $\mathrm{span}\{\alpha_x(f) : x \in G\}$ has finite dimension, L is finite on the whole $\mathrm{span}\{\alpha_x(f) : x \in G\}$. It is clear that $L(f) = \sup_{t \in T} L_t(f_t)$. So $t \mapsto L_t(f_t)$ is a real-valued function.

Let

$$D_f = \{(\alpha_x(f) - f)/l(x) : x \neq e_G\}.$$

Then $\sup_{g \in D_f} \|g\| = L(f)$. So D_f is a bounded subset of the finite-dimensional space $\mathrm{span}\{\alpha_x(f) : x \in G\}$, and hence totally bounded.

For $g \in \Gamma^{\mathbb{C}}$, let $F_g(t) = \|g_t\|$ on T. Then clearly $\|F_g - F_h\|_\infty \leqslant \|g - h\|$, and hence F as a map from $\Gamma^{\mathbb{C}}$ to $C(T)$ is Lipschitz. Therefore $F(D_f)$ is totally bounded in $C(T)$. By the Arzela–Ascoli theorem [14] $F(D_f)$ is equicontinuous. Since the supremum of a family of equicontinuous functions is continuous, we see that the function $(t \mapsto L_t(f_t)) = \sup_{g \in D_f} F(g)$ is continuous on T. $\square$

The next lemma generalizes Lemmas 8.3 and 8.4 of [38].

Lemma 10.8. *For any $\varepsilon > 0$ there is a finite subset $\mathcal{J} = \bar{\mathcal{J}}$ in $\hat{G}$, containing the class of the trivial representations, depending only on l and ε, such that for any strongly continuous action α on a complete order-unit space $\bar{A}$ and for any $a \in \bar{A}$, there is some $a' \in A_{\mathcal{J}} := A \cap (\bar{A}^{\mathbb{C}})_{\mathcal{J}}$ with*

$$\|a'\| \leqslant \|a\|, \quad L(a') \leqslant L(a) \quad \textit{and} \quad \|a - a'\| \leqslant \varepsilon L(a).$$

Proof. The complex conjugation is an isometric involution invariant under α. By [27, Lemma 4.4] it suffices to show that for any linear combination φ of finitely many characters on G we have $L \circ \alpha_\varphi \leqslant \|\varphi\|_1 \cdot L$ on $\bar{A}^{\mathbb{C}}$, where α_φ is the linear map on $\bar{A}^{\mathbb{C}}$ defined in Lemma 10.5. Notice that φ is central. Then it is easy to see that α_φ is

G-equivariant. Thus for any $b \in \bar{A}^{\mathbb{C}}$ and $x \in G$ we have

$$\begin{aligned}\|\alpha_\varphi(b) - \alpha_x(\alpha_\varphi(b))\| &= \|\alpha_\varphi(b) - \alpha_\varphi(\varphi_x(b))\| \leqslant \|\varphi\|_1 \cdot \|b - \varphi_x(b)\| \\ &\leqslant l(x)\|\varphi\|_1 L(b).\end{aligned}$$

Consequently, $L(\alpha_\varphi(b)) \leqslant \|\varphi\|_1 L(b)$. $\square$

We are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Since the conditions in Definition 6.4 and (i)–(iii) in Theorem 1.3 are all local statements, we may assume that T is compact. By Lemmas 10.6 and 10.7 the set of G-finite elements in A_t is contained in Γ_t^L. Lemma 10.8 tells us that the restriction of L_t on the set of G-finite elements determines the whole of $L_t^{\mathrm{c}} = L_t$. Thus the induced field $(\{(A_t, L_t)\}, \Gamma)$ is a continuous field of compact quantum metric spaces. (ii) $\Longrightarrow$ (i) follows from Lemma 10.6. Let $R > \int_G l(x)\,dx$. By Theorem 8.2, $R \geqslant r_{A_t}$ for all $t \in T$. We will pick a special sequence f_n in Γ in order to apply Theorem 7.1.

As indicated in Remark 8.1, G is metrizable and hence $L^2(G)$ is separable. Since every $\gamma \in \hat{G}$ appears in the left regular representation, $\hat{G}$ is countable. Then we can take an increasing sequence of finite subsets $\mathcal{J}_1 \subseteq \mathcal{J}_2 \subseteq \cdots$ of $\hat{G}$ such that $\gamma_0 \in \mathcal{J}_1$, $\cup_{k=1}^\infty \mathcal{J}_k = \hat{G}$, and $\overline{\mathcal{J}_k} = \mathcal{J}_k$ for all k. Clearly $a + ia' \in (\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}$ if and only $a - ia' \in (\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}$. Let $(A_t)_{\mathcal{J}_k} = A_t \cap (\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}$. Then $(A_t)_{\mathcal{J}_k}$ spans $(\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}$ as a complex vector space. Let $m_k = \dim((\overline{A_{t_0}}^{\mathbb{C}})_{\mathcal{J}_k})$. Let A_t^{fin} and Γ^{fin} be the set of G-finite elements in A_t and Γ respectively. Then we can find a basis $(b_1, b_2, \ldots)$ of $A_{t_0}^{\mathrm{fin}}$ such that $b_1 = e_{A_{t_0}}$ and $(b_1, \ldots, b_{m_k})$ is a basis of $(A_{t_0})_{\mathcal{J}_k}$ for all k. Let π_t be the projection $\Gamma \to \overline{A_t}$.

Lemma 10.9. *There exists a G-equivariant linear (probably unbounded) map $\varphi : A_{t_0}^{\mathrm{fin}} \to \Gamma^{\mathrm{fin}}$ such that $\varphi(e_{A_{t_0}}) = e$ and φ is a right inverse of π_{t_0},* i.e. *$\pi_{t_0} \circ \varphi$ is the identity map on $A_{t_0}^{\mathrm{fin}}$.*

Proof. Let $\phi : A_{t_0}^{\mathrm{fin}} \to \Gamma$ be a linear right inverse map of π_{t_0} with $\phi(e_{A_{t_0}}) = e$. For any $a \in A_{t_0}^{\mathrm{fin}}$ its G-orbit $\{(\alpha_{t_0})_x(b) : x \in G\}$ is contained in a finite-dimensional subspace of $A_{t_0}^{\mathrm{fin}}$. Thus $\int_G (\alpha_x \circ \phi \circ (\alpha_{t_0})_{x^{-1}})(a)\,dx$ makes sense. It is standard [7, p. 77] that $\varphi = \int_G \alpha_x \circ \phi \circ (\alpha_{t_0})_{x^{-1}}\,dx$ is a G-equivariant linear map from $A_{t_0}^{\mathrm{fin}}$ to Γ. Clearly $\varphi(e_{A_{t_0}}) = e$. For any $b \in A_{t_0}^{\mathrm{fin}}$ we have

$$\begin{aligned}\pi_{t_0}(\varphi(b)) &= \pi_{t_0}\left(\int_G (\alpha_x \circ \phi \circ (\alpha_{t_0})_{x^{-1}})(b)\,dx\right) = \int_G (\pi_{t_0} \circ \alpha_x \circ \phi \circ (\alpha_{t_0})_{x^{-1}})(b)\,dx \\ &= \int_G ((\alpha_{t_0})_x \circ \pi_{t_0} \circ \phi \circ (\alpha_{t_0})_{x^{-1}})(b)\,dx = \int_G ((\alpha_{t_0})_x \circ (\alpha_{t_0})_{x^{-1}})(b)\,dx \\ &= b.\end{aligned}$$

Thus φ is also a right inverse of π_{t_0}. $\square$

Take a dense sequence $\{a_n : n \in \mathbb{N}\}$ in $\mathcal{D}_R(A_{t_0}^{\mathrm{fin}})$ such that $\{a_n : n \in \mathbb{N}\} \cap \mathcal{D}_R((A_{t_0})_{\mathcal{J}_k})$ is dense in $\mathcal{D}_R((A_{t_0})_{\mathcal{J}_k})$ for all k. By Lemma 10.8 the set $\mathcal{D}_R(A_{t_0}^{\mathrm{fin}})$ is dense in $\mathcal{D}_R(A_{t_0})$. Consequently, so is $\{a_n : n \in \mathbb{N}\}$. For each n let $f_n = \varphi(a_n)$. Then $f_n \in \Gamma$ and $(f_n)_{t_0} = a_n$. Let $\varphi_t = \pi_t \circ \varphi$. Let $g_n = \varphi(b_n)$, and let $(V_t)_k = \varphi_t((A_{t_0})_{\mathcal{J}_k})$. Since φ_t is G-equivariant, $(V_t)_k$ is contained in $(A_t)_{\mathcal{J}_k}$. Now we apply Theorem 7.1 to these f_n's.

Suppose that (i) holds. Let $\varepsilon > 0$ be given. By Lemma 10.8 there is some k such that $\mathcal{D}_R((A_t)_{\mathcal{J}_k})$ is $\frac{\varepsilon}{2}$-dense in $\mathcal{D}_R(A_t)$ for all $t \in T$. Then there is a neighborhood $\mathcal{U}_1$ of t_0 such that $\dim((\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}) = \dim((\overline{A_{t_0}}^{\mathbb{C}})_{\mathcal{J}_k})$ and hence $\dim((A_t)_{\mathcal{J}_k}) = \dim((A_{t_0})_{\mathcal{J}_k})$ for all $t \in \mathcal{U}_1$. According to Lemma 10.4 there is some compact neighborhood $\mathcal{U}_2 \subseteq \mathcal{U}_1$ of t_0 such that $(g_1)_t, \ldots, (g_{m_k})_t$ are linearly independent in A_t for all $t \in \mathcal{U}_2$. Then $(g_1)_t, \ldots, (g_{m_k})_t$ is a basis of $(A_t)_{\mathcal{J}_k}$ for all $t \in \mathcal{U}_2$. Take a real vector space V of dimension m_k with a fixed basis $v_1, \ldots, v_{m_k}$. For each $t \in \mathcal{U}_2$ let $\psi_t : V \to (A_t)_{\mathcal{J}_k}$ be the linear isomorphism determined by $\varphi_t(v_j) = (g_j)_t$ for all $1 \leqslant j \leqslant m_k$. Then V gets an order-unit space structure and a Lip-norm for each $t \in \mathcal{U}_2$ by identifying V and $(A_t)_{\mathcal{J}_k}$ via ψ_t. Lemma 10.7 tells us that this is a continuous field of finite-dimensional order-unit spaces and Lip-norms as defined in Example 7.8. Let $\{f_{n_s}\}_{s\in\mathbb{N}}$ be the subsequence of $\{f_n\}_{n\in\mathbb{N}}$ whose image under π_{t_0} is contained in $(A_{t_0})_{\mathcal{J}_k}$. Then $\{f_{n_s}\}_{s\in\mathbb{N}}$ is dense in $\mathcal{D}_R((A_{t_0})_{\mathcal{J}_k})$, and we can apply Lemma 7.10 to $\{f_{n_s}\}_{s\in\mathbb{N}}$. So there are a neighborhood $\mathcal{U} \subseteq \mathcal{U}_2$ of t_0 and some $S \in \mathbb{N}$ such that the open $\frac{\varepsilon}{2}$-balls in $(A_t)_{\mathcal{J}_k}$ centered at $(f_{n_1})_t, \ldots, (f_{n_S})_t$ cover $\mathcal{D}_R((A_t)_{\mathcal{J}_k})$ for all $t \in \mathcal{U}$. Consequently, the open ε-balls in A_t centered at $(f_1)_t, \ldots, (f_{n_S})_t$ cover $\mathcal{D}_R(A_t)$ for all $t \in \mathcal{U}$. By Theorems 7.1 and 1.1 we get (iii).

We proceed to show that (iii) $\Longrightarrow$ (ii). Suppose that (iii) holds. Let $\gamma \in \hat{G}$. Say, $\gamma \in \mathcal{J}_k$. Let $(\alpha_t)_{\mathcal{J}_k} : \overline{A_t}^{\mathbb{C}} \to (\overline{A_t}^{\mathbb{C}})_{\mathcal{J}_k}$ be the continuous map defined in Lemma 10.5. Then $\|(\alpha_t)_{\mathcal{J}_k}\| \leqslant M_1 := \sum_{\gamma' \in \mathcal{J}_k} \dim(\gamma')\|\chi_{\overline{\gamma'}}\|_1$. Since $\mathcal{J}_k = \overline{\mathcal{J}_k}$, $(\alpha_t)_{\mathcal{J}_k}$ maps $\overline{A_t}$ into $(A_t)_{\mathcal{J}_k}$. Let ε be a positive number which we shall choose later. By Theorems 7.1 and 1.1 there is a neighborhood $\mathcal{U}$ of t_0 and some N such that the open ε-balls in A_t centered at $(f_n)_t, n = 1, \ldots, N$, cover $\mathcal{D}_R(A_t)$ for all $t \in \mathcal{U}$. Then the open $M_1\varepsilon$-balls in A_t centered at $(\alpha_t)_{\mathcal{J}_k}((f_n)_t), n = 1, \ldots, N$, cover $\mathcal{D}_R((A_t)_{\mathcal{J}_k})$ for all $t \in \mathcal{U}$. Notice that $(\alpha_t)_{\mathcal{J}_k}((f_n)_t) = \varphi_t((\alpha_{t_0})_{\mathcal{J}_k}(a_n))$ is contained in $(V_t)_k$ for all n.

Suppose that $\mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma) > \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma)$ for some t. Since φ_t is G-equivariant, we have $\mathrm{mul}((V_t)^{\mathbb{C}}, \gamma) \leqslant \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma)$. So we can find a u in $\overline{A_t}^{\mathbb{C}} \setminus (V_t)^{\mathbb{C}}$ such that the complex linear span of $\{(\alpha_t)_x(u) : x \in G\}$, the G-orbit of u, is irreducible of type γ. Say $u = u' + iu''$ with $u', u'' \in (A_{t_j})_{\mathcal{J}_k}$. Let W be the sum of $(V_t)_k$ and the real linear span of the G-orbits of u' and u''. Then $W \supsetneqq (V_t)_k$. Clearly $\mathrm{mul}(W^{\mathbb{C}}, \gamma') = 0$ for all $\gamma' \in \hat{G} \setminus \mathcal{J}_k$ and $\mathrm{mul}(W^{\mathbb{C}}, \gamma') \leqslant \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma') + 2$ for all γ' in $\mathcal{J}_k$. Let $\omega : \mathcal{J}_k \to \mathbb{N}$ be the function $\omega(\gamma') = \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma') + 2$ for all γ' in $\mathcal{J}_k$. Let M_2 be the constant $M_{\mathcal{J}_k,\omega}$ in Lemma 8.4. Then $L_t \leqslant M_2\|\cdot\|$ on $W^{\mathbb{C}}$. Pick a vector in $W/(V_t)_k$ with norm $\min(\frac{1}{M_2}, R)$ and lift it up to a vector v in W with the same norm. Then $\|v - a\| \geqslant \min(\frac{1}{M_2}, R)$ for all $a \in (V_t)_k$ and $L_t(v) \leqslant M_2\|v\| \leqslant 1$. So $v \in \mathcal{D}_R(A_t)$. Thus if we choose ε small enough so that $\min(\frac{1}{M_2}, R) > M_1 \cdot \varepsilon$, then $\mathrm{mul}(\overline{A_t}^{\mathbb{C}}, \gamma) \leqslant \mathrm{mul}(\overline{A_{t_0}}^{\mathbb{C}}, \gamma)$ throughout $\mathcal{U}$. This completes our proof of Theorem 1.3. $\square$

Remark 10.10. Based on Lemmas 10.7 and 10.8, one can also prove (i) $\Longrightarrow$ (iii) along the lines Rieffel used to prove the continuity of quantum tori [38, Theorem 9.2].

Example 10.11 (*Quantum tori*). Fix $n \geqslant 2$. Denote by Θ the space of all real skew-symmetric $n \times n$ matrices. For $\theta \in \Theta$, let $\mathcal{A}_\theta$ be the corresponding quantum torus [32,34]. It could be described as follows. Let ω_θ denote the skew bicharacter on $\mathbb{Z}^n$ defined by

$$\omega_\theta(p, q) = e^{i\pi p\cdot\theta q}.$$

For each $p \in \mathbb{Z}^n$ there is a unitary u_p in $\mathcal{A}_\theta$. And $\mathcal{A}_\theta$ is generated by these unitaries with the relation

$$u_p u_q = \omega_\theta(p, q) u_{p+q}.$$

So one may think of vectors in $\mathcal{A}_\theta$ as some kind of functions on $\mathbb{Z}^n$. The n-torus $\mathbb{T}^n$ has a canonical ergodic action α_θ on $\mathcal{A}_\theta$. Notice that $\mathbb{Z}^n$ is the dual group of $\mathbb{T}^n$. We denote the duality by $\langle p, x\rangle$ for $x \in \mathbb{T}^n$ and $p \in \mathbb{Z}^n$. Then α_θ is determined by

$$\alpha_{\theta,x}(u_p) = \langle p, x\rangle\, u_p.$$

Fix a length function on $G = \mathbb{T}^n$. Let L_θ and A_θ be as in Theorem 8.2 for the order-unit space $((\mathcal{A}_\theta)_{\mathrm{sa}}, e_{\mathcal{A}_\theta})$. Then (A_θ, L_θ) is a compact quantum metric space. Rieffel showed that for each $\theta_0 \in \Theta$ we have $\mathrm{dist}_{\mathrm{q}}(A_\theta, A_{\theta_0}) \to 0$ as $\theta \to \theta_0$ [38, Theorem 9.2]. Here we give a new proof using Theorems 1.3 and 1.1. By [33, Corollary 2.8] the sections $\theta \mapsto u_p$, where p runs through $\mathbb{Z}^n$, generate a continuous field of C^*-algebras $(\{\mathcal{A}_\theta\}, \Gamma)$ over Θ. Notice that for any $x \in \mathbb{T}^n$ and $p \in \mathbb{Z}^n$ the section $\theta \mapsto \alpha_{\theta,x}(u_p) = \langle p, x\rangle\, u_p$ is also in Γ. Then it is easy to see that $\{\alpha_\theta\}$ is a continuous field of strongly continuous ergodic actions. For each $p \in \mathbb{Z}^n = \widehat{\mathbb{T}^n}$ and $\theta \in \Theta$, the multiplicity of p in α_θ is one. Then Theorems 1.3 and 1.1 imply that $\mathrm{dist}_{\mathrm{q}}(A_\theta, A_{\theta_0}) \to 0$ as $\theta \to \theta_0$ for all $\theta_0 \in \Theta$.

Example 10.12 (*Integral coadjoint orbits*). Let G be a compact connected Lie group with a fixed length function. Choose a maximal torus of G and a Cartan–Weyl basis of the complexification of the Lie algebra of G. Then there are bijective correspondences between equivalence classes of irreducible unitary representations of G, dominant weights, and integral coadjoint orbits of G [7] [24, Section IV]. Let $\mathcal{O}_\lambda$ be an integral coadjoint orbit corresponding to a dominant weight λ. Then the restriction of the coadjoint action of G on $\mathcal{O}_\lambda$ is transitive and hence the induced action α_0 on $C(\mathcal{O}_\lambda)$, the algebra of $\mathbb{C}$-valued continuous functions on $\mathcal{O}_\lambda$, is ergodic. So we have the compact quantum metric space (A_0, L_0) defined as in Theorem 8.2. Also let H_n be the carrier space of the irreducible representation of G with highest weight $n\lambda$. Then the conjugate action $\alpha_{1/n}$ of G on $B(H_n)$, the algebra of bounded operators on H_n, is ergodic.

Let $(A_{1/n}, L_{1/n})$ be the corresponding compact quantum metric space defined as in Theorem 8.2. Using the Berezin quantization, Rieffel proved that when G is semisimple, $\mathrm{dist}_q(A_{1/n}, A_0) \to 0$ as $n \to \infty$ [39, Theorem 3.2]. This means that the matrix algebras $B(H_n)$ converge to the coadjoint orbit $\mathcal{O}_\lambda$ as $n \to \infty$. Here we give a new proof using Theorems 1.3 and 1.1. Let P_n be the rank-one projection of $B(H_n)$ corresponding to the highest weight $n\lambda$. For any $a \in B(H_n)$ its Berezin covariant symbol [3,31], σ_a, is defined by

$$\sigma_a(x) = \mathrm{tr}(a\alpha_{1/n,x}(P_n)),$$

where $x \in G$ and tr denotes the usual (un-normalized) trace on $B(H_n)$. There is a natural G-equivariant homeomorphism from the orbit GP_n of P_n (in the projective space) under $\alpha_{1/n}$ onto the coadjoint orbit $\mathcal{O}_{n\lambda}$ [24, Proposition 4]. Dividing everything in $\mathcal{O}_{n\lambda}$ by n, we may identify GP_n with $\mathcal{O}_\lambda$. It is evident that σ_a could be viewed as a continuous function on $GP_n = \mathcal{O}_\lambda$. One can check easily that $a \mapsto \sigma_a$ gives a unital, completely positive, G-equivariant linear map σ_n from $B(H_n)$ to $C(\mathcal{O}_\lambda)$. Endow $\mathcal{O}_\lambda$ with the image of the Haar measure on G, which is a probability measure invariant under α_0. Then $C(\mathcal{O}_\lambda)$ has an inner product as usual. Clearly this inner product is invariant under α_0. Using the normalized trace on $B(H_n)$, which is invariant under $\alpha_{\frac{1}{n}}$, $B(H_n)$ has the Hilbert–Schmidt inner product. Then σ_n has an adjoint operator, $\hat{\sigma}_n$, from $C(\mathcal{O}_\lambda)$ to $B(H_n)$. For any $a \in B(H_n)$ a function $f \in C(\mathcal{O}_\lambda)$ with $\hat{\sigma}_n(f) = a$ is called a Berezin contravariant symbol [3,31] for a. It is easy to see that $\hat{\sigma}_n$ is unital, completely positive and G-equivariant. Since unital completely positive maps are norm-nonincreasing [23, Lemma 5.3], $\hat{\sigma}_n$ is norm-nonincreasing. In [24] Landsman proved that the sections given by these $\hat{\sigma}_n(f)$'s, where f runs through $C(\mathcal{O}_\lambda)$, generate a continuous field of C^*-algebras over $T' = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ with fibers $B(H_n)$ at $\frac{1}{n}$ and $C(\mathcal{O}_\lambda)$ at 0. In fact, Landsman proved that this is a strict quantization of the canonical symplectic structure on $\mathcal{O}_\lambda$, though we do not need this fact here. Using the fact that $\hat{\sigma}_n$ is G-equivariant and norm-nonincreasing, it is easy to check that the $\alpha_{\frac{1}{n}}$'s and α_0 are a continuous field of strongly continuous ergodic actions of G. When G is semisimple, it is known that the maps σ_n are all injective [31] [43, Lemma A.2.1] [39, Theorem 3.1]. Then for each $\gamma \in \hat{G}$ we see that $\mathrm{mul}(B(H_n), \gamma) \leqslant \mathrm{mul}(C(\mathcal{O}_\lambda), \gamma)$. So Theorem 1.3 and 1.1 tell us that $\mathrm{dist}_q(A_{1/n}, A_0) \to 0$ as $n \to \infty$.

11. Continuity of θ-deformations

In this section we prove Theorem 1.4.

We use the notation in [27, Sections 3 and 5]. Let us explain first some convention used in the statement of Theorem 1.4. $C^\infty(M_\theta)$ is a locally convex $*$-algebra, and has a natural $*$-homomorphism Ψ_θ into $C(M_\theta)$ (see the discussion after Definition 3.9 in [27]). Let W_θ be the image of $(C^\infty(M_\theta))_{\mathrm{sa}}$ under the map Ψ_θ. [27, Theorem 1.1] tells us that $(C(M_\theta), L_\theta)$ is a C^*-algebraic compact quantum metric space. This means that $(W_\theta, L_\theta|_{W_\theta})$ is a compact quantum metric space. Since the map Ψ_θ is injective

[27, Lemma 3.10], we may identify $C^\infty(M_\theta)$ with its image $\Psi_\theta(C^\infty(M_\theta))$. In this way $(C^\infty(M_\theta))_{\mathrm{sa}}$ is identified with W_θ, and L_θ has a restriction on $C^\infty(M_\theta)$, which we still denote by L_θ in the statement of Theorem 1.4. In order to make the argument clear, in the rest of this section we shall still distinguish $C^\infty(M_\theta)$ (resp. $(C^\infty(M_\theta))_{\mathrm{sa}}$) and $\Psi_\theta(C^\infty(M_\theta))$ (resp. W_θ).

Let $(\{\mathcal{A}_\theta\}, \Gamma)$ be the continuous field of C^*-algebras over Θ in Example 10.11. We shall also see later, after Lemma 11.1, that the elements in $C(M, Cl^{\mathbb{C}}M) \otimes_{\mathrm{alg}} \Gamma$ generate a continuous field of C^*-algebras [15] over Θ with fibers $C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{A}_\theta$. Let $(\{C(M_\theta)\}, \Gamma^M)$ be the subfield with fibers $C(M_\theta)$.

Lemma 11.1. *There exist a C^*-algebra $\mathcal{B}$ and faithful $*$-homomorphisms $\phi_\theta : \mathcal{A}_\theta \hookrightarrow \mathcal{B}$ such that for every $f \in \Gamma$, the $\mathcal{B}$-valued function $\theta \mapsto \phi_\theta(f_\theta)$ is continuous over Θ.*

Proof. Every unital C^*-algebra admitting an ergodic action of $\mathbb{T}^n$ is nuclear [29, Lemma 6.2] [16, Proposition 3.1]. Thus $\mathcal{A}_\theta$ is nuclear.

Notice that $\mathcal{A}_\theta$ is isomorphic to $\mathcal{A}_{\theta+M}$ naturally for any skew-symmetric $n \times n$ matrix M with even integer entries, by identifying the corresponding u_q. So we may think of $(\{\mathcal{A}_\theta\}, \Gamma)$ as a continuous field over the quotient space of Θ by all the skew-symmetric $n \times n$ matrices with even integer entries. This quotient space is just a torus of dimension $\frac{n(n-1)}{2}$. Now our assertion follows from the result of Blanchard [5, Theorem 3.2] that every separable unital continuous field of nuclear C^*-algebras over a compact metric space has a faithful $*$-homomorphism from each fiber into the Cuntz algebra $\mathcal{O}_2$ such that the global continuous sections become continuous paths in $\mathcal{O}_2$. $\square$

Via identifying $\mathcal{A}_\theta$ with $\phi_\theta(\mathcal{A}_\theta)$ we see that the continuous field $(\{\mathcal{A}_\theta\}, \Gamma)$ becomes a subfield of the trivial field over Θ with fiber $\mathcal{B}$. Then the elements of $C(M, Cl^{\mathbb{C}}M) \otimes_{\mathrm{alg}} \Gamma$ are continuous sections of the trivial field with fiber $C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{B}$. So they generate a subfield of the trivial field with fibers $C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{A}_\theta$.

Now we need to distinguish the norms for elements of the form $\sum_{j=1}^{k} y_{q_j} \otimes u_{q_j}$ at different θ. For this we let $\|\cdot\|_\theta$ denote the norm of $C(M, Cl^{\mathbb{C}}M) \otimes \mathcal{A}_\theta$.

Let $\mathcal{J}$ be a finite subset of $\mathbb{Z}^n = \widehat{\mathbb{T}^n}$ such that $\mathcal{J} = \bar{\mathcal{J}}$ and $\gamma_0 \in \mathcal{J}$. For any $q \in \mathbb{Z}^n$ let $(C(M_\theta))_q$ be the q-isotypic component of $C(M_\theta)$ under the action $\alpha = I \otimes \tau$, and let $(C^\infty(M))_q$ be the q-isotypic component of $C^\infty(M)$ under the action σ as in [27, Section 6], where τ and σ are the actions of $\mathbb{T}^n$ on $\mathcal{A}_\theta$ and $C(M)$, respectively. Similarly we define $(C(M_\theta))_{\mathcal{J}}$ and $(C^\infty(M))_{\mathcal{J}}$. By [27, Lemma 6.2] we have

$$(C(M_\theta))_{\mathcal{J}} \cap W_\theta = \left(\sum_{q \in \mathcal{J}} (C^\infty(M))_q \otimes u_q \right)_{\mathrm{sa}}.$$

Let $V = (\sum_{q \in \mathcal{J}} (C^\infty(M))_q \otimes u_q)_{\mathrm{sa}}$, and let $e = 1_M \otimes u_{\gamma_0}$. Then (V, e) gets an order-unit space structure from $((C(M_\theta))_{\mathrm{sa}}, e)$. Clearly the restriction of $\|\cdot\|_\theta$ is exactly the order-unit norm. Denote by (V_θ, e) this order-unit space. For each $\theta \in \Theta$, by Proposition 2.3 the restriction of L_θ to V_θ is a Lip-norm with $r_{V_\theta} \leqslant r_{W_\theta}$.

Lemma 11.2. *$(V, e, \{\|\cdot\|_\theta|_V\}, \{L_\theta|_V\})$ is a uniformly continuous field of order-unit spaces and Lip-norms over Θ (see Definition 7.9). For any $v \in V$ the function $\theta \mapsto L_\theta(v)$ is continuous over Θ.*

Proof. Let $v \in V$. Say $v = \sum_{q\in\mathcal{J}} v_q \otimes u_q$. By [27, Corollary 5.7] we have

$$L_\theta = L^D \tag{12}$$

on $C(M_\theta)$, where L^D was defined in [27, Definition 5.3]. Recall that for any $f \in C^\infty(M)$ we have [25, Lemma II.5.5]

$$[D, f] = df \text{ as linear maps on } C^\infty(M, S), \tag{13}$$

where $df \in C^\infty(M, T^*M^{\mathbb{C}}) \subseteq C^\infty(M, Cl^{\mathbb{C}}M)$ acts on $C^\infty(M, S)$ via the left $C^\infty(M, Cl^{\mathbb{C}}M)$-module structure of $C^\infty(M, S)$. Then

$$\begin{aligned} L_\theta(v) &= L_\theta\left(\sum_{q\in\mathcal{J}} v_q \otimes u_q\right) \overset{(12)}{=} L^D\left(\sum_{q\in\mathcal{J}} v_q \otimes u_q\right) = \left\|\left[D^{L^2}, \sum_{q\in\mathcal{J}} v_q \otimes u_q\right]\right\|_\theta \\ &= \left\|\sum_{q\in\mathcal{J}} [D, v_q] \otimes u_q\right\|_\theta \overset{(13)}{=} \left\|\sum_{q\in\mathcal{J}} (dv_q) \otimes u_q\right\|_\theta. \end{aligned} \tag{14}$$

Therefore the function $\theta \mapsto L_\theta(v)$ is continuous over Θ. As in the proof of [27, Lemma 4.6] we have that

$$L_\theta(v_q \otimes u_q) \leqslant L_\theta(v) \quad \text{and} \quad \|v_q \otimes u_q\|_\theta \leqslant \|v\|_\theta$$

for all $q \in \mathbb{Z}^n$.

Let $\theta_0 \in \Theta$, and let $\varepsilon > 0$ be given. Since for each q the map $\theta \mapsto \phi_\theta(u_q)$ from Θ to $\mathcal{B}$ is continuous, there is some neighborhood $\mathcal{U}$ of θ_0 such that $\sum_{q\in\mathcal{J}} \|\phi_\theta(u_q) - \phi_{\theta_0}(u_q)\| < \varepsilon$ throughout $\mathcal{U}$. Let $\theta \in \mathcal{U}$. Then for any z_q's in $C(M, Cl^{\mathbb{C}}M)$ with $q \in \mathcal{J}$ and $\|z_q\| \leqslant 1$ we have

$$\begin{aligned} &\left\|(I \otimes \phi_\theta)\left(\sum_{q\in\mathcal{J}} z_q \otimes u_q\right) - (I \otimes \phi_{\theta_0})\left(\sum_{q\in\mathcal{J}} z_q \otimes u_q\right)\right\| \\ &\leqslant \sum_{q\in\mathcal{J}} \|z_q\| \cdot \|\phi_\theta(u_q) - \phi_{\theta_0}(u_q)\| \leqslant \varepsilon. \end{aligned}$$

Suppose that $\|v\|_{\theta_0} = 1$ for some $v \in V$. Say $v = \sum_{q\in\mathcal{J}} v_q \otimes u_q$. Then $\|v_q\| = \|v_q \otimes u_q\|_{\theta_0} \leqslant 1$ for each $q \in \mathcal{J}$. Thus

$$\begin{aligned} \|v\|_\theta = \|(I \otimes \phi_\theta)(v)\| &\leqslant \|(I \otimes \phi_\theta)(v) - (I \otimes \phi_{\theta_0})(v)\| + \|(I \otimes \phi_{\theta_0})(v)\| \\ &\leqslant \varepsilon + \|v\|_{\theta_0} = \varepsilon + 1. \end{aligned}$$

Similarly, $\|v\|_\theta \geqslant 1-\varepsilon$. Therefore $(1-\varepsilon)\|\cdot\|_{\theta_0} \leqslant \|\cdot\|_\theta \leqslant (1+\varepsilon)\|\cdot\|_{\theta_0}$ on V throughout $\mathcal{U}$. Now suppose that $L_{\theta_0}(w) = 1$ for some $w \in V$. Say $w = \sum_{q\in\mathcal{J}} w_q \otimes u_q$. Then by (14) $\|dw_q\| = \|dv_q \otimes u_q\|_{\theta_0} = L_{\theta_0}(v_q \otimes u_q) \leqslant 1$. Let $w' = \sum_{q\in\mathcal{J}}(dw_q)\otimes u_q \in \sum_{q\in\mathcal{J}} C(M, Cl^{\mathbb{C}}M)\otimes u_q$. Then

$$\begin{aligned} L_\theta(w) &\overset{(14)}{=} \|w'\|_\theta = \|(I\otimes\phi_\theta)(w')\| \\ &\geqslant \|(I\otimes\phi_{\theta_0})(w')\| - \|(I\otimes\phi_{\theta_0})(w') - (I\otimes\phi_\theta)(w')\| \\ &\overset{(14)}{\geqslant} L_{\theta_0}(v) - \varepsilon = 1-\varepsilon. \end{aligned}$$

Therefore $(1-\varepsilon)L_{\theta_0} \leqslant L_\theta$ on V throughout $\mathcal{U}$. $\square$

Combining [27, Lemma 4.4] and Lemma 11.2 together we see that the field $(\{(W_\theta, L_\theta|_{W_\theta})\}, (\Gamma^M)_{\mathrm{sa}})$ is a continuous field of compact quantum metric spaces. Let $R = r_M + C\int_{\mathbb{T}^n} l(x)\,dx$, where r_M is the radius of M equipped with the geodesic distance and the constant C was defined in [27, Proposition 5.5]. At the end of [27] it was proved that the radius of $(W_\theta, L_\theta|_{W_\theta})$ is no bigger than R for each θ. Let $\varepsilon > 0$ be given. Pick a finite subset $\mathcal{J} \subseteq \mathbb{Z}^n$ for ε in [27, Lemma 4.4]. Then $\mathrm{dist}^R_{\mathrm{oq}}(W_\theta, V_\theta) \leqslant \mathrm{dist}^{W_\theta}_{\mathrm{H}}(\mathcal{D}_R(W_\theta), \mathcal{D}_R(V_\theta)) \leqslant \varepsilon$ for all $\theta \in \Theta$. By Lemmas 11.2, 7.10, and Theorem 7.1 $\mathrm{dist}^R_{\mathrm{oq}}(V_\theta, V_{\theta_0}) \to 0$ as $\theta \to \theta_0$. Thus there is a neighborhood $\mathcal{U}$ of θ_0 such that $\mathrm{dist}^R_{\mathrm{oq}}(V_\theta, V_{\theta_0}) < \varepsilon$ throughout $\mathcal{U}$. Then clearly $\mathrm{dist}^R_{\mathrm{oq}}(W_\theta, W_{\theta_0}) \leqslant 3\varepsilon$ throughout $\mathcal{U}$. Therefore $\mathrm{dist}^R_{\mathrm{oq}}(W_\theta, W_{\theta_0}) \to 0$ as $\theta \to \theta_0$. By Theorem 1.1 we get $\mathrm{dist}_{\mathrm{oq}}(W_\theta, W_{\theta_0}) \to 0$. This finishes the proof of Theorem 1.4.

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