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Reducible family of height three level algebras

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ABSTRACT

Let $R = k[x_1, \dots, x_r]$ be the polynomial ring in r variables over an infinite field k, and let M be the maximal ideal of R. Here a *level algebra* will be a graded Artinian quotient A of R having socle $\mathrm{Soc}(A) = 0 : M$ in a single degree j. The Hilbert function $H(A) = (h_0, h_1, \dots, h_j)$ gives the dimension $h_i = \dim_k A_i$ of each degree-i graded piece of A for $0 \leqslant i \leqslant j$. The embedding dimension of A is h_1, and the *type* of A is $\dim_k \mathrm{Soc}(A)$, here h_j. The family $\mathrm{LevAlg}(H)$ of level algebra quotients of R having Hilbert function H forms an open subscheme of the family of graded algebras or, via Macaulay duality, of a Grassmannian.

We show that for each of the Hilbert functions $H_1 = (1, 3, 4, 4)$ and $H_2 = (1, 3, 6, 8, 9, 3)$ the family $\mathrm{LevAlg}(H)$ has several irreducible components (Theorems 2.3(A), 2.4). We show also that these examples each lift to points. However, in the first example, an irreducible Betti stratum for Artinian algebras becomes reducible when lifted to points (Theorem 2.3(B)). We show that the second example is the first in an infinite sequence of examples of type three Hilbert functions $H(c)$ in which also the number of components gets arbitrarily large (Theorem 2.10).

The first case where the phenomenon of multiple components can occur (i.e. the lowest embedding dimension and then the lowest type) is that of dimension three and type two. Examples of this first case have been obtained by the authors (unpublished) and also by J.O. Kleppe.

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1. Introduction

In Section 1.1 we give the context of this paper and summarize some recent work on level algebras. In Section 1.2 we discuss the behavior of the graded Betti numbers in a flat family of Artinian algebras, stating a result of A. Ragusa and A. Zappalá, and I. Peeva. In Section 1.3 we describe the parametrization of $\mathrm{LevAlg}(H)$. In Section 1.4 we summarize our results.

We let $R = k[x_1, \ldots, x_r]$ be the ring of polynomials in r variables over an infinite field k, and denote by $M = (x_1, \ldots, x_r)$ the irrelevant maximal ideal. We will consider graded standard Artinian algebras $A = R/I$,

$$A = A_0 \oplus A_1 \oplus \cdots \oplus A_j, \quad A_j \neq 0, \tag{1.1}$$

of socle degree j, such that $I \subset M^2$. The *socle* $\mathrm{Soc}(A)$ is $\mathrm{Soc}(A) = (0 : M)_A = \{a \in A \mid Ma = 0\}$. The Artinian algebra A is *level* if its socle lies in a single degree. Thus, A is level iff $\mathrm{Soc}(A) = A_j$. We will say that a sequence $H = (h_0, \ldots, h_j)$ is *level* if it occurs as the Hilbert function $H(A)$ for some level algebra A. We denote by $n(A)$, the *length* of A, the vector space dimension $n(A) = \dim_k A$. The *length* $n(\mathfrak{Z})$ of a punctual subscheme $\mathfrak{Z}$ of $\mathbb{P}^n$ is that of a minimal Artinian reduction of its coordinate ring. The family $\mathrm{LevAlg}(H)$ of level algebra quotients of R having Hilbert function H forms an open subscheme of the family of graded algebras [Kle1,Kle4,I2] or, via Macaulay duality, of a Grassmannian [IK,ChGe,Kle4].

1.1. Gorenstein algebras and level algebras

When the type is one, the level algebra is *Gorenstein*. Gorenstein algebras occur in many branches of mathematics, and have been widely studied (see [Mac1,BuE,Kle2,Kle3,Hu]). The level algebras of higher types $t = 2, 3, \ldots$ are a natural generalization of the concept of Artinian Gorenstein algebras.

To specify a socle degree, type and Hilbert function of a level algebra A of embedding dimension r is to specify the Hilbert function and the highest graded Betti number of A, which must be in a single degree. Graded Artinian algebras are defined by ideals I that are intersections of level ideals, so one goal of understanding level algebras is to be able to extend results about them suitably to more general Artinian algebras. For example, in [I3] certain results about level algebras of embedding dimension two are extended to general graded Artinian algebras of embedding dimension two: in particular the Hilbert functions compatible with a given type sequence $H(Soc(A))$ are specified. Thus, to understand level algebras is a step toward the study of the family of Artinian algebras with given but arbitrary graded Betti numbers.

For certain pairs (r, t), there are structure theorems for the level algebras of fixed embedding dimension r and type t. Then the sequences H that occur as Hilbert functions for such level algebras are well understood; and it is known that the family $\mathrm{LevAlg}(H)$ is irreducible, even smooth, of known dimensions.

In embedding dimension two, the Hilbert–Burch theorem states that the defining ideal of a CM quotient R/I is given as the maximal minors of a $(\nu - 1) \times \nu$ matrix. When $H = (1, 2, \ldots, t)$ is a level sequence $\mathrm{LevAlg}(H)$ is also irreducible and smooth, of known dimension [I2,I3].

The Buchsbaum–Eisenbud structure theorem shows that the defining ideal of a height three Gorenstein algebra $A = R/I$ is generated by the square roots of the ν diagonal $(\nu - 1) \times (\nu - 1)$ minors of a $\nu \times \nu$ alternating matrix $\mathcal{M}$, where ν is odd: namely $I =$ Pfaffian ideal of $\mathcal{M}$ [BuE]. Also in this case, the family $\mathrm{PGor}(H)$ of Artinian Gorenstein algebras having Hilbert function H is both irreducible and smooth (see [Di,Kle2]); and the dimension of the family $\mathrm{PGor}(H)$ is known (several authors: for a survey of these and related results see [IK, Section 4.4]). However, in embedding dimension four there is no structure theorem for Gorenstein algebras and the family $\mathrm{PGor}(H)$ is in general neither smooth nor even irreducible [Bj,IS].

For arbitrary r and types t, the maximum and minimum possible level Hilbert functions are well known—see [I1] for maximum, and [BiGe,ChoI] for minimum. Recently, several authors have studied level algebras of small lengths, and types, and embedding dimensions three or four, with the idea of delimiting the set of possible Hilbert functions that occur, given the triple (h_1, j, t) of embedding dimension, socle degree, and type [GHMS,Za1,Za2]. For example, A. Geramita et al. studied type two level algebras of embedding dimension three, and determined the Hilbert functions that occur for low socle degree, $j \leqslant 6$. They also gave some techniques for determining more generally which H occur in type 2 height three [GHMS]. Concerning the Hilbert function of height three level algebras, F. Zanello recently showed that in embedding dimensions at least three, and for high enough types t, there are level sequences H that are nonunimodal [Za2]. A. Weiss showed that there are such height three nonunimodal level H for type $t \geqslant 5$, and height four nonunimodal level H for $t \geqslant 3$ [Wei]. Other recent results concern weak Lefschetz properties of level algebras, and liftability of certain level algebras to points [BjZa,M2,M3,MMi].

In embedding dimension four, M. Brignone and G. Valla [BriV] noted that there are many Hilbert function sequences $H = (1, 4, \ldots)$, such that the poset $\beta_{\mathrm{lev}}(H)$ of minimal level Betti sequences that occur for Artinian algebras $A \in \mathrm{LevAlg}(H)$, has two or more minimal elements. By a result of A. Ragusa and G. Zappala [RZ] (see Lemma 1.1 below), this implies that such families $\mathrm{LevAlg}(H)$ have at least two irreducible components, corresponding to the minimal elements of $\beta_{\mathrm{lev}}(H)$.

However, there had been no published studies of the component structure of $\mathrm{LevAlg}(H)$ in embedding dimension three for $t \geqslant 2$ until we began the present work. Our first (in the sense of smallest socle degree, then among those, of smallest length) example of type two height three H for which $\mathrm{LevAlg}(H)$ has two irreducible components is $H = (1, 3, 6, 10, 12, 12, 6, 2)$, which is part of a series we will study elsewhere. Meanwhile, J.O. Kleppe has given an example where, proving a conjecture of the second author about the Hilbert function $H = (1, 3, 6, 10, 14, 10, 6, 2)$, he shows that $\mathrm{LevAlg}(H)$ has at least two irreducible components. He also notes that by linking one can construct further such examples [Kle4, Example 49, Remark 50(b)].

1.2. Betti numbers in flat families

We here state a preparatory result combining work of several authors. Recall that the graded Betti numbers for A are defined as follows. The minimal resolution of A has the form

$$0 \to \bigoplus_k R^{\beta_{r,k}}(-k) \xrightarrow{\delta_r} \cdots \xrightarrow{\delta_2} \bigoplus_k R^{\beta_{1,k}}(-k) \xrightarrow{\delta_1} R \xrightarrow{\delta_0} A \to 0, \tag{1.2}$$

where the collection $\beta = (\beta_{i,j})$ are the graded Betti numbers, and the ith (total) Betti number is $\beta_i = \sum_k \beta_{i,k}$. The poset $\beta(H)$ has as elements the sequences $\beta = (\beta_{ik})$ of minimal graded Betti numbers that occur for Artinian algebras of Hilbert function H; the partial order is

$$\beta \leqslant \beta' \quad \text{iff each } \beta_{i,k} \leqslant \beta'_{i,k}. \tag{1.3}$$

We denote by $\beta_{\mathrm{lev}}(H)$ the restriction of $\beta(H)$ to the subset of level Betti sequences β: namely those for which $\beta_{rk} = 0$ except for $k = j + r$. We denote the corresponding Betti stratum of $\mathrm{LevAlg}(H)$ by $\mathrm{LevAlg}_\beta(H)$. A *consecutive cancellation* is when a new sequence β' of graded Betti numbers are formed from β by choosing $\beta'_{ik} = \beta_{ik}$ except that for some one pair (i_0, k_0) of indices

$$\beta'_{i_0,k_0} = \beta_{i_0,k_0} - 1, \quad \text{and} \quad \beta'_{i_0-1,k_0} = \beta_{i_0-1,k_0} - 1. \tag{1.4}$$

The following result is shown by A. Ragusa and G. Zappalá with antecedents by M. Boratynski and S. Greco who showed that the total Betti numbers are upper semicontinuous on a postulation stratum [RZ,BoG]. The consecutive cancellation portion was shown by I. Peeva [Pe, Remark after Theorem 1.1], based on a result of K. Pardue [Par]. For further discussion see [M1, Theorem 1.1] and [Kle4, Remark 7]. By postulation of a scheme Z we mean the Hilbert function of its coordinate ring $\mathcal{O}_Z = R/\mathcal{I}_Z$.

Lemma 1.1. *Let $Y_T \to T$ be a flat family of punctual subschemes of $\mathbb{P}^{r-1}$, such that the postulation $H(\mathcal{O}_{Y_t}) \mid t \in T$ is constant, and assume that for $t \in T - t_0$, the minimal graded Betti numbers of $\mathcal{O}_{Y_t}$ are constant, equal to β. Then the minimal Betti numbers $\beta(0)$ at the special point $\mathcal{O}_{Y_{t_0}}$ satisfy $\beta(0) \geqslant \beta$, in the poset $\beta(H)$; also β may be obtained from $\beta(0)$ by a sequence of consecutive cancellations.*

If the poset $\beta_{\mathrm{lev}}(H)$ has two incomparable minimal elements β, β', then $\mathrm{LevAlg}(H)$ *has at least two irreducible components corresponding to closures of open subfamilies of* $\mathrm{LevAlg}_\beta(H)$, $\mathrm{LevAlg}_{\beta'}(H)$.

1.3. *Parametrization of the family* LevAlg(H)

We let $H = (1, h_1, h_2, \dots h_j)$, with $h_1 = r$ be a fixed level sequence. For a vector subspace $V \subset R_i$ we denote by $R_a V$ the vector subspace

$$R_a V = \langle f v, f \in R_a, v \in V \rangle \subset R_{a+i}.$$

We denote by $\mathcal{R}$ the ring of divided powers for r variables over k, upon which R acts by differentiation or "contraction" (see [IK, Appendix A], this is an avatar of Macaulay duality [Mac1]). We will use the ring $\mathcal{R}$ in the second parametrization $L(H)$ for level algebras in (B) just below. We denote by $r_i = \dim_k R_i = \binom{i+r-1}{i}$.

There are two natural ways to parametrize the family of level algebra quotients of R having Hilbert function H:

(A) LevAlg(H) is an open subscheme of GrAlg(H), the family of graded algebras quotients of R having Hilbert function H. The family GrAlg(H) is the closed subscheme of

$$\prod_{i=2}^{j} \mathrm{Grass}(r_i - h_i, r_i), \tag{1.5}$$

parametrizing those sequences of subspaces

$$(V_2, \dots, V_j), \quad V_i \subset R_i, \qquad \dim V_i = r_i - h_i, \quad \text{such that } R_1 V_2 \subset V_3, \dots, R_1 V_{j-1} \subset V_j. \tag{1.6}$$

The condition (1.6) is just that the sequence of vector spaces forms the degree two to j graded components of a graded ideal of R. For further detail see [Kle1,Kle3,Kle4], or the discussion of the "postulation Hilbert scheme" in [IK], or [I2, Definition 1.9].

(B) Closed points p_A, $A = R/I$ of LevAlg(H) correspond by Macaulay duality $1-1$ to vector subspaces $\mathcal{W} = (I_j)^\perp$ of $\mathcal{R}_j = \mathrm{Hom}(R_j, k)$, hence to points of the Grassmannian Grass$(h_j, \mathcal{R}_j)$. We define the locally closed subscheme $L(H) \subset \mathrm{Grass}(h_j, \mathcal{R}_j)$ by the "catalecticant" conditions specifying

$$\dim R_{j-1} \circ \mathcal{W} = h_1, \quad \dots, \quad \dim R_1 \circ \mathcal{W} = h_{j-1}.$$

This parametrization is introduced in the type one Gorenstein case in [IK, Section 1.1], and for general level algebras in [ChGe].

The closed points of the two parametrizations LevAlg(H) and $L(H)$ are evidently the same (in char $k = p \leqslant j$ one must use the "contraction action" of R_i on $\mathcal{R}_j$). By the universality property of the family of graded algebras [Kle1] there is a morphism $\iota : L(H)_{\mathrm{red}} \to \mathrm{LevAlg}(H)$, where $L(H)_{\mathrm{red}}$ denotes the reduced scheme structure (see [Kle3, Problem 12]). Recently J.O. Kleppe has shown that $L(H)$ and LevAlg(H) give the same topological structures [Kle4, Theorem 44], extending his earlier result that there is an isomorphism between the tangent spaces to LevAlg(H) and to $L(H)$ at corresponding closed points.

1.4. *Questions and results*

What is a good description of LevAlg(H)? From the point of view of deformations we should answer as fully as possible:

(i) What are the possible level Betti sequences β compatible with H?
(ii) What is the dimension of each Betti stratum $\mathrm{LevAlg}_\beta(H)$?
(iii) What is the closure of $\mathrm{LevAlg}_\beta(H)$?

(iv) What are the irreducible components of LevAlg(H) and of LevAlg$_\beta(H)$?
 (a) Can the component structure of LevAlg(H) be related to that of an appropriate A Hilbert scheme of points or of curves on $\mathbb{P}^{r-1}$?
 (b) Do the components of LevAlg(H) lift to the family LevPoint(T) parametrizing smooth punctual subschemes of $\mathbb{P}^r$, $\Delta T = H$?

Of course, in the absence of a structure theorem (so when $(r,t) \neq (2,t)$ or $(3,1)$), it is hopeless to answer all of these questions for all H, even in embedding dimension three. However, we can answer them for certain H, that might either be of special interest or suggest patterns that are frequent. A productive approach has been that of Question (iv)(a). [Bj,IK,Kle3,Kle4]. J.O. Kleppe in [Kle4] establishes in many cases a $1-1$ correspondence between the set of irreducible components of LevAlg(H) and those of a suitable Hilbert scheme of points.

1.4.1. Results

In this article we first answer the questions above for $H_1 = (1,3,4,4)$ and $H_2 = (1,3,6,8,9,3)$, perhaps the simplest cases in height three where LevAlg(H) has several components.

We show that for each Hilbert function $H = H_1$ or $H = H_2$ the family LevAlg(H) has several irreducible components (Theorems 2.3(A), 2.4). We determine the Betti strata and their closures for each of these two Artinian examples. We then show that each of these examples lifts to families of points having several components (Theorems 2.3(B), 2.5). However, in the first example, an irreducible Betti stratum for Artinian algebras becomes reducible when lifted to points, although the overall component structures for LevAlg(H_1) and LevPoint(T_1) correspond.

These two families LevAlg(H) have different behavior with respect to the poset $\beta_{\mathrm{lev}}(H)$ of graded level Betti number sequences compatible with H. In the first, $H_1 = (1,3,4,4)$, there are two minimal elements of $\beta(H_1)$ and the Ragusa–Zappalá result applies. In the other, $H_2 = (1,3,6,8,9,3)$ there is a unique minimum element of $\beta_{\mathrm{lev}}(H)$, and we must use a different argument to show the reducibility of LevAlg(H).

We show in Section 2.3 that the example H_2 is the first in an infinite series of height three Hilbert functions of type three, $H(c)$, $c \geqslant 3$, where $H(c)$ has socle degree $2c-1$ and satisfies

$$H(c)_i = \min\{r_i - 2r_{i-c}, 3r_{2c-1-i}\}, \quad 0 \leqslant i \leqslant 2c-1;$$

and such that the number of irreducible components of LevAlg$(H(c))$ is bounded from below by $(1-1/\sqrt{2})c$ (Theorem 2.10 and Remark 2.11). This result uses the connection mentioned above in Question (iv)(a). to the Hilbert scheme of points on $\mathbb{P}^2$. We denote by $G(c) = \mathrm{Grass}(2, R_c)$ the Grassmannian parametrizing pencils of degree c plane curves; we let $X(c)$ denote the closed subset of $G(c)$ that is the complement of the open dense set of pencils $\langle f, g\rangle$ spanned by a CI. Then $X(c)$ is the union of irreducible components $G(c)_a$, $1 \leqslant a \leqslant c-1$ corresponding to the degree a of the base component of the pencil. A delicate issue is whether the coordinate ring of the variety that is a union of the base curve C_a and a general enough complete intersection of bidegree $(c-a, c-a)$, has type three Artinian quotients of Hilbert function $H(c)$: we show this using the uniform position property of general enough complete intersections (Lemma 2.9).

We plan to study elsewhere further series of height three Hilbert functions, for which LevAlg(H) has several irreducible components; one such series begins with $H = (1,3,6,10,12,12,6,2)$ of type two, but at the time of writing we have been not able to show that the series is infinite.

2. Families LevAlg(H) having several irreducible components

We now state and prove our main results, outlined above. Henceforth we let $R = k[x,y,z]$ so $r_i = \binom{i+2}{i}$. Note that k is algebraically closed in Section 2.1, infinite in Section 2.2, and algebraically closed of characteristic zero in Section 2.3.

Table 2.1
Graded Betti numbers $\beta(1)$ for $H_1 = (1,3,4,4)$, CI related Artinian algebra.

Total	1	6	9	4
0:	1	–	–	–
1:	–	2	–	–
2:	–	–	1	–
3:	–	4	8	4

2.1. *The family* $\mathrm{LevAlg}(H_1)$, $H_1 = (1,3,4,4)$

We first consider the Artinian case. Let $R = k[x,y,z]$ with k algebraically closed, and consider $\mathrm{LevAlg}(H_1)$, $H_1 = (1,3,4,4)$. The subfamily C_1 parametrizes type 4, socle degree three level quotients A of complete intersections $B = R/J$ where $J = (f_1, f_2)$ has generator degrees $(2,2)$. Thus, $A = R/I$, where the defining ideal satisfies (here $m = (x,y,z)$)

$$I = \left(f_1, f_2, m^4\right). \tag{2.1}$$

As a variety, $C_1 \cong CI(2,2)$, the variety parametrizing complete intersections $B = R/J$, of generator degrees $(2,2)$ for J; evidently C_1 may be regarded as a dense open subset of $\mathrm{Grass}(2, R_2)$, so satisfies

$$\dim C_1 = 2 \cdot 4 = 8.$$

It is easy to check that the monomial ideal

$$I(1) = \left(x^2, y^2, m^4\right) \tag{2.2}$$

defines $A(1) = R/(I(1))$ in this component, and has the minimal resolution $\beta(1)$ of all general enough elements of C_1, given in Table 2.1 (we use the standard “Macaulay” notation). Note that it is not possible to split off the redundant term $R(-4)$ in the minimal free resolution $\beta(1)$ of $A(1)$, as the two quadrics need such a syzygy: thus $\beta(1)$ is a minimal element of the poset $\beta(H)$.

The subfamily C_2 parametrizes type 4 socle degree three quotients $A = R/I$ of the coordinate ring B of a line union a point in $\mathbb{P}^2$: that is, we let $B = R/(I_2)$, where $I_2 \cong \langle xy, xz\rangle$ or $I_2 \cong \langle x^2, xy\rangle$ up to $\mathrm{Gl}(3)$ linear map (coordinate change). Then $R_1 I_2$ has vector space dimension five, and also codimension five in R_3, because of the one linear relation on I_2. We have

$$\dim C_2 = 8. \tag{2.3}$$

This is $2+2$ for the choice of a point and a line, plus $1 \cdot 4$ for the choice of a cubic form, an element of the quotient vector space $R_3/R_1 I_2$. Thus, for $A = R/I \in C_2$ we have, after a linear coordinate change,

$$I \cong (xy, xz, f, W), \quad f \in R_3,\ W \subset R_4,\ \dim_k W = 3, \tag{2.4}$$

or similar generators with $I_2 = (x^2, xy)$. A monomial ideal $I(2)$ determining an Artinian algebra $A(2)$ in C_2 is

$$I(2) = \left(x^2, xy, z^3, y^4, y^2z^2, y^3z\right), \tag{2.5}$$

whose minimal resolution $\beta(2)$ is that of Table 2.2. Again, although there is a redundant term $R(-3)$ in the minimal free resolution, it cannot be split off, so $\beta(2)$ is also a minimal element of the poset $\beta(H)$.

Table 2.2
Graded Betti numbers $\beta(2)$ for $H_1 = (1, 3, 4, 4)$, Artinian algebra related to a line.

Total	1	6	9	4
0:	1	–	–	–
1:	–	2	1	–
2:	–	1	–	–
3:	–	3	8	4

Table 2.3
Graded Betti numbers $\beta(3)$ for $H_1 = (1, 3, 4, 4)$, Artinian algebras in $\overline{C_1} \cap \overline{C_2}$.

Total	1	7	10	4
0:	1	–	–	–
1:	–	2	1	–
2:	–	1	1	–
3:	–	4	8	4

We now show that there is only one other Betti sequence possible for Artinian algebras in $\mathrm{LevAlg}(H_1)$, namely $\beta(3)$, the supremum in $\beta(H_1)$ of $\beta(1)$ and $\beta(2)$ (see Table 2.3). A monomial ideal $I(3)$ defining an algebra $A(3) = R/I(3)$ in $C_3 := \mathrm{LevAlg}_{\beta(3)}(H_1)$ is

$$\left(x^2, xy, y^3, xz^3, y^2z^2, yz^3, z^4\right). \tag{2.6}$$

Since we show in Theorem 2.3 that the Betti stratum $C_3 = \overline{C_1} \cap \overline{C_2}$, the algebra $A(3)$ is a simple example of an obstructed level algebra.

Lemma 2.1. *The Betti stratum $C_3 = \mathrm{LevAlg}_{\beta(3)}(H_1)$ is irreducible of dimension seven. There are no further Betti strata in $\mathrm{LevAlg}(H_1)$, other than $\beta(1)$, $\beta(2)$, and $\beta(3)$.*

Proof. Let $A = R/I$ be a level algebra of Hilbert function $H(A) = H_1$. In order to have a linear relation among the two degree two generators of I, we must have

$$I_2 \cong \langle xy, xz\rangle \text{ or } I_2 \cong \left\langle x^2, xy\right\rangle, \tag{2.7}$$

up to isomorphism.

We now prove that $\dim \mathrm{LevAlg}_{\beta(3)}(H_1) = 7$. Suppose that we are in the first case of (2.7). In order to have a further relation in degree four among the degree two generators xy, xz and the degree three generator f, we must have

$$\ell \cdot f = q_1 \cdot xy + q_2 \cdot xz, \quad \text{where } \ell \in R_1,\ q_1, q_2 \in R_2, \tag{2.8}$$

whence x divides ℓ or f. If x divides f, then f mod (xy, xz) satisfies $f = ax^3$, $a \in k^*$, and it follows that $R_1 \cdot x^2 \subset I$, implying $x^2 \in \mathrm{Soc}(A)$, contradicting the assumption that A is level of socle degree three. Thus, x divides ℓ, and $f \in (y, z)$. Since we may now assume that f, which may be taken mod (xy, xz), has no terms involving x, we have that $f \in R'_3$ where $R' = k[y, z]$. Since k is algebraically closed, up to an isomorphism of R', we may assume one of $f = y^3$, $f = y^2z$, or $f = yz(y + z)$, the last being the generic case, specializing to the two others. The dimension count for this subfamily is seven: two for the choice of $x \in R_1$, two for the choice of $\langle y, z\rangle \subset R$, determining R', and three for the choice of an element f of $R'_3 = \langle y^3, y^2z, yz^2, z^3\rangle$ mod k^*.

Next let us assume that $I_2 = \langle x^2, xy\rangle$, the second case of (2.7). Then (2.8) is replaced by

$$\ell \cdot f = q_1 \cdot x^2 + q_2 \cdot xy, \quad \text{where } \ell \in R_1,\ q_1, q_2 \in R_2. \tag{2.9}$$

If x divides f, then $f \bmod (x^2, xy)$ satisfies $f = axz^2, a \in k^*$, whence $R_1 \cdot xz \subset I$ contradicting that A is level. Hence x divides ℓ, $f \in (x, y)$, and $f \bmod (x^2, xy)$ satisfies

$$f \in \langle xz^2, y^3, y^2z, yz^2 \rangle. \tag{2.10}$$

The dimension count for this subfamily (where $I_2 \cong \langle x^2, xy\rangle$), is six: two for the choice of $x \in R_1$, one for the choice of $\langle x, y\rangle$ containing x, and three for the choice of f up to scalar from a four-dimensional vector space. This completes the proof that $\dim \mathrm{LevAlg}_{\beta(3)}(H_1) = 7$.

We now show that $C_3 = \mathrm{LevAlg}_{\beta(3)}(H_1)$ is irreducible. Assume that $A = R/I \in \mathrm{LevAlg}_{\beta(3)}(H)$, and that $\ell = x$ in (2.9). We deform (x^2, xy) to $(x(x + t(z - x)), xy), t \in k$, and correspondingly deform f to $f(t)$ satisfying

$$f(t) = q_1 \cdot \big(x + t(x - z)\big) + q_2 \cdot y.$$

Then the relation (2.9) deforms to

$$x \cdot f(t) = q_1 \cdot \big(x^2 + tx(x - z)\big) + q_2 \cdot xy.$$

This gives a deformation of $A = R/I$ to $A(t) = R/I(t)$ whose fiber over $t \neq 0$ is an algebra with $I(t))_2 \cong \langle xy, xz\rangle$. Defining for $t \in k$, $I(t) = (x(x+t(x-z)), xy, f(t), m^4)$, we have constant length, hence flat, family of Artinian algebras. Thus C_3 is irreducible.

We next show that there are no further level Betti sequences for H_1. Suppose by way of contradiction that $A = R/I \in \mathrm{LevAlg}(H_1)$ and that there are two or more relations in degree four, among the three generators of the ideal I having degrees two and three. More than two is not possible by Macaulay's theorem. If there are exactly two such relations, the algebra $A' = R/(I_{\leqslant 3})$ would have Hilbert function $H(A') = (1, 3, 4, 4, 5, \ldots)$; the growth of $H(A')$ from 4 to 5 in degrees three and four is the maximum possible by Macaulay's Hilbert function theorem [Mac2], (see [BrH, pp. 155–156]) since $4_3 = \binom{4}{3}$ so the maximum possible for $H(A')_4$ is $4_3' = \binom{5}{4}$. It follows from the Gotzmann Hilbert scheme theorem ([Gotz], see also [IKl]) that $I_3 = (I_{\mathfrak{Z}})_3$, where $\mathfrak{Z}$ is a projective variety with Hilbert polynomial $t + 1$, and regularity degree one, the number of terms in the Macaulay expansion. Thus, $\mathfrak{Z}$ is a line. It follows from I level that $I_{\leqslant 3} = (I_{\mathfrak{Z}})_{\leqslant 3}$, implying $H(A) = (1, 2, 3, 4)$, a contradiction. We have shown that there is at most one relation in degree four among the generators of $(I_{\leqslant 3})$.

Since $A = R/I \in \mathrm{LevAlg}(H_1)$ implies that the minimal generators of I have degrees at most four, we have ruled out any Betti sequence greater than $\beta(3)$ for level algebras of Hilbert function H. This completes the proof of Lemma 2.1. □

Remark 2.2. It is easy to see that here as for most H, $\beta(H) \neq \beta_{\mathrm{lev}}(H)$, as the known maximal graded Betti numbers are rarely level. Here, taking $I = \mathrm{Ann}(X^3, Y^3, (X + Y)^3, (X + 2Y)^3, Z^2)$, with $X, Y, Z \in S$ gives such a maximum β in $\beta(H_1) - \beta_{\mathrm{lev}}(H_1)$.

The first component of $\mathrm{LevAlg}(H_1)$ in the theorem below is $\overline{C_1}$, the closure in $\mathrm{LevAlg}(H_1)$ of the subfamily C_1 for which I_2 defines a complete intersection. Recall that the algebras in C_1 comprise the Betti stratum $\beta(1)$ of Table 2.1. The second component is $\overline{C_2}$, the closure in $\mathrm{LevAlg}(H_1)$ of the subfamily C_2 for which I_2 determines a point union a line. Recall that the algebras in C_2 comprise the Betti stratum $\beta(2)$ of Table 2.2, and that those in C_3 comprise the $\beta(3)$ stratum of $\mathrm{LevAlg}(H_1)$.

We let $T_1 = (1, 4, 8, 12, 12, \ldots)$ with first difference $\Delta T_1 = H_1$, and now define corresponding subfamilies C_1', C_2', C_3' of $\mathrm{LevPoint}(T_1)$, the family of smooth length $n(\mathfrak{Z}) = 12$ punctual subschemes $\mathfrak{Z}$ of $\mathbb{P}^3$ having postulation T_1 (equivalently, having h-vector H_1). We let C_1' parametrize the punctual schemes $\mathfrak{Z}$ lying over C_1, in the sense that some minimal reduction of $R/I_{\mathfrak{Z}}$ (some quotient A of $R/I_{\mathfrak{Z}}$ by a linear element $\ell \in A_1$, such that $\dim_k A = n(\mathfrak{Z})$) is an Artinian algebra in C_1. We similarly define C_2' lying over C_2, and C_3' over C_3.

We denote by D_a', D_b' and D_{ab}' the following three subfamilies of C_3'. In describing, say, eight points on a plane cubic, we allow certain degenerate configurations—as a length 8 punctual scheme on the cubic—provided that the resulting punctual scheme remains in C_3': that is, it has an Artinian quotient that is in C_3. In particular the Artinian quotient must have the Hilbert function H_1, a condition which limits the amount of degeneracy. (We do not here attempt to delimit the allowable degenerations.) We let D_a' parametrize the subfamily of punctual subschemes of $\mathbb{P}^3$ that are unions of nine points on an irreducible plane cubic, and three points on a line meeting the cubic. Let D_b' be the subfamily parametrizing subschemes that are unions of eight general enough points on a plane and four points on a line not in the plane. We let D_{ab}' parametrize general enough subschemes that are unions of nine points on a plane cubic, of which one is the intersection of the line and cubic, union three more points on the line; or a degeneration with a double point at the intersection of the line and cubic.

Theorem 2.3.

(a) LevAlg(H_1), $H_1 = (1, 3, 4, 4)$ *has the two irreducible components* $\overline{C_1}, \overline{C_2}$ *defined above, each the closure of a Betti stratum. Both components have dimension eight. We have*

$$\overline{C_1} \cap \overline{C_2} = C_3 := \mathrm{LevAlg}_{\beta(3)}(H_1). \tag{2.11}$$

(B) LevPoint(T_1), $T_1 = (1, 4, 8, 12, 12, \ldots)$ *has two irreducible components, each the closure of a subfamily* C_1', C_2' *lying above* C_1, C_2*, respectively. Algebras in* C_1' *are the coordinate rings of schemes comprising* 12 *points lying on the complete intersection curve of two quadric surfaces in* $\mathbb{P}^3$*. Elements of* C_2' *parametrize* (*an open subfamily of*) *schemes comprised of nine points lying on a plane cubic, union three points on a line in* $\mathbb{P}^3$*. Each subfamily* C_1' *and* C_2' *has dimension* 28*. The Betti stratum* C_3' *lies in* $\overline{C_1'}$ *and has two irreducible components, the closures of* D_a' *and* D_b'*, each of dimension* 27*. The intersection*

$$\overline{C_1'} \cap \overline{C_2'} = \overline{D_a'} \subset C_3'. \tag{2.12}$$

The intersection of $\overline{D_a'}$ *and* $\overline{D_b'}$ *is* D_{ab}'*, and has dimension* 26*.*

Proof. To show that LevAlg(H_1) has two irreducible components, it suffices by the Ragusa–Zappalá Lemma 1.1 to note that the Betti resolutions of Tables 2.1 and 2.2, which occur for the monomial ideals given in (2.2), (2.5), are each minimal in $\beta(H_1)$. This is true as the two degree two generators of I must have a quadratic relation as in Table 2.1 unless they have a linear relation as in Table 2.2 (or see Lemma 2.1.)

The subfamily C_1 is isomorphic to $CI(2, 2)$, an open set in Grass$(2, R_2) \cong$ Grass$(2, 6)$, so it is irreducible. The family C_2 has fiber an open set in Grass$(1, R_3/R_1I_2) \cong$ Grass$(1, 5)$, over the variety $\mathbb{P}_2^\vee \times \mathbb{P}_2$ parametrizing a line union a point in $\mathbb{P}^2$, so C_2 is irreducible, also of dimension eight.

We now show that the intersection $\overline{C_1} \cap \overline{C_2} = C_3$. Let $A = R/I$ be in the intersection $\overline{C_1} \cap \overline{C_2}$. Then by Lemma 1.1 the graded Betti numbers of A are at least the supremum $\beta(3)$ of $\beta(1), \beta(2)$ in Tables 2.1 and 2.2, so by Lemma 2.1 they are $\beta(3)$. Thus, $I_2 = \langle f_1, f_2\rangle$ has a "linear" relation in degree 3; also, there is an extra degree 4 relation among the degree three generator f, and f_1, f_2. Thus, for $A \in \overline{C_1} \cap \overline{C_2}$ we have that up to isomorphism I satisfies,

$$I = \big(xV, f = hv, m^4\big), \quad V \subset R_1,\ \dim V = 2,\ v \in V,\ h \in R_2. \tag{2.13}$$

We may rule out $x \mid f$ since, as in the proof of Lemma 2.1, this would imply that A is not level. Likewise, if $x \in V$ then $v \notin \langle x\rangle$. Hence, up to isomorphism (after a change of basis for R) we have,

$$\begin{aligned} &I = \big(xy, xz, yh, m^4, h \in \langle y^2, yz, z^2\rangle\big) \text{ or} \\ &I = \big(x^2, xy, yh, m^4, h \in \langle y^2, xz, yz, z^2\rangle\big). \end{aligned} \tag{2.14}$$

We now show that all ideals of the form (2.14), thus all in C_3 lie in the intersection $\overline{C_1} \cap \overline{C_2}$. First, to deform those of form (2.14) into C_2 is easy, as we need only deform yh to a degree three form having no linear factor. To show there is a deformation to C_1, we note that for $I \supset (xy, xz)$ and general enough satisfying the first case of (2.14) we may assume after a further change of variables in R (see Lemma 2.1), that

$$I = \bigl(xy, xz, yz(y+z), m^4\bigr). \tag{2.15}$$

We now consider the one-parameter family of ideals in $R[t]_t$,

$$I(t) = \bigl(xy, xz + tz(y+z), m^4\bigr), \quad t \neq 0. \tag{2.16}$$

For each $t \neq 0$, we have $I(t) \in C_1$, as $(xy, zx + zt(y+z))$ is a CI. Note that each $I(t)$, $t \neq 0$, contains

$$\frac{1}{t}\bigl[y\bigl(xz + tz(y+z)\bigr) - z(xy)\bigr] = yz(y+z),$$

hence the limit

$$I(0) := \lim_{t\to 0} I(t) = \bigl(xy, xz, yz(y+z), m^4\bigr). \tag{2.17}$$

Thus, we have shown that the general element (2.15) of C_3 is in $\overline{C_1}$ and in $\overline{C_2}$. Since C_3 is irreducible by Lemma 2.1, we have shown $\overline{C_1} \cap \overline{C_2} = C_3$.

Since C_1, C_2 are each irreducible Betti strata, since C_3 is also irreducible, and since by Lemma 2.1, $\beta(1), \beta(2), \beta(3)$ are the only Betti sequences that occur for level algebras of Hilbert function H_1, it follows that there can be no further irreducible components of $\mathrm{LevAlg}(H_1)$. A check of tangent space dimensions by MACAULAY showed that the tangent space dimension is eight for general points of C_1 or C_2 but nine for the algebra $A(3)$ in $\overline{C_1} \cap \overline{C_2}$, defined by a monomial ideal, given by (2.6).

We now consider $\mathrm{LevPoint}(T_1)$, $T_1 = (1, 4, 8, 12, 12, \ldots)$, for which $\Delta T_1 = H_1$, the sequence above. Recall that C_1', C_2' lie over $C_1, C_2 \subset \mathrm{LevAlg}(H_1)$, and that there are monomial ideals in each of C_1, C_2, as well as in $\overline{C_1} \cap \overline{C_2}$ (Eqs. (2.2), (2.5), (2.6)); thus each of $C_1', C_2', \overline{C_1'} \cap \overline{C_2'}$ is nonempty, and the graded Betti numbers for generic elements of these subsets agree with those that occur for the corresponding subfamilies of $\mathrm{LevAlg}(H_1)$ (see Tables 2.1–2.3). That C_1', C_2' arise as stated in the theorem is evident.

We first show that each of the two irreducible subfamilies C_1', C_2' of $\mathrm{LevPoint}(T_1)$ has dimension 28. Let $S = k[x, y, z, w]$, and suppose that $J \subset S$ is the defining ideal of the punctual scheme. The dimension count for C_1' is $\dim C_1' = 16 + 12 = 28$ as $I_2 = \langle f_1, f_2\rangle \in \mathrm{Grass}(2, 10)$, has dimension $16 = 2 \cdot 8$, and 12 points are chosen on the curve $\{f_1 = 0\} \cap \{f_2 = 0\}$. We have $\dim C_2' = 28$ as follows: 3 for the choice of a plane, and $4 = \dim \mathrm{Grass}(2, 4)$ for the choice of a line in $\mathbb{P}^3$, the plane union a line determining $I_2 \cong \langle xy, xz\rangle$. Then 12 for the choice of a cubic f in $S_3/(R_1 I_2)$, a vector space of dimension 13 because of the linear relation among the two generators of I_2. The cubic intersect the line determines three of the 12 points; the last 9 must lie on the cubic curve defined by the cubic surface $\{f = 0\}$ intersection the plane $\{x = 0\}$.

Since a punctual scheme must be defined by an algebra S/J having a linear nonzero divisor, it follows from our classification of all the Betti sequences for level algebras of Hilbert function H that there can be no other irreducible components of $\mathrm{LevPoint}(T_1)$.

We now show that $C_3' = \mathrm{LevPoint}_{\beta(3)}(T_1)$ has two irreducible components, the closures of D_a' and D_b', each of dimension 27. Evidently, D_a' and D_b' are irreducible. For D_a' (9 points on the plane cubic, three on the line), the choice of a plane cubic and a line meeting the cubic requires a $15 = 3 + 9 + 3$-dimensional family, and the choice of the 12 points distributed $(9, 3)$ gives $\dim D_a' = 27$. For D_b', (8 points on a plane cubic, 4 on a line), the eight points may be chosen as a general set of 8 points on the plane, having $3 + 16 = 19$ parameters; choosing a line requires 4 parameters, and, there is one plane cubic through the eight points meeting the line. One now chooses four points on the line so $\dim D_b' = 19 + 4 + 4 = 27$. The family D_{ab}' restricts one point to be the intersection of the line and the cubic, so has dimension 26.

It remains to show that there are no other components of C_3'. The two generators of degree two have a common factor, so any punctual scheme with this resolution has to lie on a plane union a line. The twelve points can be distributed differently between the two components, but if there are less than eight points in the plane, the Hilbert function is at most $7+4=11$ in degree three. If there are at most two points on the line, the first part of the ideal is the ideal of these points union the plane, in which case we get socle in degree one or two. Thus, the only distributions that give the correct Betti numbers are the ones described in D_a' and D_b'.

We now need to show that each point $p_3 \in C_3'$, p_3 a punctual scheme, lies in the closure of C_1'. But, since the first part of the minimal resolution of the coordinate ring $\mathcal{O}_3$ is that of an ACM curve, one can deform that to a complete intersection. We now show that $\overline{C_2'} \cap \mathrm{LevPoint}_{\beta(3)}(T) = \overline{D_a'}$. But, considering a generic point of D_a', it suffices to move the plane cubic and its nine points so that it does not intersect the line with its three points, in order to deform to C_2'. On the other hand a subfamily of C_2' cannot converge to twelve points, of which eight are on a cubic, and four are on a line, unless one of the points is the line intersection the cubic (so a scheme in D_{ab}'). $\square$

Note. Tangent space calculations using Macaulay2 [GrSt] show that the tangent space at a general point of C_1', C_2' or D_b' has dimension 28, whereas the tangent space at a general point of D_a' or of D_{ab}' has dimension 29.

The strong Lefschetz (SL) condition is that $\times\ell$ for ℓ generic in R_1 acting on A has Jordan partition $P(H)$ given by the lengths of the rows of the bar graph of H. (This definition generalizes that in use for H unimodal symmetric [HarW].) It conceivably might be used to distinguish components of $\mathrm{LevAlg}(H)$, as SL is an open condition. However, the algebra $A(3)$ of (2.6) satisfies the strong Lefschetz condition, by calculation. Since $A(3) \in C_3 = \overline{C_1} \cap \overline{C_2}$, the general elements of C_1, C_2 are SL.

2.2. *The family* $\mathrm{LevAlg}(H_2)$, $H_2 = (1,3,6,8,9,3)$

We let k be an infinite field, and consider $\mathrm{LevAlg}(H_2)$, $H_2 = (1,3,6,8,9,3)$. The subfamily $C_1 = \mathrm{LevAlg}_{\beta(1)}(H_2)$ parametrizes type 3, socle degree five level quotients A of complete intersections $B = R/J$ where $J = (f_1, f_2)$ has generator degrees $(3,3)$, and having the graded Betti numbers $\beta(1)$ given by omitting the degree four generator-relation pair from $\beta(2)$ in Table 2.4. Thus the algebras $A = R/I$ in C_1 satisfy

$$I_3 = \langle f_1, f_2 \rangle, \tag{2.18}$$

where f_1, f_2 define a complete intersection in $\mathbb{P}^2$. The family $CI(3,3) \subset \mathrm{Grass}(2, R_3)$ of complete intersections $B = R/(f_1, f_2)$ is open dense in $\mathrm{Grass}(2, R_3)$ and has dimension $2 \cdot 8 = 16$. The subfamily C_1 is fibred over $CI(3,3)$ by an open set in the Grassmannian $\mathrm{Grass}(3, R_5/R_2 I_3) \cong \mathrm{Grass}(3,9)$ parametrizing the choice of a type three, socle degree five quotient of $B = R/(I_3)$: we may take

$$A = B/(W), \quad W = \langle h_1, \ldots, h_6 \rangle \subset B_5, \tag{2.19}$$

where W is a six-dimensional, general enough subspace of the nine-dimensional space B_5. Thus we have

$$\dim C_1 = 16 + 18 = 34.$$

One needs of course to check that such level quotients having the Hilbert function H_2 exist: this is easy to do, using the special case $B(0) = R/(x^3, y^3)$.

The graded Betti numbers $\beta(1)$ for Artinian algebras $A \in C_1$ are the minimum consistent with the Hilbert function H: there are besides the eight generators of I already mentioned, ten relations in degree six, and the three relations among the relations in degree eight.

The second component $\overline{C_2}$ is the closure of the Betti stratum $C_2 = \mathrm{LevAlg}_{\beta(2)}(H_2)$ given by Table 2.4, and parametrizes ideals whose initial portion $(I_{\leqslant 4})$ defines a quadric union a point. More precisely, C_2 consists generically of quotients $A = R/I$ such that

Table 2.4
Graded Betti numbers $\beta(2)$ for $H_2 = (1, 3, 6, 8, 9, 3)$, $(I_{\leqslant 4}) = (x^3, x^2y, z^4)$.

Total	1	9	11	3
0:	1	–	–	–
1:	–	–	–	–
2:	–	2	1	–
3:	–	1	–	–
4:	–	6	10	–
5:	–	–	–	3

$$I_3 = \xi \cdot V, \quad \xi \in R_2,\ V \subset R_1,\ \dim V = 2,$$
$$I_4 = \langle R_1 I_3, f \rangle \tag{2.20}$$

and having no degree five relations. Thus A is the quotient of an Artinian algebra $B = R/(\xi \cdot V, f)$ where $R/(\xi \cdot V)$ defines a quadric $\xi = 0$ union the point $V = 0$ in $\mathbb{P}^2$.

$$A = B/(W), \quad W = \langle h_1, \ldots, h_6 \rangle \subset B_5, \tag{2.21}$$

where as before W is a six-dimensional, general enough subspace of the nine-dimensional space B_5.

We have

$$\dim C_2 = 2 + 5 + 9 + 18 = 34,$$

as follows. First, the choice of $\xi \cdot V$, $V \subset R_2$ requires $7 = 2 + 5$ parameters, since the quadric requires five and the point two parameters. Since $\dim_k R_4/R_1I_3 = 10$, the choice of $f \in R_4/R_1I_3$ is from an open set in $\mathbb{P}^9$; and the choice of $W \subset B_5$ is that of a point in Grass(6, 9).

The graded Betti numbers $\beta(2)$ for an element $A \in C_2$ are those of Table 2.4: these graded Betti numbers are attained for $B = R/(x^3, x^2y, z^4))$, and W general enough.

There are two more Betti sequences that occur for level algebras of Hilbert function $H_2 = (1, 3, 6, 8, 9, 3)$. The first, $\beta(3)$ corresponds to ideals I having an extra relation and generator in degree five compared to the graded Betti numbers for algebras in C_1. There must then be a total of seven degree five generators. Here the ideal $J = (I_{\leqslant 4})$ determines an algebra $B = R/J$ defining a line union four points, $J = (J_3)$, and

$$I_3 = J_3 = \langle \ell g, \ell h \rangle, \quad \ell \in R_1,\ g, h \in R_2.$$

It is easy to see that this stratum has dimension 31, as follows. We count 2 for the choice of a line $\{\ell = 0\}$, 8 for the choice of a CI (g, h) of degree two forms (four points), and (7)(3) for the choice of a three-dimensional quotient A_5 of R_5/R_2I_3, a ten-dimensional vector space. It is also easy to see that such ideals are in the first component $\overline{C_1}$: writing $J(t) = (\ell g + tf_1, \ell h + tf_2)$ we find that $gf_2 - hf_1 \in J(t)$, hence is also in the limit $J(0)$, but this is no restriction, as it merely requires $I(0)_5 \cap (g, h)_5 \neq 0$, and a codimension 3 space I_5 always intersects a codimension four space $(g, h)_5 = R_3(g, h)$ nontrivially in the dimension 21 vector space R_5.

The last Betti stratum $\beta(4)$ is like the previous one, except for having the extra relation and generator pair in degree four, as for algebras in C_2. This corresponds to $(I_{\leqslant 4}) = (\ell g, \ell h, \ell q_3)$, where (g, h) have a common factor, a degeneration of the previous case. Thus,

$$B = R/J, \quad J = (I_{\leqslant 4}) = (\ell \rho x, \ell \rho y, \ell q_3), \quad \ell, \rho \in R_1,\ q_3 \in (x, y) \cap R_3.$$

Here it is not hard to check that this Betti stratum forms a thirty-dimensional family in the closure of the previous stratum, and as well in $\overline{C_1} \cap \overline{C_2}$.

The proof below that there are two irreducible components of $\mathrm{LevAlg}(H_2)$ depends primarily on the dimension count, since the poset $\beta_{\mathrm{lev}}(H)$ has a minimum element.

Theorem 2.4. *The family* LevAlg(H_2), $H_2 = (1,3,6,8,9,3)$ *has two irreducible components* $\overline{C_1}, \overline{C_2}$, *whose open dense subsets* C_1, C_2 *corresponding to two different Betti strata for* H_2 *are described above* (*see* (2.19), (2.20)), *each of dimension* 34. *Also* $\overline{C_1} \cap C_2 = V_2$, *a codimension one subset of* C_2 *parametrizing quotients* $A = R/I$,

$$V_2\colon\ I = (hW_1, f_4), \quad W_1 \subset R_1,\ \dim_k W_1 = 2,\ h \subset R_2,\ f_4 \in R_3 W_1. \tag{2.22}$$

The third Betti stratum $\mathrm{LevAlg}_{\beta(3)}(H_2)$ *lies in* $\overline{C_1}$ *and has dimension* 31. *The fourth, most special Betti stratum* $\mathrm{LevAlg}_{\beta(4)}(H_2)$ *is a subscheme of* LevAlg(H_2) *having dimension* 30, *and lies in* $\overline{C_1} \cap \overline{C_2}$.

Proof. By Lemma 1.1 it is not possible to specialize from a subfamily of $\overline{C_2}$ to a point A of C_1, so $\overline{C_2} \cap C_1 = \emptyset$: this can also be seen readily since for $B = R/J \in \overline{C_2}$, the two-dimensional vector space J_3 has a common quadratic factor, while I_3 does not have a common factor for points $A = R/I$ of C_1. Since the open dense $C_1 \subset \overline{C_1}$ comprises the Artinian algebras having the unique minimum possible graded Betti numbers compatible with H, there can be no larger irreducible subfamily of LevAlg(H_2) specializing to both C_1 and C_2. The family C_2 we have described is exactly the subfamily having the graded Betti numbers of Table 2.4. Thus, that $\overline{C_2}$ has the same dimension as $\overline{C_1}$ shows that both are irreducible components of LevAlg(H_2).

To identify the intersection $\overline{C_1} \cap C_2$ as V_2 from (2.22), for the moment take $W_1 = \langle x, y\rangle$ and consider the following flat family converging to a point $B(0)$ of GrAlg(H'), $H' = (1,3,6,8,9,9,\ldots)$ where $B(0) = R/J(0)$ satisfies $I(0)_3$ has a degree-two common factor h:

$$B(t) = R/J(t), \quad J(t) = (xh + tf_1, yh + tf_2).$$

Each element of the family for $t \neq 0$ contains $xf_2 - yf_1$, thus the limit $J(0) = \lim_{t\to 0} J(t)$ satisfies $J(0) = (xh, yh, xf_2 - yf_1)$. Here f_1, f_2 may be chosen arbitrarily. The corresponding subfamily of $\overline{C_1}$ is fibred by an open dense subset in Grass$(6, R_5/J(t)_5)$ over $B(t)$, parametrizing a vector space $V(t)$, such that $I(t) = (J(t), V(t))$ defines $A(t) = R/I(t) \in \overline{C_1}$. This shows that each element of the right side of (2.22) occurs in the closure of C_1, and vice versa. That V_2 has codimension one in C_2 is a consequence of "$f_4 \in R_3W_1$" being a codimension one condition, as R_3W_1 has vector space codimension one in R_4. $\square$

Note. The general element of $\mathrm{LevAlg}_{\beta(4)}(H_2)$ is strong Lefschetz by calculation, and it follows here as for H_1 that the general elements of C_1, C_2 are also strong Lefschetz.

We now consider the sum function $T_2 = (1,4,10,18,27,30,30,\ldots)$ of H_2, and we determine components $\overline{C_1'}, \overline{C_2'}$ of LevPoint(T_2), lying over the components $\overline{C_1}, \overline{C_2}$, respectively, of LevAlg(H_2). Here an open dense subset C_1' of $\overline{C_1'}$ is comprised of thirty distinct points in $\mathbb{P}^3$ that are general enough, lying on a CI curve C that is the intersection of two cubic surfaces. An open dense subset C_2' of $\overline{C_2'}$ is constructed as follows: intersect the union of a quadratic surface K and a line L with a general enough quartic surface Q. Choose twenty six general enough distinct points on the curve $K \cap Q$, in addition to the four points comprising $L \cap Q$. The proof of the following Theorem is simplified by the fact that H_2 is the h-vector of 30 points in generic position lying on the intersection of two cubics.

We denote by $S = k[x,y,z,w]$, the coordinate ring of $\mathbb{P}^3$.

Theorem 2.5. *The scheme* LevPoint(T_2), $\Delta T_2 = H_2 = (1,3,6,8,9,3)$ *has two irreducible components,* $\overline{C_1'}$, $\overline{C_2'}$, *each of dimension* 66. *Their tangent spaces each have dimension* 66.

Proof. The putative components have open dense subsets that are Betti strata, and $\beta(1) < \beta(2)$ in $\beta_{\mathrm{lev}}(H_2)$, so it suffices to verify the dimension calculations. Here $CI(3,3)$ has dimension $2(18) = 36$, and the choice of thirty points on the CI curve gives a total of 66 for $\dim C_1'$.

For $\overline{C_2'}$, the choice of a line in $\mathbb{P}^3$ is that of a point in $\mathrm{Grass}(2, S_1)$ gives 4 dimensions, and the choice of a quadric surface $C : c = 0, c \in S_2$ up to k^* multiple gives 9 more. The choice of a quartic surface $Q\colon\ q = 0$ in the twenty eight-dimensional vector space $[S/(\ell_1 c, \ell_2 c)]_4$, $\ell_i \in S_1$ gives 27; the choice of 26 further points on $Q \cap C$ gives a total of 66. This shows that $\overline{C_1}$ and $\overline{C_2}$ are components.

In both cases, calculations in Macaulay2 show that the tangent space at a general point of C_1' or of C_2' has dimension 66. $\square$

2.3. *An infinite sequence of examples of type three*

We will now show that the example $H_2 = (1, 3, 6, 8, 9, 3)$ is the first in an infinite sequence of examples where also the number of components gets arbitrarily large. The idea is to start with the Hilbert function we get by taking a general level quotient of type three and socle degree $2c - 1$ of a complete intersection of type (c, c), where $c \geqslant 3$. We will see that we can get other level algebras with the same Hilbert function, which are not specializations of the ones coming from complete intersections. In order for us to use a result on the uniform position property [Ha], we have to assume in this section that k is algebraically closed of characteristic 0.

Definition 2.6. For $c \geqslant 3$, we define the Hilbert function $H(c)$ of socle degree $j = 2c - 1$ by

$$H(c)_i = \min\{r_i - 2r_{i-c}, 3r_{2c-1-i}\}, \quad 0 \leqslant i \leqslant 2c - 1, \tag{2.23}$$

where r_i is the Hilbert function of the polynomial ring $R = k[x, y, z]$. The transition between the two phases occurs in degree $\alpha(c) := 2c - \sqrt{(c^2 - 1)/2}$. Note that this is usually not an integer (cf. Remark 2.11).

We now look at the different possibilities for the generators in degree c of an ideal with Hilbert function $H(c)$.

Definition 2.7. For $c \geqslant 1$, let $G(c) = \mathrm{Grass}(2, R_c)$ parametrize two-dimensional subspaces of the space R_c of forms of degree c in $R = k[x, y, z]$. For $a = 0, 1, 2, \ldots, c - 1$, let $G(c)_a$ be the subset parametrizing subspaces $V \subseteq R_c$ where the greatest common divisor has degree a. This is a stratification of $G(c)$ into disjoint semi-closed subsets.

$$G(c) = \bigcup_{a=0}^{c-1} G(c)_a.$$

The open stratum $G(c)_0$ corresponds to vector spaces V which define a complete intersection of type (c, c) in $\mathbb{P}^2$. The following lemma tells us about the complement of this open stratum. Recall that $r_a = \dim_k R_a$.

Lemma 2.8. *The dimension of the stratum $G(c)_a$ is $r_a + 2r_{c-a} - 5$, for $0 \leqslant a < c$ and the closures of the smaller strata, $\overline{G(c)_1}, \overline{G(c)_2}, \ldots, \overline{G(c)_{c-1}}$ are the irreducible components of the variety $X(c)$ defined as the complement of the open stratum, $G(c)_0$, in $G(c) = \mathrm{Grass}(2, R_c)$.*

Proof. In the closure of the stratum $G(c)_a$ there is always a common factor of degree a, but there could also be a common factor of higher degree. In that case, this common factor has a factor of degree a. We get a surjective map

$$\mathbb{P}(R_a) \times \mathrm{Grass}(2, R_{c-a}) \to \overline{G(c)_a}$$

by sending $(f, \langle g, h\rangle)$ to $\langle fg, fh\rangle$ and we get the dimension of $G(c)_a$ as $G(c)_a = r_a - 1 + 2(r_{c-a} - 2) = r_a + 2r_{c-a} - 5$.

The variety $X(c) \subset \mathrm{Grass}(2, R_c)$, defined as the complement of the open stratum $G(c)_0$, parametrizes two-dimensional subspaces with a common factor. If the greatest common divisor of such a space $V \subseteq R_c$ is irreducible of degree a, we are in the stratum $G(c)_a$ but not in the closure of any of the other strata $G(c)_i$, $i \neq 0, a$. Thus the closures of the smaller strata are not contained in each other, but their union is all of $X(c)$. $\square$

Now, we will show that the we can get level algebras with Hilbert function $H(c)$ as quotients of ideals generated by two forms in degree c.

Lemma 2.9. *Let $\langle f, g\rangle \subseteq R_c$ be a general element in $G(c)_a$ and, if $a > 0$, let $I = (f, g, h)$, where h is a general form of degree $2c - a$.*

If $a = 0$ or $\sqrt{(c^2 - 1)/2} \leqslant a < c$, then a sufficiently general type three level quotient of R/I with socle in degree $2c - 1$ has Hilbert function $H(c)$.

Proof. We first look at the quotient of R/I with the ideal $(x, y, z)^{2c}$. With $\bar{I} = I + (x, y, z)^{2c}$ we get that $R/\bar{I}$ is a level algebra with Hilbert function

$$H(R/\bar{I})_i = \begin{cases} r_i - 2r_{i-c}, & 0 \leqslant i < 2c, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, we have that $H(R/\bar{I})_{2c-2} = H(R/\bar{I})_{2c-1} = c^2$. The ideal I is clearly uniquely determined by $\bar{I}$. In the parameter space of algebras having the same Hilbert function as $\bar{I}$, we can specialize to an ideal $\overline{I'}$ and if the general level quotient of $R/\overline{I'}$ has the desired Hilbert function, so does the general level quotient of $R/\bar{I}$.

Since $\langle f, g\rangle$ is assumed to be general, the zero-set defined by the ideal (f, g) consists of a curve of degree a and a complete intersection of type $(c - a, c - a)$. We can specialize I into $I' = (f, g, h')$ by asserting that h' vanishes on the complete intersection. This is a condition of codimension $(c - a)^2$. Now, in order to show that the Hilbert function of a general type three level quotient of R/I is $H(c)$, it suffices to show that this is the case for the specialized ideal I', which is an ideal of a set of c^2 points in the plane. This set of points is the union of a complete intersection of type $(a, 2c - a)$ and a complete intersection of type $(c - a, c - a)$.

We will now go on to show that if these complete intersections are chosen general enough, the Hilbert function of a general level quotient of type 3 and socle degree $2c - 1$ is the expected $H(c)$.

Let $N = \max\{H(c)_i \mid 0 \leqslant i \leqslant c\}$ and consider a set of N points in the plane which is a subset of the union of a complete intersection of type $(a, 2c - a)$ and a complete intersection of type $(c - a, c - a)$. We want to partition this set into three parts of sizes differing by at most one, such that each of the parts has the Hilbert function of a generic set of points.

Let W be a complete intersection in $\mathbb{P}^2$ of type $(c - a, c - a)$ having the uniform position property, i.e., all its subsets of the same cardinality have the same Hilbert function. Let $W = W_1 \cup W_2 \cup W_3$ be a partition such that $|W_1| \leqslant |W_2| \leqslant |W_3| \leqslant |W_1| + 1$.

For any positive integer $m < r_a$ let $\mathcal{X}_m \subseteq \mathrm{Hilb}^m(\mathbb{P}^2)$ be the subscheme parametrizing reduced sets of m points with the generic Hilbert function, i.e., $h_i = \min\{m, r_i\}$, for $i \geqslant 0$. Let $\mathcal{Y}$ be the subscheme of $\mathrm{Hilb}^{a(2c-a)}(\mathbb{P}^2)$ parametrizing complete intersections of type $(a, 2c - a)$ having the uniform position property. This is an open dense subset in the parameter space of complete intersections [Ha].

Consider the correspondence given by

$$\mathcal{C}_m = \{(X, Y) \mid X \subseteq Y\} \subseteq \mathcal{X}_m \times \mathcal{Y}$$

with the two maps $\mathcal{C}_m \to \mathcal{X}_m$ and $\mathcal{C}_m \to \mathcal{Y}$ induced by the projections.

The fibers of the map $\mathcal{C}_m \to \mathcal{X}_m$ are all irreducible of the same dimension since we only have to pick general forms of degree a and $2c - a$ in the ideal of X. Hence $\mathcal{C}_m$ is irreducible. The map $\mathcal{C}_m \to \mathcal{Y}$ is finite, since a reduced complete intersection only has a finite number of subschemes. Let $\mathcal{Z}_m \subseteq \mathcal{X}_m$ be the subscheme parameterizing sets of points, $X \subseteq \mathbb{P}^2$ such that the Hilbert function of one of the sets $X \cup W_1$, $X \cup W_2$ and $X \cup W_3$ is not generic. Then $\mathcal{Z}_m \subseteq \mathcal{X}_m$ is a closed proper subset, since a generic set of points in the plane yields a generic Hilbert function for each of the three sets $X \cup W_1$, $X \cup W_2$ and $X \cup W_3$. Thus the dimension of the inverse image of $\mathcal{Z}_m$ in $\mathcal{C}_m$ is less than the dimension of $\mathcal{C}_m$ and hence the general element Y in $\mathcal{Y}$ has the property that all of its subschemes of length m lie outside of $\mathcal{Z}_m$. Since this is true for all integers m, we can now take such a general element Y in $\mathcal{Y}$ and let $V \subseteq Y$ be a subset of size $N - (c-a)^2$. We can partition V into subsets $V_1 \cup V_2 \cup V_3$, where $|V_1| \geqslant |V_2| \geqslant |V_3| \geqslant |V_1| - 1$. In this way we have a subscheme $U = V \cup W \subseteq \mathbb{P}^2$ such that $U_1 = V_1 \cup W_1$, $U_2 = V_2 \cup W_2$ and $U_3 = V_3 \cup W_3$ all have generic Hilbert functions. In order for this to work, we have to make sure that the sizes of the sets V_1, V_2 and V_3 are all less than r_a. It suffices to show that $N \leqslant 3r_a - 3$.

We can get an upper bound for the number of points N by looking at the value of the Hilbert function $H(c)$ in degree $\alpha(c)$ where we get

$$N \leqslant H(c)_{\alpha(c)} = r_{\alpha(c)} - 2r_{\alpha(c)-c} = \frac{3(c^2-1) + 3\sqrt{2(c^2-1)}}{4} < \frac{3}{4}(c+1)^2,$$

for $c \geqslant 1$. By assumption, $\sqrt{(c^2-1)/2} \leqslant a < c$, so we have that

$$r_a = \frac{a^2+3a+2}{2} > \frac{c^2-1}{4} + \frac{3(c-1)}{2\sqrt{2}} + 1 \geqslant \frac{c^2-1}{4} + (c-1) + 1 = \frac{(c+1)^2 + 2(c-1)}{4},$$

$$r_a = \frac{a^2+3a+2}{2} > \frac{c^2-1}{4} + \frac{3(c-1)}{2\sqrt{2}} + 1 \geqslant \frac{c^2-1}{4} + (c-1) + 1 = \frac{(c+1)^2 + 2(c-1)}{4}.$$

Comparing these two inequalities, we get $N \leqslant 3r_a - 3$, whenever $c \geqslant 3$.

The Hilbert function of the set U is given by $r_i - 2r_{i-c}$ in degrees $i \leqslant \alpha(c)$ and N in degrees $i > \alpha(c)$.

Suppose that an ideal $I \subset R$ satisfies $I_j \neq R_j$. We may construct a "general" Gorenstein ideal $J \supset I$ with R/J Artinian of socle degree j by choosing first a general enough codimension one vector subspace $J_j \subset R_j$ satisfying $J_j \supset I_j$; and then taking J to be the largest ideal satisfying

$$J \cap M^j = (J_j) \cap M^j.$$

Thus, J is the *ancestor ideal* of J_j [I2,IK]. Equivalently, we choose a generic element $w \in (I_j)^\perp \cap \mathcal{R}_j$—that is, w is annihilated by the contraction action of I_j on R_j—and let $J = \{f \in R \mid f \circ w = 0\}$.

Now let $J(1)$, $J(2)$ and $J(3)$ be general Gorenstein ideals containing $I(U_1)$, $I(U_2)$ and $I(U_3)$, respectively, and whose quotients $R/J(1), R/J(2), R/J(3)$ each are Artinian of socle degree $2c-1$. By [IK, Lemma 1.17 or Theorem 4.1A] we have that the Hilbert function of $R/J(\ell)$ is given by

$$\max\{r_i, |U_\ell|, r_{2c-1-i}\}.$$

In degrees where the Hilbert function of U equals N, the coordinate ring of U is a direct sum of the coordinate rings of U_1, U_2 and U_3. Thus the intersection $J = J_1 \cap J_2 \cap J_3$ gives a level algebra R/J whose Hilbert function, in these degrees, is the sum of the Hilbert functions of the Gorenstein quotients R/J_1, R/J_2 and R/J_3. The sum is equal to $3r_{2c-1-i}$ as long as this number is less than or equal N. In degrees at most $\alpha(c)$, the Hilbert function of R/J will equal the Hilbert function of U, i.e., $r_i - 2r_{i-c}$ since the initial degrees of the three Gorenstein ideals in the coordinate rings of the parts are higher than $\alpha(c)$. Thus we have shown that the Hilbert function of R/J is $H(c)$ and hence the general level quotient has Hilbert function $H(c)$. $\square$

After having established that there are level algebras with Hilbert function $H(c)$ with different degrees of the common divisor in degree c, we can now state the main theorem of this section.

Theorem 2.10. *For $a = 0$ and $\sqrt{(c^2-1)/2} \leqslant a < c$, the family $\mathcal{F}_a$ of all level algebras in* $\mathrm{LevAlg}(H(c))$ *whose degree c part lies in $G(c)_a$ is a nonempty open set in a component of* $\mathrm{LevAlg}(H(c))$*. The dimension of this component is $4c^2 + 3c - 11$ for $a = 0$ and $4c^2 + 3c - 12 + (c-a)^2$ otherwise.* $\mathrm{LevAlg}(H(c))$ *is reducible for $c \geqslant 3$.*

Proof. Consider the map

$$\Phi : \mathrm{LevAlg}\big(H(c)\big) \to \mathrm{Grass}(2, R_c)$$

given by sending the level ideal I to its degree c component I_c.

By Lemma 2.9 we know that $\mathcal{F}_a$ is nonempty, since Φ has a nonempty fiber over the general point of $G(c)_a$. The general point in $\mathrm{Grass}(2, R_c)$ lies in the image and we conclude that Φ is dominant. There has to be at least one component of $\mathrm{LevAlg}(H(c))$ dominating the image, and since the general fiber of Φ is irreducible of dimension $3(c^2-3)$, we conclude that there is a single such component, $\overline{\mathcal{F}}_0$, of dimension

$$2r_c - 4 + 3\big(c^2 - 3\big) = 4c^2 + 3c - 11.$$

Let $X(c)$ be the variety defined as the complement of G_0 in $\mathrm{Grass}(2, R_c)$. Lemma 2.8 gives us the dimensions of the components, $\overline{G}(c)_a$, of $X(c)$. Now we know from Lemma 2.9 that for $\sqrt{(c^2-1)/2} \leqslant a < c$, the fiber over the general point of $G(c)_a$ is nonempty. Thus we know that there has to be a component of $\mathrm{LevAlg}(H(c))$ dominating the component $\overline{G}(c)_a$ of $X(c)$. The fiber of Φ over the general point in $G(c)_a$ is irreducible of dimension $r_{2c-a} - 2r_{c-a} + 3c^2 - 9$ since we have to add a form of degree $2c - a$ to the ideal generated by the two forms of degree c due to their common factor of degree a. Observe that since $a \geqslant \sqrt{(c^2-1)/2}$ we have that $2c - a \leqslant \alpha(c)$ and the form of degree $2c - a$ is determined up to scalar multiples by the level ideal. Thus the component, $\overline{\mathcal{F}_a}$, dominating $\overline{G}(c)_a$ is irreducible of dimension

$$2r_{c-a} + r_a - 5 + r_{2c-a} - 2r_{c-a} + 3c^2 - 9 = r_a + r_{2c-a} + 3c^2 - 14 = 4c^2 + 3c - 12 + (c-a)^2.$$

Since this is greater than or equal to the dimension of the component spanned by $\mathcal{F}_0$, we have that $\mathcal{F}_a$ cannot be contained in the closure of $\mathcal{F}_0$, and we have shown that they are different components. For different $a \neq 0$, the components spanned by $\mathcal{F}_a$ are different since they map onto different components of $X(c)$. □

Remark 2.11. The first few Hilbert functions in the series $H(c)$ are

$$H(3) = (1, 3, 6, 8, 9, 3),$$
$$H(4) = (1, 3, 6, 10, 13, 15, 9, 3),$$
$$H(5) = (1, 3, 6, 10, 15, 19, 22, 18, 9, 3),$$
$$H(6) = (1, 3, 6, 10, 15, 21, 26, 30, 30, 18, 9, 3),$$
$$H(7) = (1, 3, 6, 10, 15, 21, 28, 34, 39, 43, 30, 18, 9, 3),$$

and the first time we get more than two components is for $c = 7$. The number of components given by Theorem 2.10 is $c + 1 - \lceil\sqrt{(c^2-1)/2}\rceil$, which is bounded from below by $(1 - 1/\sqrt{2})c$.

The values of c for which the degree $\alpha(c) = 2c - \sqrt{(c^2-1)/2}$ is an integer correspond to solutions of Pell's equation, $c^2 - 2d^2 = 1$, which in turn are related to the continued fraction expansion of $\sqrt{2}$, namely $2 = (1, 2, 2, \ldots)$. Considering the approximants p_k/q_k, and letting $c = p_k, d = q_k$ for k odd, one obtains all such solutions. The first five are $(p_k, q_k) = (3, 2), (17, 12), (99, 70), (577, 408), (3383, 2378)$, for $k = 1, 3, 5, 7, 9$. (See [Ro, Theorem 13.11].)

It would be interesting to know whether the infinite series of Artinian examples of Theorem 2.10 lifts to an infinite series of examples with level set of points in a similar way as in the first example in the series. This question is still open and the problem is that we are lacking general results that guarantee that the Hilbert function of a general enough level set of points on a given curve has the expected Hilbert function.

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References

[BiGe] A. Bigatti, A. Geramita, Level algebras, lex segments, and minimal Hilbert functions, Comm. Algebra 33 (3) (2003) 1427–1451.

[Bj] M. Boij, Components of the space parametrizing graded Gorenstein Artin algebras with a given Hilbert function, Pacific J. Math. 187 (1999) 1–11.

[BjZa] M. Boij, F. Zanello, Level algebras with bad properties, Proc. Amer. Math. Soc. 135 (9) (2007) 2713–2722.

[BoG] M. Boratynski, S. Greco, Hilbert functions and Betti numbers in a flat family, Ann. Mat. Pura Appl. (4) 142 (1985) 277–292 (1986).

[BriV] M. Brignone, G. Valla, On the resolution of certain level algebras, Comm. Algebra 32 (11) (2004) 4221–4245.

[BrH] W. Bruns, J. Herzog, Cohen–Macaulay Rings, Cambridge Stud. Adv. Math., vol. 39, Cambridge Univ. Press, Cambridge, UK, 1993, revised paperback edition, 1998.

[BuE] D. Buchsbaum, D. Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for codimension three, Amer. J. Math. 99 (1977) 447–485.

[ChGe] J. Chipalkatti, A. Geramita, On parameter spaces for Artin level algebras, Michigan Math. J. 51 (1) (2003) 187–207.

[ChoI] Y. Cho, A. Iarrobino, Hilbert functions of level algebras, J. Algebra 241 (2001) 745–758.

[Di] S.J. Diesel, Some irreducibility and dimension theorems for families of height 3 Gorenstein algebras, Pacific J. Math. 172 (1996) 365–397, 77–1585.

[GHMS] A. Geramita, T. Harima, J. Migliore, Y.S. Shin, The Hilbert Function of a Level Algebra, Mem. Amer. Math. Soc., vol. 186, Amer. Math. Soc., Providence, RI, 2007, 139 p.

[Gotz] G. Gotzmann, Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes, Math. Z. 158 (1) (1978) 61–70.

[GrSt] D. Grayson, M. Stillman, Macaulay 2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/.

[HarW] T. Harima, J. Watanabe, The central simple modules of Artinian Gorenstein algebras, J. Pure Appl. Algebra 210 (2) (2007) 447–463.

[Ha] J. Harris, Curves in Projective Space, Semin. Math. Supérieures, vol. 85, Presses de l'Université de Montréal, 1982, 138 pp.

[Hu] C. Huneke, Hyman Bass and ubiquity: Gorenstein rings, in: Algebra, K-Theory, Groups, and Education, New York, 1997, in: Contemp. Math., vol. 243, Amer. Math. Soc., Providence, RI, 1999, pp. 55–78.

[I1] A. Iarrobino, Compressed algebras: Artin algebras having given socle degrees and maximal length, Trans. Amer. Math. Soc. 285 (1984) 337–378.

[I2] A. Iarrobino, Ancestor ideals of vector spaces of forms, and level algebras, J. Algebra 272 (2004) 530–580.

[I3] A. Iarrobino, Betti strata of height two ideals, J. Algebra 285 (2) (2005) 835–855.

[IK] A. Iarrobino, V. Kanev (Eds.), Power Sums, Gorenstein Algebras, and Determinantal Loci, Lecture Notes in Math., vol. 1721, Springer, Heidelberg, 1999, 345+xxvii pp.

[IKl] A. Iarrobino, S.L. Kleiman, The Gotzmann theorems and the Hilbert scheme, in: A. Iarrobino, V. Kanev (Eds.), Power Sums, Gorenstein Algebras, and Determinantal Loci, in: Lecture Notes in Math., vol. 1721, Springer, Heidelberg, 1999, pp. 289–312, Appendix C.

[IS] A. Iarrobino, H. Srinivasan, Artinian Gorenstein algebras of embedding dimension four: Components of PGOR(H) for $H = (1, 4, 7, \dots, 1)$, J. Pure Appl. Algebra 201 (2005) 62–96.
[Kle1] J.O. Kleppe, Deformations of graded algebras, Math. Scand. 45 (1979) 205–231.
[Kle2] J.O. Kleppe, The smoothness and the dimension of PGOR(H) and of other strata of the punctual Hilbert scheme, J. Algebra 200 (1998) 606–628.
[Kle3] J.O. Kleppe, Maximal families of Gorenstein algebras, Trans. Amer. Math. Soc. 358 (7) (2006) 3133–3167.
[Kle4] J.O. Kleppe, Families of Artinian and one-dimensional algebras, J. Algebra 311 (2) (2007) 665–701.
[Mac1] F.H.S. Macaulay, The Algebra of Modular Systems, Cambridge Univ. Press, Cambridge, UK, 1916, reprinted with a foreword by P. Roberts, Cambridge Univ. Press.
[Mac2] F.H.S. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. London Math. Soc. 26 (1927) 531–555.
[M1] J. Migliore, Families of reduced zero-dimensional schemes, Collect. Math. 57 (2) (2006) 173–192.
[M2] J. Migliore, The Geometry of Hilbert functions, in: I. Peeva (Ed.), Syzygies and Hilbert functions, in: Lect. Notes Pure Appl. Math., vol. 254, CRC Press, 2007, pp. 179–208.
[M3] J. Migliore, The geometry of the weak Lefschetz property, and level sets of points, Canad. J. Math. 60 (2) (2008) 391–411.
[MMi] J. Migliore, R. Miró-Roig, Ideals of general forms and the ubiquity of the weak Lefschetz property, J. Pure Appl. Algebra 182 (1) (2003) 79–107.
[Par] K. Pardue, Deformation classes of graded modules and maximal Betti numbers, Illinois J. Math. 40 (1996) 564–585.
[Pe] I. Peeva, Consecutive cancellation in Betti numbers, Proc. Amer. Math. Soc. 132 (2) (2004) 3503–3507.
[RZ] A. Ragusa, G. Zappalá, On the reducibility of the postulation Hilbert scheme, Rend. Circ. Mat. Palermo (2) 53 (3) (2004) 401–406.
[Ro] K. Rosen, Elementary Number Theory and Its Applications, fourth ed., Addison–Wesley–Longman, Reading, MA, 2000.
[Wei] A. Weiss, Some new nonunimodal level algebras, PhD thesis, Tufts Univ., 2006, arXiv: 0708.3354 [math.AC].
[Za1] F. Zanello, Level algebras of type 2, Comm. Algebra 34 (2) (2006) 691–714.
[Za2] F. Zanello, A non-unimodal codimension 3 level h-vector, J. Algebra 305 (2) (2006) 949–956.