Optimal Harvesting for a Nonlinear Age-Dependent Population Dynamics

Sebastian Anița

Faculty of Mathematics, University “Al. I. Cuza,” Iași 6600, Romania

Submitted by Mimmo Iannelli
Received March 28, 1997

Here we investigate an optimal harvesting problem for a nonlinear age-dependent population dynamics. Existence and uniqueness of a positive solution for the model are demonstrated. The structure of the solution is also investigated. We establish the existence of the optimal control and the convergence of a certain fractional step scheme. For some approximating problems we obtain the optimal controllers in feedback form via the dynamic programming method.

1. INTRODUCTION AND SETTING OF THE PROBLEM

We consider the following nonlinear population dynamics model,

\[
\begin{align*}
 p_t + p_a + \mu(a)p + \Phi(P(t))p &= -u(t)p, \quad (a, t) \in Q, \\
p(0, t) &= \int_0^{a_t} \beta(a)p(a, t)da, \quad t \in (0, T), \\
p(a, 0) &= p_0(a), \quad a \in (0, a_t) \\
P(t) &= \int_0^{a_t} p(a, t)da, \quad t \in (0, T),
\end{align*}
\]

where \( Q = (0, a_t) \times (0, T) \ (a_t, T \in (0, +\infty)). \)

The model (1) is describing the evolution of an age-structured population subject to harvesting. Namely, in (1) \( p(a, t) \) is the population density of age \( a \) at the moment \( t \), \( \beta(a) \) and \( \mu(a) \) are the vital rates (and are depending only on age), and \( u(t) \) is the harvesting rate.

* E-mail address: sanita@uaic.ro.

0022-247X /98 $25.00
Copyright © 1998 by Academic Press
All rights of reproduction in any form reserved.
Note that $P(t)$ is the total population, so that in (1) the term $\Phi(P(t))$ stands for an external mortality rate which does not depend on age (see [11]).

Throughout this article we suppose that the following assumptions hold

\[
\beta \in L^r(0, a_1), \quad \beta(a) \geq 0, \quad \text{a.e. in } (0, a_1), \quad (A1)
\]

\[
\mu \in L^1_{loc}([0, a_1]), \quad \mu(a) \geq 0, \quad \text{a.e. in } (0, a_1),
\]

\[
\int_0^{a_1} \mu(a) \, da = +\infty, \quad (A2)
\]

\[
\Phi : [0, +\infty) \to [0, +\infty) \text{ is continuously differentiable}, \quad (A3)
\]

and the initial density $p_0$ satisfies

\[
p_0 \in L^r(0, a_1), \quad p_0(a) > 0, \quad \text{a.e. in } (0, a_1). \quad (A4)
\]

For more details concerning the biological significance of the terms in (1) and the meaning of the hypotheses we refer to [14, 16]. Concerning the harvesting rate $u$ we suppose that it belongs to the following set of controllers,

\[
\mathscr{U} = \{ u \in L^r(0, T) ; 0 \leq u(t) \leq L \text{ a.e. in } (0, T) \},
\]

($L \in (0, +\infty)$) so that denoting by $p^u$ the solution of (1) we consider the problem,

\[
\text{Maximize } \int_0^T \int_0^{a_1} u(t) g(a) p^u(a, t) \, da \, dt, \quad (P_u)
\]

subject to $u \in \mathscr{U}$.

Here $g$ satisfies the assumption,

\[
g \in L^1(0, a_1), \quad g(a) > 0, \quad \text{a.e. in } (0, a_1). \quad (A5)
\]

We recall that problem (1) belongs to the class of "separable" models considered in [8]. Actually here we are dealing with a slightly different situation because of the harvesting term $u(t)p(a, t)$, where $u$ belongs to $\mathscr{U}$. However, by a similar procedure as in [8] we can get a solution of (1) under the separated form

\[
p(a, t) = y(t) \tilde{p}(a, t), \quad (2)
\]
where $\tilde{p}$ is the solution of the “free” problem,

$$
\begin{cases}
\tilde{p}_t + \tilde{p}_a + \mu(a)\tilde{p} = 0, & (a, t) \in Q, \\
\tilde{p}(0, t) = \int_0^a \beta(a)\tilde{p}(a, t) \, da, & t \in (0, T), \\
\tilde{p}(a, 0) = p_0(a), & a \in (0, a_1),
\end{cases}
$$

(3)

that we know has a unique (and strictly positive) solution in the following sense,

$$
\tilde{p} \in L^\infty(Q) \quad \text{and} \\
\begin{cases}
D\tilde{p}(a, t) = -\mu(a)\tilde{p}(a, t), & \text{a.e. in } Q, \\
\lim_{h \to 0^+} \tilde{p}(h, t) = \int_0^a \beta(a)\tilde{p}(a, t) \, da, & \text{a.e. in } t \in (0, T), \\
\tilde{p}(a, 0) = p_0(a), & \text{a.e. in } (0, a_1),
\end{cases}
$$

(4)

(by $D\tilde{p}$ we denote the directional derivative,

$$
D\tilde{p}(a, t) = \lim_{h \to 0^+} \frac{1}{h} \left[ \tilde{p}(a + h, t) - \tilde{p}(a, t) \right].
$$

Note that by (4), a solution of $\tilde{p}$ of (3) must be an absolutely continuous function almost on every line of equation $a - t = k, (a, t) \in Q, k \in \mathbb{R}$, so that (4)$_2$ is meaningful.

Actually, plugging (2) into (1) we get the following conditions on $y$,

$$
[y'(t) + \Phi(P_0(t)y(t))y(t) + u(t)y(t)]\tilde{p}(a, t) = 0, \quad \text{a.e. } (a, t) \in Q, \\
y(0) = 1,
$$

so that we reduce to the following problem,

$$
\begin{cases}
y'(t) + \Phi(P_0(t)y(t))y(t) + u(t)y(t) = 0, & \text{a.e. } t \in (0, T), \\
y(0) = 1,
\end{cases}
$$

(5)

where $P_0(t) = \int_0^a \tilde{p}(a, t) \, da, t \in [0, T]$ is known. Moreover, denoting by $y^u$ the solution of (5) problem (P$_0$) is reduced to the following one on $y^u$,

$$
\text{Maximize } \int_0^T m(t)u(t)y^u(t) \, dt,
$$

subject to $u \in \mathcal{U}$, where $m(t) = \int_0^a g(a)\tilde{p}(a, t) \, da, t \in [0, T]$ is known (the assumption on $g$ implies that $m \in L^\infty(0, T), m(t) > 0$ a.e. in $(0, T)$).
In conclusion, our study of problem (P_0)-(1) can be reduced to (P)-(5) via the representation formula (2), because

$$\int_0^T \int_0^{a_t} u(t) g(a) p^u(a, t) \, da \, dt = \int_0^T m(t) y^u(t) \, dt,$$

thus our attention is focused on this one and any result in this article can be easily translated into a result for the original problem.

We notice that (P)-(5) actually depends on the initial datum p_0(a) via the term P_0(t).

We mention that the optimal harvesting problem for other age structured population with some special assumptions on the structure of the problem is previously studied in [7, 12, 13, 15]. Our treatment is however technically different and allows us to indicate an algorithm in order to approximate an optimal control.

A precise description of the optimal harvesting effort for periodic linear age-dependent population dynamics was obtained in [2].

The article is organized as follows. Section 2 is devoted to the existence and uniqueness of the solutions for (1) and (5), respectively. In Section 3 we obtain the existence of an optimal control for (P). Section 4 concerns a fractional step scheme for problem (P) and finally in Section 5 we obtain the dynamic programming equations and the optimal controllers in a feedback form for some approximating problems.

### 2. THE ANALYSIS OF SYSTEM (1)

This section is devoted to the study of existence, uniqueness, and positivity of solution to system (1), assuming that u ∈ 𝒰 is fixed.

By a solution to (1) we mean a function p ∈ L^∞(Q) such that:

$$\begin{align*}
Dp(a_t, t) + μ(a)p(a, t) + Φ(P(t))p(a, t) &= -u(t)p(a, t), \\
&\quad \text{a.e. in } Q,

\lim_{h \to 0} p(h, t + h) &= \int_0^{a_t} β(a)p(a, t) \, da, \\
&\quad \text{a.e. } t ∈ (0, T),

p(a, 0) &= p_0(a), \quad \text{a.e. in } (0, a_t),
\end{align*}$$

(6)

where P(t) = ∫_0^t p(a, t) \, da, a.e. t ∈ (0, T).
It is known that system (3) has an unique solution, which is strictly positive, a.e. in $Q$ because $p_0$ is (see [14]). Denote this solution by $\tilde{p}$ and by

$$P_0(t) = \int_0^t \tilde{p}(a, t) \, da, \quad a.e. \, t \in (0, T).$$

It is obvious that $P_0 \in L^\infty(0, T)$ and $P_0(t) > 0$ a.e. in $(0, T)$.

Consider the following problem,

$$\begin{cases}
y'(t) + \Phi(P_0(t) y(t)) y(t) = -u(t) y(t), & t \in (0, T), \\
y(0) = y_0 \in (0, +\infty). 
\end{cases}$$

(7)

Actually we have

**Lemma 2.1.** System (7) has an unique Carathéodory solution.

**Proof.** Denote by $\tilde{y}$ the Carathéodory solution of

$$\begin{cases}
y'(t) = -u(t) y(t), & t \in (0, T), \\
y(0) = y_0,
\end{cases}$$

(8)

and

$$C = \{ v \in L^\infty(0, T); \, 0 \leq v(t) \leq \tilde{y}(t) \, a.e. \, t \in (0, T) \}.$$ 

It is easy to prove via a Banach fixed point theorem that $\mathcal{F}: C \to C$ defined by $\mathcal{F} h = y_h$, where $y_h$ is the Carathéodory solution of

$$\begin{cases}
y'(t) + \Phi(P_0(t) h(t)) y(t) = -u(t) y(t), & t \in (0, T), \\
y(0) = y_0,
\end{cases}$$

has an unique fixed point. So, we may conclude that (7) has an unique Carathéodory solution.

Let us now prove a result which gives the structure of a certain solution for (1).

**Theorem 2.2.** The function $p \in L^\infty(Q)$ given by

$$p(a, t) = y(t) \tilde{p}(a, t), \quad a.e. (a, t) \in Q,$$

(9)

where $y$ is the Carathéodory solution of (7) with $y_0 := 1$, is a solution of (1).

**Proof.** Because (3) has an unique solution, which is strictly positive a.e. in $Q$, we may conclude via Lemma 2.1 that $p$ given by (9) satisfies $p \in L^\infty(Q)$, $p(a, t) > 0$ a.e. in $Q$ and (6) hold.
Now using the same argument as in [14-III] we finally obtain the next

**Theorem 2.3.** System (1) has an unique solution.

Notice that Theorems 2.2 and 2.3 imply the strictly positivity of the solution of (1).

3. THE EXISTENCE OF THE OPTIMAL HARVESTING EFFORT

We study here the existence of the optimal control for the optimal harvesting problem,

\[
\text{Maximize } \int_0^T m(t) u(t) y''(t) \, dt, \quad (P)
\]

subject to \( u \in \mathcal{U} \) and \( y'' \) is the Carathéodory solution of (7).

The main result in this section is Theorem 3.1.

**Theorem 3.1.** There exists at least one optimal control for (P)-(7).

We use as the main ingredient the following

**Lemma 3.2.** If \( \{ u_n \} \subset \mathcal{U} \) satisfies \( u_n \rightharpoonup u \) weakly in \( L^2(0, T) \), then

\[
y^{u_n} \to y^u, \quad \text{in } L^2(0, T).
\]

**Proof of Lemma 3.2.** Because \( y^{u_n} \) is the Carathéodory solution of (7) with \( u := u_n \) we infer that

\[
y^{u_n}(t) = \exp \left[ - \int_0^t \left( u_n(s) + \Phi(P_0(s) y^{u_n}(s)) \right) \, ds \right] \cdot y_0,
\]

for any \( t \in [0, T] \) and this implies

\[0 \leq y^{u_n}(t) \leq \tilde{y}(t), \quad \text{for any } t \in [0, T],\]

(\( \tilde{y} \) is defined by (8)). The last relation allows us to conclude that the sequence \( \{ v_n \} \) given by

\[v_n(t) = \Phi(P_0(t) y^{u_n}(t)), \quad \text{a.e. } t \in (0, T),\]

satisfies

\[0 \leq v_n(t) \leq M, \quad \text{a.e. } t \in (0, T),\]
where \( M \in (0, +\infty) \) is a constant. On a subsequence (also denoted by \((v_n)\)) we have
\[
v_n \rightharpoonup v \text{ weakly in } L^2(0, T).
\] (10)
Because \( u_n \rightharpoonup u \text{ weakly in } L^2(0, T) \), we conclude that
\[
y''_n \rightharpoonup \tilde{y}^u, \quad \text{in } L^2(0, T),
\]
where \( \tilde{y}^u \) is the Carathéodory solution to
\[
\begin{cases}
y'(t) + v(t)y(t) = -u(t)y(t), & t \in (0, T), \\
y(0) = y_0.
\end{cases}
\]
From (10) we get
\[
v_n(\cdot) = \Phi(P_0(\cdot) u''_n(\cdot)) \rightharpoonup \Phi(P_0(\cdot) \tilde{y}^u(\cdot)), \quad \text{in } L^2(0, T).
\]
So, the conclusion is that \( v(\cdot) = \Phi(P_0(\cdot) \tilde{y}^u(\cdot)) \) and \( \tilde{y}^u \equiv y^u \).

**Proof of Theorem 3.1.** Consider now
\[
d = \sup_{u \in \mathcal{U}} \int_0^T m(t)u(t)y''_n(t) \, dt.
\]
It is obvious that \( d \in [0, +\infty) \). Let \( u_n \in \mathcal{U} \) be such that
\[
d - \frac{1}{n} < \int_0^T m(t)u_n(t)y''_n(t) \, dt \leq d.
\]
However, as a consequence we obtain that on a subsequence (also denoted by \((u_n)\)) we have
\[
u_n \rightharpoonup u^* \text{ weakly in } L^2(0, T).
\] (11)
By Lemma 3.2 we obtain that
\[
y''^u \rightharpoonup y''^u, \quad \text{in } L^2(0, T),
\]
and consequently,
\[
my''_n \rightharpoonup my''^u, \quad \text{in } L^2(0, T),
\]
(because \( m \in L^\infty(0, T) \)) and so
\[
\int_0^T m(t)u_n(t)y''_n(t) \, dt \to \int_0^T m(t)u^*(t)y''^u(t) \, dt,
\] (12)
and (from (11) and (12)),
\[ d = \int_0^T m(t)u^*(t)y^{u^*}(t) \, dt. \]

We may infer now that \((u^*, y^{u^*})\) is an optimal pair for \((P)-(7)\).

4. AN APPROXIMATING SCHEME

The main result of this section amounts to saying that problem \((P)-(7)\) can be approximated for \(\varepsilon \to 0^+\) by the following sequence of optimal control problems,

\[
\text{Maximize } \int_0^T m(t)u(t)y^{u}_{\varepsilon}(t) \, dt, \quad (P_{\varepsilon})
\]

subject to \(u \in \mathcal{U}\) and \(y^{u}_{\varepsilon}\) the Carathéodory solution of

\[
\begin{cases}
  y'(t) + \gamma(t)y(t) = -u(t)y(t), & t \in (i\varepsilon, (i + 1)\varepsilon), \\
  y(i\varepsilon) = \theta((i + 1)\varepsilon - i\varepsilon, y(y(i\varepsilon))), & i = 0, 1, \ldots, N - 1, \varepsilon = T/N \\
  y(0-) = y_0 \in (0, +\infty),
\end{cases}
\]

where \(\theta(t; i\varepsilon, x)\) is the Carathéodory solution of

\[
\begin{cases}
  \theta'(t) + \Phi(P_0(t)\theta(t))\theta(t) = \gamma(t)\theta(t), & t \in (i\varepsilon, (i + 1)\varepsilon), \\
  \theta(i\varepsilon) = x.
\end{cases}
\]

Problem \((P_{\varepsilon})\) is of course much simpler than \((P)-(7)\), because we have “split” (7) into two simpler problems: (13) and (14).

Here we assume in addition that

\[
\gamma \in C([0, T]).
\]

For other results concerning some fractional step schemes we refer to [1, 4–6].

Using an analogous argument as in the previous section it is possible to prove that \((P_{\varepsilon})\) has at least one optimal pair.

We establish for the beginning the following technical result.

**Lemma 4.1.** If \(u_{\varepsilon} \rightharpoonup u \text{ weakly in } L^2(0, T)\) for \(\varepsilon \to 0^+\) \((u_{\varepsilon} \in \mathcal{U})\), then

\[ y^{u}_{\varepsilon} \rightharpoonup y^u, \quad \text{in } \mathcal{BV}([0, T]), \]

for \(\varepsilon \to 0^+\).
Proof. From (13) and (14) we obtain that for any $i \varepsilon < t < (i + 1) \varepsilon$ we have

$$y_{\varepsilon}^{u}(t) = \exp \left[ - \int_{i \varepsilon}^{t} (u_{\varepsilon} + \gamma)(s) \, ds \right] \cdot \theta((i + 1) \varepsilon - ; i \varepsilon, y_{\varepsilon}^{u}(i \varepsilon -))$$

$$= \exp \left[ - \int_{i \varepsilon}^{t} (u_{\varepsilon} + \gamma)(s) \, ds + \int_{i \varepsilon}^{(i+1) \varepsilon} (\gamma - v_{\varepsilon})(s) \, ds \right] y_{\varepsilon}^{u}(i \varepsilon -) \, ds,$$

(15)

where

$$v_{\varepsilon}(s) = \Phi(P_{0}(s) \theta(s; i \varepsilon, y_{\varepsilon}^{u}(i \varepsilon -))), \text{ a.e. } s \in (0, T).$$

From (14) and using the positivity of $v_{\varepsilon}$, we get the boundedness of the sequence $(v_{\varepsilon})$ in $L^{0}(0, T)$. It is obvious now that (15) and the positivity of $v_{\varepsilon}$ imply the existence of a constant $M \in (0, +\infty)$ such that

$$|y_{\varepsilon}^{u}(t)| \leq M, \text{ for all } t \in [0, T].$$

From (13), we also get

$$\int_{0}^{T} |(y_{\varepsilon}^{u}(t))' dt = \int_{0}^{T} |\gamma(t) + u(t)| \cdot |y_{\varepsilon}^{u}(t)| dt$$

$$\leq M \cdot T \cdot (L + \|\gamma\|_{C[0,T]}),$$

(16)

and

$$|y_{\varepsilon}^{u}(i \varepsilon +) - y_{\varepsilon}^{u}(i \varepsilon -)|$$

$$= \left| \exp \left[ \int_{i \varepsilon}^{(i+1) \varepsilon} (\gamma - v_{\varepsilon})(s) \, ds \right] - 1 \right| \cdot |y_{\varepsilon}^{u}(i \varepsilon -)|$$

$$\leq M \cdot \int_{i \varepsilon}^{(i+1) \varepsilon} (\gamma(s) - v_{\varepsilon}(s)) \, ds.$$  (17)

By (17) we have

$$\sum_{i=0}^{N} |y_{\varepsilon}^{u}(i \varepsilon +) - y_{\varepsilon}^{u}(i \varepsilon -)| \leq M_{1},$$

(18)

(where $M_{1} \in (0, +\infty)$ is a constant) and now (16) and (18) imply that the total variation of $y_{\varepsilon}^{u}$ on $[0, T]$ is bounded by the same constant (indepen-
dent of ε). We conclude via Helly’s theorem that on a subsequence (also denoted by \((y^n)\)) we have
\[
y^n \to y, \quad \text{in } BV([0, T]),
\]
and consequently,
\[
y^n(t) \to y(t), \quad \text{for all } t \in [0, T],
\]
and
\[
y^n \to y, \quad \text{in } L^2(0, T).
\]
Let us prove that \(y^n = y\). Indeed, from (15) we obtain that for any \(i \varepsilon \leq t < (i + 1) \varepsilon\) we have
\[
y^n(t) = \exp\left(- \int_0^t u_n(s) \, ds - \int_0^{(i+1) \varepsilon} v_n(s) \, ds - \int_{i \varepsilon}^t \gamma(s) \, ds\right) \cdot y_0. \tag{19}
\]
Because
\[
\exp\left[- \int_0^t u_n(s) \, ds\right] \to \exp\left[- \int_0^t u(s) \, ds\right], \quad \text{in } C([0, T]),
\]
\[
\exp\left[\int_{i \varepsilon}^t \gamma(s) \, ds\right] \to 1, \quad \text{in } BV([0, T]),
\]
and
\[
\exp\left[- \int_0^{(i+1) \varepsilon} v_n(s) \, ds\right] \to \exp\left[- \int_0^{(i+1) \varepsilon} v(s) \, ds\right], \quad \text{in } BV([0, T]),
\]
(where, on a subsequence also denoted by \((v_n)\) we have \(v_n \to \tilde{v}\) weakly in \(L^2(0, T)\)) we obtain by passing to the limit in (19) that \(y\) is the Carathéodory solution of
\[
\begin{cases}
y'(t) + \tilde{v}y(t) = -u(t)y(t), & t \in (0, T), \\
y(0) = y_0.
\end{cases} \tag{20}
\]
The convergence
\[
v_n(t) = \Phi(P_0(t) y^n(t)) \to \Phi(P_0(t) y(t)), \quad \text{a.e. in } (0, T),
\]
for $\varepsilon \to 0^+$ implies
\[
\tilde{v}(t) = \Phi(P_0(t)y(t)), \quad \text{a.e. in } (0,T),
\]
and from (20) we get that $y^\varepsilon u = y$.

The proof of Lemma 4.1 is now complete.

Consider $\varphi, \varphi_\varepsilon: \mathcal{U} \to [0, +\infty)$ defined by
\[
\varphi(u) = \int_0^T m(t)u(t)y^u(t) \, dt,
\]
and
\[
\varphi_\varepsilon(u) = \int_0^T m(t)u(t)y^\varepsilon u(t) \, dt,
\]
and $u^\varepsilon u$ an optimal control for $\langle P_\varepsilon \rangle$.

We conclude this section with the following main result.

**Theorem 4.2.** If $u^\varepsilon$ is a weak limit point of $\{u^\varepsilon\}$ in $L^2(0,T)$, then $u^\varepsilon$ is an optimal control for $\langle P \rangle$ and in addition,
\[
\lim_{\varepsilon \to 0^+} \varphi(u^\varepsilon) = \varphi(u^\ast), \tag{21}
\]
and
\[
\lim_{\varepsilon \to 0^+} \varphi_\varepsilon(u^\varepsilon) = \varphi(u^\ast). \tag{22}
\]

**Proof.** Because
\[
\varphi_\varepsilon(u^\varepsilon) = \int_0^T m(t)u^\varepsilon u(t)y^\varepsilon u(t) \, dt \geq \int_0^T m(t)u(t)y^u(t) \, dt,
\]
for any $u \in \mathcal{U}$ and using the previous lemma we conclude that
\[
\int_0^T m(t)u^\ast(t)y^\varepsilon u(t) \, dt \geq \int_0^T m(t)u(t)y^u(t) \, dt,
\]
for any $u \in \mathcal{U}$. This implies that $u^\varepsilon$ is an optimal control for $\langle P \rangle$.

Now, because the following convergences hold
\[
u^\varepsilon u \to u^\ast \text{ weakly in } L^2(0,T),
\]
and
\[
y^\varepsilon u \to y^u, \quad \text{in } L^2(0,T),
\]
we may conclude that (22) holds.
Now using the convergence,

\[ y^{u^n} \rightarrow y^u, \quad \text{in } L^2(0,T), \]

(see Section 3) we obtain relation (21).

5. THE DYNAMIC PROGRAMMING METHOD

Consider the following problems,

\[
\begin{align*}
\text{Maximize} & \quad \int_t^T m(s)u(s)y^u(s; t, x) \, ds, \\
\text{subject to} & \quad u \in \mathcal{U} = \{ v \in L^r(t, T); \ 0 \leq v(s) \leq L \ \text{a.e.} \} \quad \text{and} \quad y^u \quad \text{the Carathéodory solution of} \\
\left\{ \begin{array}{l}
y'(t) + \gamma(t)y(t) = -u(t)y(t), \\
y(i\epsilon +) = \theta((i + 1)\epsilon - i\epsilon, y^u(i\epsilon - t, x)), \\
y(t^-) = x.
\end{array} \right.
\end{align*}
\]

Here \( \tau(t, x) \in [0, T] \times (0, +\infty) \) and \( \theta(s; t, x) \) is the Carathéodory solution of

\[
\begin{align*}
\left\{ \begin{array}{l}
\theta'(s) + \Phi(P_0(s)\theta(s))\theta(s) = \gamma(s)\theta(s), \\
\theta(t^+) = x.
\end{array} \right.
\end{align*}
\]

However, as in the third section the existence of an optimal pair \( (u^\ast; \cdot; t, x), y^\ast(\cdot; t, x)) \) for \( (P_{\varepsilon, 1}) \) follows.

Suppose that in addition the following assumption holds

\[
m \in C^1([0, T]),
\]

(which is fulfilled under certain additional assumptions on \( p_0 \); for example, if \( p_0 \in W^{1, 2}(0, a) \) and \( p_0(0) = \int_0^a \beta(a) p_0(a) \, da \)—see [14]) and

\[
\gamma - \frac{m'}{m} \text{ cannot be a constant on a subset of positive measure},
\]

(\( \gamma \) is chosen in order to satisfy this condition and (A 6)).
Denote by

$$\psi^\varepsilon(t, x) = \max_{u \in \mathcal{U}_t} \int_t^T m(s)u(s)y_\varepsilon^u(s; t, x) \, ds.$$  

It is obvious that for all $0 \leq t \leq s \leq T$ and $x \in (0, +\infty)$, $\psi^\varepsilon(t, x)$ is the optimal value for

$$\text{Maximize} \int_t^s m(\tau)u(\tau)y_\varepsilon^u(\tau; t, x) \, d\tau + \psi^\varepsilon(s, y_\varepsilon^u(s; t, x)), \quad (P_{\varepsilon, t, s})$$

subject to $u \in L^\infty(t, s), \ 0 \leq u(\tau) \leq L$ a.e. (and $y_\varepsilon^u$ is the Carathéodory solution of (23) on $[t, s]$) and $(u_\varepsilon \mid_{[t, s]}, y_\varepsilon \mid_{[t, s]})$ is an optimal pair for this last maximization problem.

It is also clear that

$$\psi^\varepsilon(i \varepsilon - , x) = \psi^\varepsilon(i \varepsilon +, \theta((i + 1) \varepsilon - ; i \varepsilon, x)), \quad i = 0, 1, \ldots, N - 1,$$

and $\psi^\varepsilon$ is locally Lipschitz on $(i \varepsilon, (i + 1) \varepsilon) \times (0, +\infty)$, $i = 0, 1, \ldots, N - 1$ (see, e.g., [5]).

In virtue of Rademacher’s theorem, the function $\psi^\varepsilon$ is almost everywhere derivable on $[0, T] \times (0, +\infty)$.

The optimality conditions for $(P_{\varepsilon, t, (i + 1) \varepsilon})$ (where $i \varepsilon < t < (i + 1) \varepsilon$) are given by

**Theorem 5.1.** There exists an absolutely continuous function $q$ on $[t, (i + 1) \varepsilon)$, which is a Carathéodory solution to

$$\begin{cases}
q'(s) - \gamma(s)q(s) + \frac{m'(s)}{m(s)}q(s) = u_\varepsilon(s; t, x)(1 + q(s)), \\
s \in (t, (i + 1) \varepsilon), \\
q((i + 1) \varepsilon - ) \geq - \frac{1}{m((i + 1) \varepsilon)} \\
\times \partial_x \psi^\varepsilon((i + 1) \varepsilon - , y_\varepsilon((i + 1) \varepsilon - ; t, x)),
\end{cases} \quad (24)$$

and

$$u_\varepsilon(s; t, x) = \begin{cases} 0, & \text{if } 1 + q(s) < 0, \\
L, & \text{if } 1 + q(s) > 0. \end{cases} \quad (25)$$

Here $\partial_x \psi^\varepsilon(\cdot, \cdot)$ denotes the Clarke generalized gradient with respect to the second variable.
Proof. The proof is standard. We sketch it only under the additional assumption \( \psi((i + 1)\varepsilon, \cdot) \in C^1(0, +\infty) \).

In this case \( \partial_x \psi = \partial \psi / \partial x \).

Because \( (u, y(x, t), y_x(x, t, x)) \) is an optimal pair, then

\[
\int_t^{(i+1)\varepsilon} m(s)u_x(s)y_x(s; t, x) \, ds + \psi^\varepsilon((i + 1)\varepsilon - , y_x((i + 1)\varepsilon - ; t, x)) \\
\geq \int_t^{(i+1)\varepsilon} m(s)(u_x + \eta v)(s)y_{x+\eta v}^\varepsilon(s; t, x) \, ds \\
+ \psi^\varepsilon((i + 1)\varepsilon - , y_{x+\eta v}^\varepsilon((i + 1)\varepsilon - ; t, x)),
\]

for any \( v \in T_{u^\varepsilon}(u, x) \), \( \forall \eta > 0 \) and consequently,

\[
\int_t^{(i+1)\varepsilon} mu_x^\varepsilon y_x^\varepsilon - y_x^\varepsilon(s; t, x) \, ds + \int_t^{(i+1)\varepsilon} m y_{x+\eta v}^\varepsilon(s; t, x) \, ds \\
+ \frac{1}{\eta} [\psi^\varepsilon((i + 1)\varepsilon - , y_{x+\eta v}^\varepsilon((i + 1)\varepsilon - ; t, x)) \\
- \psi^\varepsilon((i + 1)\varepsilon - , y^\varepsilon((i + 1)\varepsilon - ; t, x))] \leq 0,
\]

(Here \( T_{u^\varepsilon}(u) \) denotes the normal cone to \( u^\varepsilon \) in \( u \) in the space \( L^2(t, T) \)).

Passing to the limit \( (\eta \rightarrow 0^+) \) we obtain

\[
\int_t^{(i+1)\varepsilon} m(s)(u_x z + vy_x)(s; t, x) \, ds + \frac{\partial}{\partial x} \psi^\varepsilon((i + 1)\varepsilon - , y_x((i + 1)\varepsilon - ; t, x)) \cdot z((i + 1)\varepsilon) \leq 0, \tag{26}
\]

for any \( v \in T_{u^\varepsilon}(u, x) \), where \( z \) is the Carathéodory solution of

\[
\begin{cases}
  z'(s) + \gamma(s)z(s) = -u_x(s)z(s) - v(s)y_x(s; t, x), \quad s \in (t, (i + 1)\varepsilon], \\
  z(t) = 0.
\end{cases} \tag{27}
\]

Let \( q \) be the solution of (24). Multiplying (24) by \( m(s)z(s) \) and integrating on \( [t, (i + 1)\varepsilon] \) we obtain

\[
\int_t^{(i+1)\varepsilon} q'(s)m(s)z(s) \, ds - \int_t^{(i+1)\varepsilon} \gamma(s)q(s)m(s)z(s) \, ds \\
+ \int_t^{(i+1)\varepsilon} m(s)q(s)z(s) \, ds \\
= \int_t^{(i+1)\varepsilon} u_x(s)(1 + q(s))m(s)z(s) \, ds.
\]
After an easy calculation involving (24) and (27) we obtain

\[
- \frac{\partial}{\partial x} \psi^\varepsilon((i + 1)\varepsilon, y_x((i + 1)\varepsilon - ; t, x)) z((i + 1)\varepsilon) \\
+ \int_t^{(i+1)\varepsilon} q(s) m(s) v(s) y_x(s; t, x) \, ds \\
= \int_t^{(i+1)\varepsilon} u_x(s) m(s) z(s) \, ds,
\]

\( v \in T_m(u_x). \) The last relation and (26) imply

\[
\int_t^{(i+1)\varepsilon} m(s) v(s) (1 + q(s)) y_x(s) \, ds \leq 0,
\]

for any \( v \in T_m(u_x), \) i.e.,

\[
u_x(s; t, x) = \begin{cases}
0, & \text{if } m(s)(1 + q(s)) y_x(s; t, x) < 0, \\
L, & \text{if } m(s)(1 + q(s)) y_x(s; t, x) > 0.
\end{cases}
\]

Because \( m(s) > 0 \) and \( y_x(s; t, x) > 0 \) a.e. in \((0, T)\) we may conclude that (25) holds.

**Remark.** From Theorem 5.1 and hypothesis (A8) we conclude that the optimal control \( u_x \) is of bang-bang type.

In the same manner as in Theorem 5.1 it is possible to prove:

**Lemma 5.2.** We have

\[
q(s) \equiv - \frac{1}{m(s)} \cdot \partial_y \psi^\varepsilon(s, y_x(s; t, x)), \text{ a.e. } s \in (t, (i + 1)\varepsilon),
\]

(where \( q \) is given by Theorem 5.1).

For any \( i\varepsilon \leq t \leq s \leq (i + 1)\varepsilon \) we have

\[
\psi^\varepsilon(t, x) - \psi^\varepsilon(s, y_x(s; t, x)) = \int_t^s m(\tau) u_x(\tau; t, x) y_x(\tau; t, x) \, d\tau,
\]

and consequently,

\[
- \frac{\partial \psi^\varepsilon}{\partial t}(t, s) - \frac{\partial \psi^\varepsilon}{\partial x}(t, x) y_x(t; t, x) = m(t) u_x(t; t, x) y_x(t; t, x), \tag{28}
\]
a.e. in \((0, T) \times (0, +\infty)\). Relations (28) and (25) imply

\[
- \frac{\partial \psi^e}{\partial t}(t, x) + \frac{\partial \psi^e}{\partial x}(t, x)(\gamma(t) + L \cdot H(1 + q(t))) x \\
= m(t)L \cdot H(1 + q(t)) x,
\]

a.e. in \((0, T) \times (0, +\infty)\) (here \(H\) is the Heaviside function).

Now we may conclude this section by the following result.

**Theorem 5.3.** The optimal value function \(\psi^e\) satisfies the following Hamilton–Jacobi equation,

\[
\frac{\partial \psi^e}{\partial t}(t, x) - \gamma(t) \frac{\partial \psi^e}{\partial x}(t, x) + L \cdot \left( m(t) - \frac{\partial \psi^e}{\partial x}(t, x) \right)^+ x = 0,
\]

a.e. \((t, x) \in (0, T) \times (0, +\infty),\)

\[
\psi^e(i\varepsilon^-, x) = \psi^e(i\varepsilon^+, \theta((i + 1)\varepsilon^- ; i\varepsilon^+, x)),
\]

\[
i = 0, 1, \ldots, N - 1,
\]

\[
\psi^e(T, x) = 0, \quad \forall x \in (0, +\infty),
\]

and

\[
u^e(t; 0, x) \in L \cdot H(m(t) - \frac{\partial \psi^e}{\partial x}(t, y^e(t; 0, x))).
\]

In fact we found the optimal control in a feedback form.

Numerical algorithms for the approximation of the optimal controller \(u^e\) for \((P)-(7)\) with \(y_0 := 1\) (i.e., \((P)-(5))\) can be developed from Theorem 5.3. For numerical results and numerical tests concerning some Hamilton–Jacobi equations we refer to [3, 9, 10].

**References**