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A bound for the Castelnuovo–Mumford regularity of log canonical varieties

Wenbo Niu

Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, 851 South Morgan Street, Chicago, IL 60607-7045, USA

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ABSTRACT

In this note, we give a bound for the Castelnuovo–Mumford regularity of a homogeneous ideal I in terms of the degrees of its generators. We assume that I defines a local complete intersection with log canonical singularities.

1. Introduction

Let I be a homogeneous ideal in a polynomial ring $R = k[x_0, \ldots, x_n]$ over a field k of characteristic zero. Consider the minimal free resolution of R/I as a graded R-module,

$$\cdots \to \oplus_j R(-d_{i,j}) \to \cdots \to \oplus_j R(-d_{1,j}) \to R \to R/I \to 0.$$

The *Castelnuovo–Mumford regularity*, or simply *regularity*, of R/I is defined by

$$\operatorname{reg} R/I = \max_{i,j}\{d_{i,j} - i\}.$$

The regularity of I is defined by $\operatorname{reg} I = \operatorname{reg} R/I + 1$. It measures the complexity of the ideal I and its syzygies. For more discussion of regularity, see the book of Eisenbud [7] or the survey of Bayer–Mumford [1].

Suppose that I is generated by homogeneous polynomials of degrees $d_1 \geq d_2 \geq \cdots \geq d_t$ and defines a projective subscheme $X = \operatorname{Proj} R/I$ in $\mathbb{P}^n$ of codimension r. It has been shown that there is a doubly exponential bound for the regularity of ideal I in terms of the degrees of its generators. An interesting question is whether one can find better bounds under some reasonable conditions on X, for instance on its singularities.

If I_X is the saturation of I, then $\operatorname{reg} I_X$ is equal to the regularity of the ideal sheaf $\mathscr{I}_X$ and $\operatorname{reg} \mathscr{I}_X$ is defined as the minimal number m such that $H^i(\mathbb{P}^n, \mathscr{I}_X(m-i)) = 0$ for all $i > 0$.

The first surprising result was worked out by Bertram et al. [2] when X is a nonsingular variety. They found a bound for the regularity of I_X which depends linearly on the degrees of the generators of I; namely

$$\operatorname{reg} R/I_X \leq \sum_{i=1}^{r} d_i - r.$$

This bound is sharp when X is a complete intersection. Chardin and Ulrich [3] use generic linkage to prove the above bound in the case when X is a local complete intersection with rational singularities. Recently, applying multiplier ideal sheaves

E-mail address: wniu2@uic.edu.

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and Nadel's vanishing theorem, deFernex and Ein [4] proved that this bound holds in the much more general situation when the pair $(\mathbb{P}^n, rX)$ is log canonical.

On the other hand, one can try to bound the regularity of I. If X is a local complete intersection with at most isolated irrational singularities, Chardin and Ulrich [3] gave the following bound:

$$\operatorname{reg} R/I \leq \frac{(\dim X + 2)!}{2}\left(\sum_{i=1}^{r} d_i - r\right), \tag{1.1}$$

which also depends linearly on the degrees of generators. Recently, in his paper [10], Fall improved (1.1) to

$$\operatorname{reg} R/I \leq (\dim X + 1)!\left(\sum_{i=1}^{r} d_i - r\right).$$

Starting from this formula and using Bertini's Theorem, Fall also gave an estimate for the regularity of the defining ideal of any projective subscheme X.

Local complete intersections with rational singularities are canonical. In light of the work of deFernex and Ein [4] and Chardin and Ulrich [3], it is natural to ask whether the bound (1.1) holds for log canonical singularities. A scheme of finite type over k is *local complete intersection log canonical* if it is a local complete intersection with log canonical singularities. In this note, we give an affirmative answer to this question in the following theorem (as an easy corollary of Theorem 4.2).

Theorem 1.1. *Let $R = k[x_0, \dots, x_n]$, and let $I = (f_1, \dots, f_t)$ be a homogeneous ideal, generated in degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$ of codimension r. Assume that $X = \operatorname{Proj} R/I$ is local complete intersection log canonical and $\dim X \geq 1$. Then*

$$\operatorname{reg} R/I \leq \frac{(\dim X + 2)!}{2}\left(\sum_{i=1}^{r} d_i - r\right).$$

Our main idea relies on the generic linkage method used in [3]. By constructing a generic link Y of X, we are able to reduce the problem to the intersection divisor $Z = Y \cap X$ and then proceed by induction on the dimension. However, for this approach there are two main problems we need to understand. First we need to know how to pass singularities from X to Z. This is the hard part of our approach and leads to the study of a flat family of log canonical singularities. Second we need to control the number and degrees of the defining equations of Z, for which there is a standard method already.

This note is organized as follows. We explore flat families of log canonical singularities in Section 2. By using Inversion of Adjunction due to Ein and Mustaţă [5], we prove the following theorem.

Theorem 1.2. *Let $f : Y \to X$ be a flat morphism of schemes of finite type over k. Assume that X and all fibers of f are local complete intersection log canonical. Then Y is local complete intersection log canonical.*

In Section 3, we use the generic residual intersection theory developed by Huneke and Ulrich [8] to pass the log canonical singularities from X to the intersection divisor Z. This is encoded in the following result.

Proposition 1.3. *Let $S = \operatorname{Spec} R$ be a regular affine scheme over k and $X \subset S$ be a subscheme defined by $I = (z_1, \dots, z_t)$ of codimension r. Construct a generic linkage J of I as follows: let $M = (U_{ij})_{t\times r}$ be a matrix of variables, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \dots, \alpha_r) = (z_1, \dots, z_t) \cdot M$ and $J = [\alpha : IR']$. Let Z be a subscheme of $\operatorname{Spec} R'$ defined by the ideal $J + IR'$. If X is local complete intersection log canonical, then Z is also local complete intersection log canonical.*

In the last section, we use induction to obtain the bound of regularity. The main idea comes from Chardin and Ulrich [3].

Some natural questions are pointed out by referees. The first question is whether it is possible to pass singularities from X to the link $Y = \operatorname{Spec} R'/J$ in Proposition 1.3. The main difficulty here is that there is no natural morphism from Y to X and therefore we do not know how to pass singularities from X to Y. We may use the morphism from Y to S, but we do not know what kind of fiber it will have. However, it is a really interesting question and we may propose a conjecture on it.

Conjecture 1.4. *Assume the hypothesis of Proposition 1.3. Set $Y = \operatorname{Spec} R'/J$. If X is local complete intersection log canonical, then Y is also local complete intersection log canonical.*

The second question is, comparing with the work of Chardin, Ulrich and Fall, can we allow X to have some non-log canonical points and get a similar bound to Fall's results? Unfortunately, the method we use in this note seems unable to solve this problem. Admitting some non-log canonical points on X, we cannot show the intersection divisor Z has the same property as X; this would be an obstruction to Fall's method. But if we could show Z also admits non-log canonical points, we may reduce the number of defining equations of Z by one, and this will lead to Fall's sharper bound. Nevertheless, we believe the answer of this question could be positive, and there will be a better bound under weaker assumptions. Here we make a conjecture in this direction.

Conjecture 1.5. *Let $R = k[x_0, \dots, x_n]$, and $I = (f_1, \dots, f_t)$ be an homogeneous ideal, generated in degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$ of codimension r. Assume that, except for some isolated points, $X = \operatorname{Proj} R/I$ is local complete intersection log canonical and $\dim X \geq 1$. Then*

$$\operatorname{reg} R/I \leq (\dim X + 1)! \left(\sum_{i=1}^{r} d_i - r \right).$$

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2. Flat family of log canonical singularities

In the present section, we study a flat family of local complete intersection log canonical singularities. We begin by recalling the definitions of minimal log discrepancy and log canonical singularities. We mainly follow the approach in [6, Section 7].

Consider a pair (X, Y), where X is a normal, $\mathbb{Q}$-Gorenstein variety and Y is a formal finite sum $Y = \sum_i q_i \cdot Y_i$ of proper closed subschemes Y_i of X with nonnegative rational coefficients q_i.

Let X' be a nonsingular variety which is proper and birational over X. If E is a prime divisor on X', then E defines a divisor *over* X. The image of E on X is called the *center* of E, denoted by $c_X(E)$.

Given a divisor E over X, we choose a proper birational morphism $\mu : X' \to X$ with X' nonsingular such that E is a divisor on X', and such that all the scheme-theoretic inverse images $\mu^{-1}(Y_i)$ are divisors. The *log discrepancy* $a(E; X, Y)$ is defined such that the coefficient of E in $K_{X'/X} - \sum_i q_i \cdot \mu^{-1}(Y_i)$ is $a(E; X, Y) - 1$. This number is independent of the choice of X'.

Let W be a nonempty closed subset of X. The *minimal log discrepancy* of the pair (X, Y) on W is defined by

$$\operatorname{mld}(W; X, Y) = \inf_{c_X(E) \subseteq W} \{a(E; X, Y)\}.$$

If $\operatorname{mld}(p; X, Y) \geq 0$ for a closed point $p \in X$, we say that the pair (X, Y) is *log canonical* at p. If (X, Y) is log canonical at each closed point of X, we then say that the pair (X, Y) is log canonical. If $Y = 0$, we just write the pair (X, Y) as X.

One important theorem on minimal log discrepancy is Inversion of Adjunction. It is proved for local complete intersection varieties by Ein and Mustaţă. Since it is used very often in our proofs, we state it here for the convenience of the reader.

Inversion of Adjunction (*[5, Theorem1.1]*). *Let X be a normal, local complete intersection variety, and $Y = \sum_i q_i \cdot Y_i$, where $q_i \in \mathbb{R}_+$ and $Y_i \subset X$ are proper closed subschemes. If $D \subset X$ is a normal effective Cartier divisor such that $D \nsubseteq \cup_i Y_i$, then for every proper closed subset $W \subset D$, we have*

$$\operatorname{mld}(W; X, D + Y) = \operatorname{mld}(W; D, Y|_D).$$

The local complete intersection log canonical singularities behave well in flat families. More specifically, consider a flat family over a local complete intersection log canonical scheme, where all fibers are also local complete intersection log canonical. Then we show that the total space itself is local complete intersection log canonical.

We start with the case where the flat family has a nonsingular base.

Proposition 2.1. *Let $f : Y \to X$ be a flat morphism of schemes of finite type over k. Assume that X is nonsingular and each fiber of f is local complete intersection log canonical. Then Y is local complete intersection log canonical.*

Proof. Since X and all fibers are normal and local complete intersections, by flatness of f, we see that Y is normal and a local complete intersection ([11, Section23]). By choosing an irreducible component of Y and its image, we may assume that Y is a variety and f is surjective. The question is local. We may assume that $X = \operatorname{Spec} A$ is affine. Choosing $x \in X$, a closed point defined by a maximal ideal m, $\mathcal{O}_{X,x}$ is a regular local ring with a maximal ideal $m_x = (t_1, t_2, \dots, t_n)$ generated by a regular system of parameters, where $n = \dim X$. Shrinking X if necessary, we can extend t_i to X and therefore may assume that $m = (t_1, \dots, t_n) \subset A$ generated by a regular sequence. Set $I_i = (t_1, \dots, t_i)$. Note that $\mathcal{O}_{X,x}/(t_1, \dots, t_i)$ is regular. By shrinking X if necessary, we may assume further that A/I_i is regular for each $i = 1, \dots, n$. Let $X_i = \operatorname{Spec} A/I_i$ be subschemes of X and consider the following fiber product

$$\begin{array}{ccc} Y_i & \longrightarrow & Y \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ X_i & \longrightarrow & X \end{array}$$

By the flatness of f_i and the assumption that each fiber of f_i is a local complete intersection and normal, we obtain that Y_i is a local complete intersection and normal for each $i = 1, \dots, n$.

Choose a closed point y on the fiber $Y_x = Y_n$. By the flatness of f, $(t_1, \dots, t_n)$ is a regular sequence in $\mathcal{O}_{Y,y}$ and therefore the t_i's define divisors $D_1, \dots, D_n$ around y in Y such that

$$Y_i = D_1 \cap D_2 \cap \cdots \cap D_i, \quad \text{for } i = 1, \dots, n.$$

Now by Inversion of Adjunction, we have

$$\begin{aligned}\mathrm{mld}(y; Y_n) &= \mathrm{mld}(y; Y_{n-1}, D_n|_{Y_{n-1}})\\ &= \mathrm{mld}(y; Y_{n-2}, D_n|_{Y_{n-2}} + D_{n-1}|_{Y_{n-2}})\\ &= \dots\\ &= \mathrm{mld}(y; Y, D_1 + \cdots + D_n).\end{aligned}$$

From the assumption that $\mathrm{mld}(y; Y_n) \geq 0$, we get that $\mathrm{mld}(y; Y) \geq 0$, i.e. Y is log canonical at y, which proves the proposition. □

In the general case in which the flat family has a singular base, we first resolve the singularities of the base, and then base change to the situation of nonsingular base. However, after base change, some extra divisors could be introduced on the new flat family. This means that we need to consider singularities of pairs on the new flat family.

Theorem 2.2. *Let $f : Y \to X$ be a flat morphism of schemes of finite type over k. Assume that X and all fibers of f are local complete intersection log canonical. Then Y is local complete intersection log canonical.*

Proof. As in the proof of Proposition 2.1, we may assume that X and Y are varieties and Y is normal and a local complete intersection. We need to show Y is log canonical. Take a log resolution of X, $\mu : \widetilde{X} \to X$, and construct the fiber product $\widetilde{Y} = Y \times_X \widetilde{X}$:

$$\begin{array}{ccc}\widetilde{Y} & \longrightarrow & Y\\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f}\\ \widetilde{X} & \xrightarrow{\mu} & X\end{array}$$

By Proposition 2.1, $\widetilde{Y}$ is local complete intersection log canonical. Since X is log canonical, we can write the relative canonical divisor $K_{\widetilde{X}/X} = P - N$, where P and N are effective divisors supported in the exceptional locus of μ, so that $N = \sum E_i$ where E_i are prime divisors with simple normal crossings. By base change for relative canonical divisors, we have $K_{\widetilde{Y}/Y} = g^*K_{\widetilde{X}/X}$ and therefore $K_{\widetilde{Y}/Y} = g^*(P) - g^*(N)$.

Denoting the F_j's as distinct irreducible components of the $g^*(E_i)$ (note that $g^*(E_i) = g^{-1}(E_i)$ as scheme-theoretical inverse image of E_i), we have $g^*(N) = \sum F_j$. This will be shown in detail at the beginning of the proof of Lemma 2.3 below.

Now we let $\pi : Y' \to \widetilde{Y}$ be a log resolution of $\widetilde{Y}$ such that

$$\begin{aligned}K_{Y'/Y} &= K_{Y'/\widetilde{Y}} + \pi^*K_{\widetilde{Y}/Y}\\ &= A - B + \pi^*g^*P - \sum \pi^*F_i\end{aligned}$$

where A is the positive part of $K_{Y'/\widetilde{Y}}$ and B is the negative part of $K_{Y'/\widetilde{Y}}$ and all prime divisors in the above formula are simple normal crossings. In order to show Y is log canonical, it is enough to show that the coefficient of each prime divisor in $B + \sum \pi^*F_i$ is 1. This is equivalent to showing that the pair $(\widetilde{Y}, g^{-1}N)$ is log canonical, which is shown in the following Lemma 2.3. □

Lemma 2.3. *Let $f : Y \to X$ be a flat morphism of varieties such that X is nonsingular and each fiber of f is local complete intersection log canonical. Assume that $E_1, \dots, E_r$ are prime divisors on X with simple normal crossings. Then the pair $(Y, \sum_{i=1}^r f^{-1}(E_i))$ is log canonical, where f^{-1} means scheme-theoretical inverse image.*

Proof. From Proposition 2.1, Y is local complete intersection log canonical. Also for each divisor E_i, the scheme-theoretical inverse $f^{-1}(E_i)$ is local complete intersection log canonical. This implies that

$$\sum_{i=1}^{r} f^{-1}(E_i) = \sum_{j=1}^{s} F_j$$

where F_j are distinct irreducible components of the subschemes $f^{-1}(E_i)$. Note that since f is flat, each F_j only appears in one $f^{-1}(E_i)$, and if some F_j's are in the same $f^{-1}(E_i)$ then they are disjoint. Furthermore each F_j is a Cartier normal divisor on Y with local complete intersection log canonical singularities. We need to show the pair $(Y, \sum F_j)$ is log canonical.

We prove this by induction on the dimension of X. First assume that $\dim X = 1$. Then $E_1, \dots, E_r$ are distinct points and $F_1, \dots, F_s$ are pairwise disjoint. It is enough to show that for each j, $\mathrm{mld}(F_j; Y, F_j) \geq 0$. Choosing a closed point $p \in F_j$ of Y, by Inversion of Adjunction and the fact that F_j has log canonical singularities, we have $\mathrm{mld}(p; Y, F_j) = \mathrm{mld}(p; F_j) \geq 0$.

Assume X has any dimension. Since Y is log canonical, it is enough to show that for each j, $\mathrm{mld}(F_j; Y, \sum_{t=1}^s F_t) \geq 0$. Without loss of generality, we prove this for F_1 and assume that $F_1 \subseteq f^{-1}(E_1)$. Choosing any closed point $p \in F_1$ of Y, by Inversion of Adjunction, we have

$$\mathrm{mld}\left(p; Y, F_1 + \sum_{t=2}^{s} F_t\right) = \mathrm{mld}\left(p; F_1, \sum_{t=2}^{s} F_t|_{F_1}\right).$$

For $i = 2, \dots, r$, we set $D_i = E_1 \cap E_i$ and note that $\sum_{t=2}^{s} F_t|_{F_1} = \sum_{i=2}^{r} f^{-1}(D_i)$, where f^{-1} means scheme-theoretical inverse image. Now we are in the situation

$$f : F_1 \to E_1,$$

where E_1 is nonsingular and $D_2, \dots, D_r$ are divisors on E_1 with simple normal crossings. Then applying induction on F_1, we get that the pair $(F_1, \sum_{t=2}^{s} F_t|_{F_1})$ is log canonical and therefore $\mathrm{mld}(p; F_1, \sum_{t=2}^{s} F_t|_{F_1}) \geq 0$ which proves the lemma. □

If $f : Y \to X$ is a surjective smooth morphism, then we can move singularities freely from Y to X. Using the notion of jet schemes, we have a quick proof for this.

Given any scheme X, we can associate the mth jet scheme X_m for any positive integer m. The properties of jet schemes are closely related to the singularities of X. We may use jet schemes to describe local complete intersection log canonical singularities. The work of Ein and Mustaţă shows that if X is a normal local complete intersection variety, then X has log canonical singularities if and only if X_m is equidimensional for every m. For more information on jet schemes and their application to singularities, we refer the reader to [6].

Proposition 2.4. *Let $f : Y \to X$ be a smooth surjective morphism of schemes of finite type over k. Then X is local complete intersection log canonical if and only if Y is local complete intersection log canonical.*

Proof. First note that since f is smooth, we have X is normal and a local complete intersection if and only if Y is normal and a local complete intersection. Since f is smooth and surjective, for every m we have an induced morphism between m-jet schemes $f_m : Y_m \to X_m$, which is smooth and surjective [6, Remark 2.10]. Then Y_m is equidimensional if and only if X_m is equidimensional. Now by [5, Theorem 1.3], we get the proposition. □

Remark 2.5. In the proof, if f is smooth but not surjective, we can only get $f_m : Y_m \to X_m$ is smooth. Then equidimensionality of X_m will imply that Y_m is equidimensional. This means that for a smooth morphism $f : Y \to X$, if X is local complete intersection log canonical then Y is also local complete intersection log canonical singularities. This is a quick proof for a special case of Theorem 2.2.

3. Log canonical singularities in a generic linkage

In this section, we study the log canonical singularities in a generic linkage. This could be compared to the work in [3] studying rational singularities in a generic linkage. The s-generic residual intersection theory can be found in [8]. Throughout this section, all rings are assumed to be Noetherian k-algebras and a point on a scheme means a point locally defined by a prime ideal, not necessarily maximal. All fiber products are over the field k unless otherwise stated.

Let $S = \operatorname{Spec} R$ be an affine scheme and $X \subset S$ be a codimension r subscheme defined by an ideal $I = (z_1, \dots, z_t)$. For an integer $s \geq 0$, let $M = (U_{ij})_{t\times s}$ be a $t \times s$ matrix of variables and $R' = R[U_{ij}]$ be the polynomial ring over R obtained by adjoining the variables of M. Define $S' = S \times \mathbb{A}^{t\times s} = \operatorname{Spec} R'$, which has a natural flat projection $\pi : S' \to S$. Let $X' = \pi^{-1}(X)$ be defined by the ideal IR'. Construct an ideal α in R' generated by $\alpha_1, \dots, \alpha_s$ as follows:

$$\alpha = (\alpha_1, \dots, \alpha_s) = (z_1, \dots, z_t) \cdot M;$$

and set $J = [\alpha : IR']$. The subscheme Y' of S' defined by J is called an *s-generic residual intersection* of X.

We define Z to be the scheme-theoretical intersection of X' and Y'. Its defining ideal is $I_Z = J + IR'$. We equip Z with a restricted projection morphism $\pi : Z \to X$ and call Z an *intersection divisor* of an s-generic residual intersection of X.

Note that if $s < r$, then α is generated by a regular sequence and therefore $J = \alpha, Z = X'$. The interesting case is when $s \geq r$. In particular, when $s = r$, Y' is called a *generic linkage* of X. Correspondingly, we call Z an *intersection divisor* of a generic linkage of X.

Under the assumption that X is a local complete intersection, the morphism $\pi : Z \to X$, and in particular its fibers, can be understood very well. This offers us an opportunity to pass singularities from X to Z.

We start with a lemma which describes the fibers of π when X is a complete intersection.

Lemma 3.1. *Let $S = \operatorname{Spec} R$ be a Gorenstein integral affine scheme and X be a complete intersection subscheme defined by a regular sequence $I = (z_1, \dots, z_r)$ in R. For $s \geq 0$, let $M = (U_{ij})_{r\times s}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \dots, \alpha_s) = (z_1, \dots, z_r) \cdot M$ and $J = [\alpha : IR']$. Assume that Z is defined by $J + IR'$ and consider the natural morphism $\pi : Z \to X$. We have*

(1) *If $s < r$, then $Z \cong X \times \mathbb{A}^{r\times s}$ and π is the projection to X.*
(2) *If $s \geq r$, then $\pi : Z \to X$ is a flat morphism and for any point $p \in X$,*

$$\pi^{-1}(p) \cong k(p)[U_{ij}]/I_r(M)$$

where $I_r(M)$ is the $r \times r$ minors ideal of M.
(3) *In particular, if $s = r$, then $\pi : Z \to X$ is a flat morphism such that each fiber is a local complete intersection with rational singularities.*

Proof. (1) is trivial because in this case $J = \alpha$ and Z is defined by IR' so that $Z = \pi^{-1}(X) \cong X \times \mathbb{A}^{r\times s}$.

For (2) and (3), picking $\mathfrak{q} \in X \subset S$ and passing to $R_{\mathfrak{q}}$, we may assume R is local. By [8, Example 3.4], $J = (\alpha_1, \ldots, \alpha_s, I_r(M))$. Z is then defined by $I_Z = J + IR' = (I, I_r(M))$.

Note that

$$R[U_{ij}]/(I, I_r(M)) = R/I \otimes_R R[U_{ij}]/I_r(M).$$

This means $\pi : Z \to X$ can be constructed from the fiber product

$$\begin{array}{ccc} Z & \longrightarrow & \operatorname{Spec} R[U_{ij}]/I_r(M) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\theta} \\ X & \longrightarrow & S = \operatorname{Spec} R \end{array}$$

Since θ is flat, we obtain π is flat. The fiber of π at $p \in X$ is

$$\begin{aligned} F &= k(p) \otimes_{R/I} R[U_{ij}]/(I, I_r(M)) \\ &= k(p)[U_{ij}]/I_r(M). \end{aligned}$$

In particular, if $s = r$, we see that F is a local complete intersection with rational singularities. □

Now we turn to the case where X is a local complete intersection.

Proposition 3.2. *Let $S = \operatorname{Spec} R$ be a Gorenstein integral affine scheme and X be a subscheme defined by an ideal $I = (z_1, \ldots, z_t)$ in R. For $s \geq 0$, let $M = (U_{ij})_{t\times s}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_t) \cdot M$, and $J = [\alpha : IR']$. Let Z be defined by $J + IR'$ and consider the natural morphism $\pi : Z \to X$. Let $\mathfrak{p} \in X$ be a point of S and assume that $I_{\mathfrak{p}}$ is generated by a regular sequence of length r. Then there is an affine neighborhood of $\mathfrak{p}$ over which π can be factored as follows*

$$\begin{array}{ccc} Z & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\pi'} & \\ X & \xleftarrow[g]{} & P \end{array}$$

such that $P = X \times \mathbb{A}^{(t-r)\times s}$ with g the projection to X, and Z can be viewed as an intersection divisor of an s-generic residual intersection of P.

Note that the above diagram is local. More precisely, there is an affine neighborhood U of $\mathfrak{p}$ and the morphism $\pi : Z \to X$ in the above diagram really means the restriction of π over U, i.e. $\pi : \pi^{-1}(U) \cap Z \to U \cap X$.

Proof. By assumption, we may replace S by an affine neighborhood of $\mathfrak{p}$ such that I is generated by a regular sequence, say $z_1, \ldots, z_r$. Then

$$\begin{cases} z_{r+1} = a_{1,r+1}z_1 + a_{2,r+1}z_2 + \cdots + a_{r,r+1}z_r \\ z_{r+2} = a_{1,r+2}z_1 + a_{2,r+2}z_2 + \cdots + a_{r,r+2}z_r \\ \quad\cdots \\ z_t = a_{1,t}z_1 + a_{2,t}z_2 + \cdots + a_{r,t}z_r \end{cases} \tag{3.1}$$

where $a_{ij} \in R$. Set $A = (a_{ij})_{r\times(t-r)}$. We can write $(z_{r+1}, \ldots, z_t) = (z_1, \ldots, z_r) \cdot A$. Denote $M = \begin{pmatrix} C \\ B \end{pmatrix}$, where

$$C = \begin{pmatrix} U_{11} & U_{12} & \cdots & U_{1s} \\ U_{21} & U_{22} & \cdots & U_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ U_{r1} & U_{r2} & \cdots & U_{rs} \end{pmatrix}, \qquad B = \begin{pmatrix} U_{r+1,1} & U_{r+1,2} & \cdots & U_{r+1,s} \\ U_{r+2,1} & U_{r+2,2} & \cdots & U_{r+2,s} \\ \cdots & \cdots & \cdots & \cdots \\ U_{t1} & U_{t2} & \cdots & U_{ts} \end{pmatrix}.$$

Using the equations in (3.1), we can rewrite $(\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_t) \cdot M$ as

$$(\alpha_1, \ldots, \alpha_s) = (z_1, \ldots, z_r) \cdot (A \cdot B + C).$$

Set $N = (V_{lm})_{r\times s} = A \cdot B + C$. Then the ring extension of R to R' can be obtained by extending twice as follows

$$R \to R_1 = R[U_{ij} | i > r] \to R' = R_1[V_{pq}].$$

The first extension $R \to R_1$ gives the morphism $g : \operatorname{Spec} R_1 = S \times \mathbb{A}^{(t-r)\times s} \to S$. Let $P = g^{-1}(X) = X \times \mathbb{A}^{(t-r)\times s}$ defined by IR_1 which is the complete intersection generated by the regular sequence $(z_1, \ldots, z_r)$ in R_1. Restricting g to P, we get a projection $g : P \to X$. In the second extension, $R_1 \to R'$, we see that Z can be viewed as an intersection divisor of an s-generic residual intersection of P with morphism $\pi' : Z \to P$. □

Since Z is a generic intersection divisor of X, the fibers of the morphism $\pi : Z \to X$ are local complete intersections with rational singularities and they are log canonical. So the morphism $\pi : Z \to X$ provides us a flat family of log canonical singularities, to which results of the previous section can be applied.

Proposition 3.3. *Let $S = \operatorname{Spec} R$ be a regular affine scheme and X be a subscheme defined by an ideal $I = (z_1, \ldots, z_t)$ with codimension r in S. Construct a generic linkage J of I as follows: let $M = (U_{ij})_{t\times r}$, $R' = R[U_{ij}]$, $\alpha = (\alpha_1, \ldots, \alpha_r) = (z_1, \ldots, z_t)\cdot M$, and $J = [\alpha : IR']$. Let Z be a subscheme of $\operatorname{Spec} R'$ defined by the ideal $J + IR'$ and consider the natural morphism $\pi : Z \to X$. If X is local complete intersection log canonical, then Z is also local complete intersection log canonical.*

Proof. Choose any point $\mathfrak{p} \in X$. By the assumption, $I_{\mathfrak{p}}$ is generated by a regular sequence with length $l \geq r$. By Proposition 3.2, there is an affine neighborhood of $\mathfrak{p}$, over which we can factor $\pi : Z \to X$ as follows

$$\begin{array}{ccc} Z & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\pi'} & \\ X & \xleftarrow[g]{} & P \end{array}$$

such that $P \cong X \times \mathbb{A}^{(t-l)\times r}$, which is defined by a regular sequence of length l in $S \times \mathbb{A}^{(t-l)\times r}$, and Z is an intersection divisor of a r-generic residual intersection of P.

There are two possibilities.

If $l = r$, then by Lemma 3.1(3), $\pi' : Z \to P$ is a flat morphism whose fibers are locally complete intersection log canonical. Now by Proposition 2.4 and Theorem 2.2 we obtain that Z is local complete intersection log canonical.

If $l > r$, then by Lemma 3.1(1), $Z \cong P \times \mathbb{A}^{l\times r}$. Using Proposition 2.4, we have that Z is local complete intersection log canonical. □

We have passed the singularities from X to Z in above proposition. As we mentioned in the Introduction, we need to understand the generators of Z. Since Z is defined by $J + IR'$, basically, we need to know the generators of the generic linkage J. The method we will use here is quite standard in [3] and we shall be brief.

Lemma 3.4. *Let $X \subset \mathbb{P}^n$ be a equidimensional Gorenstein subscheme with log canonical singularities. Then*

$$\operatorname{reg}\omega_X = \dim X + 1,$$

where ω_X is the canonical sheaf of X.

Proof. By assumption, ω_X is a direct sum of the canonical sheaves of each irreducible component of X. We may assume that X is irreducible. Since X is log canonical, Kodaira vanishing holds for X [9, Corollary 1.3], i.e.

$$H^i(X, \omega_X(k)) = 0, \quad \text{for all } k > 0 \text{ and } i > 0.$$

Note that $H^{\dim X}(X, \omega_X) \neq 0$. Then we see $\operatorname{reg} \omega_X = \dim X + 1$. □

Proposition 3.5. *Let $X \subset \mathbb{P}^n$ be a equidimensional Gorenstein subscheme with log canonical singularities and codimension r. Assume that $Y \subset \mathbb{P}^n$ is direct linked with X by forms of degrees $d_1, \ldots, d_r$. Denote by J the defining ideal of Y and write $\sigma = \sum_{i=1}^{r}(d_i - 1)$. Then $J = (J)_{\leq\sigma}$.*

Proof. Let $I \subset R = k[x_0, \ldots, x_n]$ be the defining ideal of X and $d = \dim R/I$. Let $b = I \cap J$ be generated by forms in degrees $d_1, \ldots, d_r$ and ω be the canonical module of R/I. If $d = 2$, i.e., X is a nonsingular curve, then $(\omega)_{\leq d} = \omega$ by [3, Proposition 1.1]. If $d > 2$, i.e., $\dim X > 1$, then $\operatorname{reg} \omega = \operatorname{reg} \omega_X = d$ by Lemma 3.4 and therefore we have $(\omega)_{\leq d} = \omega$.

Observe that

$$J/b = \operatorname{Hom}_R(R/I, R/b) = \operatorname{Ext}_R^r(R/I, R)[-d_1 - \cdots - d_r] = \omega[d - \sigma].$$

Hence $(J/b)_{\leq\sigma} = (\omega[d-\sigma])_{\leq\sigma} = (\omega)_{\leq d}[d - \sigma] = \omega[d - \sigma]$. From the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (b)_{\leq\sigma} & \longrightarrow & (J)_{\leq\sigma} & \longrightarrow & (J/b)_{\leq\sigma} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & b & \longrightarrow & J & \longrightarrow & J/b & \longrightarrow & 0, \end{array}$$

we see $(J)_{\leq\sigma} = J$. □

4. Bounds for Castelnuovo–Mumford regularity

Applying the results we have established, we are able to give a bound for the Castelnuovo–Mumford regularity of a homogeneous ideal which defines a local complete intersection log canonical scheme. This partially generalizes the work of Chardin and Ulrich [3] and gives a new geometric condition under which a reasonable bound can be obtained. For the convenience of the reader, we follow the construction from [3] and keep the same notations.

Proposition 4.1. *Let $R = k[x_0, \dots, x_n]$ and $I \subset R$ be a homogeneous ideal of codimension r generated by forms $f_1, \dots, f_t$ of degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$. Let*

$$a_{ij} = \sum_{|\mu|=d_j-d_i} U_{ij\mu} x^{\mu}, \quad \text{for } r+1 \leq i \leq t,\ 1 \leq j \leq r,$$

where $U_{ij\mu}$ are variables. Denote $A = (a_{ij})$, $K = k(U_{ij\mu})$, $R' = R \otimes_k K$ and define

$$(\alpha_1, \dots, \alpha_r) = (f_1, \dots, f_t)\begin{pmatrix} I_{r\times r} \\ A \end{pmatrix},$$

$J = [(\alpha_1, \dots, \alpha_r)R' : IR']$. Assume that $X = \operatorname{Proj} R/I$ is local complete intersection log canonical. Then $Z = \operatorname{Proj} R'/IR' + J$ is local complete intersection log canonical.

Proof. Reduce the question to standard affine covers of $\mathbb{P}^n_k$. Without loss of generality, we focus on one affine cover $U = \operatorname{Spec} R_{(x_0)}$, where $R_{(x_0)}$ means the degree zero part of the homogeneous localization of R with respect to x_0, which is canonically isomorphic to $k[x_1/x_0, \dots, x_n/x_0]$. Set $V = \pi^{-1}(U)$, where π is the natural morphism $\pi : \mathbb{P}^n_K \to \mathbb{P}^n_k$. Note that $V = \operatorname{Spec} R'_{(x_0)}$. For simplicity, we reset our notations as follows. Replace R by $R_{(x_0)}$, R' by $R'_{(x_0)}$, f_i by $f_i/x_0^{d_i}$, and I by $I_{(x_0)}$. Then on the affine open set U, X is generated by $I = (f_1, \dots, f_t)$ in R. We redefine elements of the matrix A by setting $a_{ij} = \sum U_{ij\mu} x^{\mu}/x_0^{|\mu|}$. We can see that on V, Z is defined by the ideal $J + IR'$, where $J = [\alpha : IR']$ and $\alpha = (\alpha_1, \dots, \alpha_r)$ is defined by the equations in the assumption. Note that a_{ij}'s now become variables over R and therefore A is a matrix of variables over R. We restrict to this affine case in the following proof.

Consider ring extensions $R[a_{ij}] \to R[U_{ij\mu}] \to R' = R \otimes_k K$. The first one is given by adjoining variables. The second one is the localization of $R[U_{ij\mu}]$ by the multiplicative set $W = k[U_{ij\mu}] \setminus \{0\}$. They give morphisms ϕ_1 and ϕ_2 respectively:

$$\operatorname{Spec} R' \xrightarrow{\phi_2} \operatorname{Spec} R[U_{ij\mu}] \xrightarrow{\phi_1} \operatorname{Spec} R[a_{ij}].$$

In $R[a_{ij}]$, set $J_1 = [\alpha : IR[a_{ij}]]$ and define a subscheme $Z_1 = \operatorname{Spec} R[a_{ij}]/(J_1 + IR[a_{ij}])$, so that $Z = (\phi_0 \circ \phi_1)^{-1}(Z_1)$. To show Z has the desired singularities, we just need to show Z_1 has the desired singularities. This is because ϕ_1 is smooth and it passes singularities from Z_1 to $\phi_1^{-1}(Z_1)$ by Proposition 2.4. Our singularities are preserved by localization and so ϕ_2 will continue passing singularities from $\phi_1^{-1}(Z_1)$ to Z. Hence all we need is to prove the proposition for Z_1 in $\operatorname{Spec} R[a_{ij}]$.

To this end, we introduce a new matrix of variables $B = (b_{lm})_{r\times r}$ and set

$$C = \begin{pmatrix} B \\ A \cdot B \end{pmatrix} = (c_{uv})_{t\times s},$$

which is also a matrix of variables over R. In the ring $R[c_{uv}]$, we construct an intersection divisor Z' of X as follows: let $\alpha' = (\alpha'_1, \dots, \alpha'_r) = (f_1, \dots, f_t) \cdot C$, $J' = (\alpha' : IR[c_{uv}])$ and define $Z' = \operatorname{Spec} R[c_{uv}]/(J' + IR[c_{uv}])$. Then consider the diagram

$$\begin{array}{ccc} \operatorname{Spec} R[a_{ij}] & \xleftarrow{\ q\ } & \operatorname{Spec} R[a_{ij}] \otimes_k k(b_{lm}) \\ \downarrow & & \downarrow p \\ \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} R[c_{uv}] \end{array}$$

where q is induced by the base field extension $R[a_{ij}] \to R[a_{ij}] \otimes_k k(b_{lm})$, and p is induced by $R[c_{uv}] \to R[a_{ij}] \otimes_k k(b_{lm})$, which is the localization of $R[c_{uv}]$ with respect to the multiplicative set $k[b_{lm}] \setminus \{0\}$. We note that $p^{-1}(Z') = q^{-1}(Z_1) = Z_1 \otimes_k k(b_{lm})$. By Proposition 3.3, Z' is local complete intersection log canonical. Since p is induced by localization, we obtain that $p^{-1}(Z')$ is also local complete intersection log canonical. Finally because q is the base field change of Z_1 from k to $k(b_{lm})$, it is easy to see that Z_1 is local complete intersection log canonical if and only if $q^{-1}(Z_1) = Z_1 \otimes_k k(b_{lm})$ is local complete intersection log canonical. This proves the proposition. □

Theorem 4.2. *Let $R = k[x_0, \dots, x_n]$ and $I = (f_1, \dots, f_t)$ be a homogeneous ideal, not a complete intersection, generated in degrees $d_1 \geq d_2 \geq \cdots \geq d_t \geq 1$ of codimension r. Assume that $X = \operatorname{Proj} R/I$ is local complete intersection log canonical and $\dim X \geq 1$. Then*

$$\operatorname{reg} R/I \leq \frac{(\dim X + 2)!}{2}\left(\sum_{i=1}^{r} d_i - r - 1\right),$$

unless $R = k[x_0, x_1, x_2]$ *and* $I = lH$ *with* l *a linear form and* H *a complete intersection of* 3 *forms of degree* $d_1 - 1$, *in which case* $\operatorname{reg} R/I = 3d_1 - 5$.

Proof. We construct R', $\alpha = (\alpha_1, \dots, \alpha_r)$, J and Z as in Proposition 4.1 and write $\sigma = \sum_{i=1}^{r}(d_i - 1)$ and $d = \dim R/I$.

By the assumption that I is not a complete intersection, we may assume that $d_2 \geq 2$. Also we note that if $\sigma = 1$, then $\operatorname{ht} I = 1$ and there is a linear form l and a homogeneous ideal H such that $f_i = lh_i$ and $H = (h_1, \dots, h_t)$, where h_i are all linear forms, so we get $\operatorname{reg} R/I = \operatorname{reg} R/(l) + \operatorname{reg} R/H = 0$. Then we may assume in the following proof that $\sigma \geq 2$.

We consider the codimension r in two cases.

Case of $r \geq 2$. We proceed by induction on d. For $d = 2$, we have $n \geq 3$. Applying [3, Proposition 2.2], we have $\operatorname{reg} R/I \leq \frac{(\dim X+2)!}{2}(\sigma - 1)$.

Assume that $d \geq 3$. Let $X' = \operatorname{Proj} R'/IR'$ which is local complete intersection log canonical. Let $(IR')^{top}$ be the unmixed part of IR'; it defines an equidimensional subscheme X'^{top} which is local complete intersection log canonical and J is directly linked with $(IR')^{top}$ by α. By Proposition 3.5, $J = (J)_\sigma$. Set $Z' = \operatorname{Proj} R'/(IR')^{top} + (J)_\sigma$ which is a Cartier divisor on X'^{top}, then in the ring $R'/(IR')^{top}$, $\overline{J}$ is generated by d forms $\overline{\beta_1}, \dots, \overline{\beta_d}$ of degrees at most σ, which give forms $\beta_1, \dots, \beta_d$ in J of degrees at most σ such that $Z' = \operatorname{Proj} R'/(IR')^{top} + (\beta_1, \dots, \beta_d)$, and therefore we obtain $Z = \operatorname{Proj} R'/IR' + (\beta_1, \dots, \beta_d)$. Let $J' = (\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_d)$. We have an exact sequence

$$0 \to R'/IR' \cap J' \to R'/IR' \oplus R'/J' \to R'/IR' + J' \to 0.$$

From this, we get

$$\operatorname{reg} R/I = \operatorname{reg} R'/IR' \leq \max\{\operatorname{reg}(R'/IR' \cap J'), \operatorname{reg}(R'/IR' + J')\}.$$

Since $IR' \cap J' = (\alpha_1, \dots, \alpha_r)$ is a complete intersection, $\operatorname{reg}(R'/IR' \cap J') = \sigma$. We just need to bound $\operatorname{reg}(R'/IR' + J')$. Note that $IR' + J' = (f_1, \dots, f_t, \beta_1, \dots, \beta_d)$ and $\operatorname{ht}(IR' + J') = r + 1$. By assumption of $d_1 \geq 2$, we have $\sigma \geq d_{r+1}$.

If $IR' + J'$ is a complete intersection, then some $r + 1$ generators will be a regular sequence. Assume that $f_{i_1}, \dots, f_{i_p}, \beta_{j_1}, \dots, \beta_{j_q}$ are such generators where $p + q = r + 1$. Then

$$\operatorname{reg} R'/IR' + J' = \sum_{\eta=1}^{p}(\deg f_{i_\eta} - 1) + \sum_{\mu=1}^{q}(\deg \beta_{i_\mu} - 1).$$

If $p \leq r$, then we can get $\operatorname{reg} R'/IR' + J' \leq \sigma + d(\sigma - 1) \leq \frac{(d+1)!}{2}(\sigma - 1)$. Otherwise $p = r + 1$, then we still have $\operatorname{reg} R'/IR' + J' \leq \sigma + \sigma - 1 \leq \frac{(d+1)!}{2}(\sigma - 1)$.

If $IR' + J'$ is not a complete intersection, then let $f_{i_1}, \dots, f_{i_p}, \beta_{j_1}, \dots, \beta_{j_q}$ be $r + 1$ highest degree generators. By Proposition 4.1, $Z = \operatorname{Proj} R'/IR' + J'$ is local complete intersection log canonical, then we use induction for $IR' + J'$ to get

$$\operatorname{reg} R'/IR' + J' \leq \frac{d!}{2}\left(\sum_{\eta=1}^{p}(\deg f_{i_\eta} - 1) + \sum_{\mu=1}^{q}(\deg \beta_{i_\mu} - 1) - 1\right).$$

If $p \leq r$, then the left part of the equality is $\leq \frac{d!}{2}(\sigma + d(\sigma - 1) - 1) \leq \frac{(d+1)!}{2}(\sigma - 1)$. If $p = r + 1$, then the left part is $\leq \frac{d!}{2}(\sigma + d_{r+1} - 1 - 1) \leq \frac{d!}{2}(\sigma + \sigma - 1 - 1) \leq \frac{(d+1)!}{2}(\sigma - 1)$. Hence we still obtain

$$\operatorname{reg} R'/IR' + J' \leq \frac{(d+1)!}{2}(\sigma - 1).$$

This proves the result for the case $r \geq 2$.

Case of $r = 1$. There is an homogeneous form l and an homogeneous ideal H such that $f_i = lh_i$, $I = lH$ and $H = (h_1, \dots, h_t) = [I : l]$. Since X is a local complete intersection and normal, $\operatorname{ht} H \geq n$. Also by assumption of $d \geq 2$ we have $n \geq 2$. We consider the following two cases for n.

$n = 2$, then $R = k[x_0, x_1, x_2]$, $\operatorname{ht} I = 1$. Applying [3, Proposition 2.2], we get $\operatorname{reg} R/I \leq 3(\sigma - 1)$, unless $R = k[x_0, x_1, x_2]$, l is a linear form and H a complete intersection of 3 forms of degree $d_1 - 1$, in which case $\operatorname{reg} R/I = 3d_1 - 5$.

$n \geq 3$, then $d = n$. We first note that we have the inequality

$$\deg l + \sum_{i=1}^{n+1}(\deg h_i - 1) \leq \frac{(n+1)!}{2}(\sigma - 1).$$

If $\operatorname{ht} H = n + 1$, then $\dim R/H = 0$, and thus we have $\operatorname{reg} R/H \leq \sum_{i=1}^{n+1}(\deg h_i - 1)$, from which we get $\operatorname{reg} R/I = \operatorname{reg} R/(l) + \operatorname{reg} R/H \leq \frac{(n+1)!}{2}(\sigma - 1)$. If $\operatorname{ht} H = n$ and H is a complete intersection, it is easy to see $\operatorname{reg} R/I \leq \frac{(n+1)!}{2}(\sigma - 1)$. If $\operatorname{ht} H = n$ and H is not a complete intersection, then by [3, Proposition 2.1], $\operatorname{reg} R/H \leq \sum_{i=1}^{n+1}(\deg h_i - 1)$. So we still obtain $\operatorname{reg} R/I \leq \frac{(n+1)!}{2}(\sigma - 1)$. □

Remark 4.3. It is well known that if I is a complete intersection, then $\operatorname{reg} R/I \leq \sigma$. Including the situation of a complete intersection in the theorem above, we get Theorem 1.1 in the Introduction.

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