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Segment endpoint visibility graphs are Hamiltonian

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Abstract

We show that the *segment endpoint visibility graph* of any finite set of disjoint line segments in the plane admits a simple Hamiltonian polygon, if not all segments are collinear. This proves a conjecture of Mirzaian.

Keywords: Line segments; Visibility graph; Hamiltonian graph

1. Introduction

The *segment endpoint visibility graph* $\mathrm{Vis}(S)$ is defined for a set S of n disjoint closed line segments in the plane. Its vertices are the $2n$ segment endpoints; two vertices a and b are connected by an edge, if and only if the corresponding line segment ab is either in S (which we call *segment edges*) or if the open segment ab does not intersect any (closed) segment from S (*visibility edges*). See Fig. 1 for an example. Note that this graph is different from the *segment visibility graph*, where vertices correspond to segments and an edge connects two vertices, if and only if some points of the two segments are mutually "visible".

Visibility graphs of disjoint objects or vertices/sides of polygons are fundamental structures in computational geometry [2,11]. They have applications in shortest path computation, motion planning, art gallery problems, but also in VLSI design, and computer graphics. The characterization and recognition problem of visibility graphs are also of independent interest. Visibility concerning disjoint line segments in the plane is basic, and problems for more complex objects can often be reduced to or approximated by this structure.

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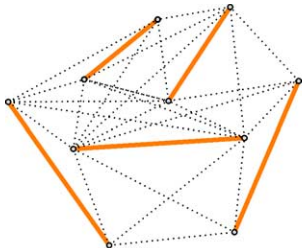

Fig. 1. A segment endpoint visibility graph; visibility edges are drawn as dotted segments.

1.1. Previous works and main theorem.

Segment endpoint visibility graphs have been subject to extensive research. The number of edges [14, 18], the computational complexity [7,9,13,15,20], storage space [1,5], and on-line updates [6] have been studied for this class of graphs over the past decade.

We are interested in the following problem that was originally formulated by Mirzaian [10] and later reposed by Bose [4]: How short can the longest circuit be in a segment endpoint visibility graph? More precisely, what is the maximal number $f(n)$ such that any segment endpoint visibility graph on n segments has a circuit of size $f(n)$?

If all segments lie on one line then, clearly, $f(n) = 0$. Otherwise, one can show using triangulations that $f(n) = \Omega(\sqrt{n})$, but no non-trivial upper bound was known so far. In fact, it was conjectured [10] that $f(n) = 2n$, i.e., there is always a Hamiltonian circuit in a segment endpoint visibility graph. We prove in this paper the following stronger version of the conjecture.

Theorem 1. *For any set of pairwise disjoint line segments, not all in a line, there exists a Hamiltonian polygon.*

Here, for a given set S of pairwise disjoint line segments, a *Hamiltonian polygon* is a simple polygon whose vertices are exactly the endpoints of the line segments and whose sides correspond to edges of $\mathrm{Vis}(S)$.

Previously, Theorem 1 was shown to hold for a few special cases: Mirzaian [10] proved it for *convexly independent* segments, that is, where every line segment has at least one endpoint on the boundary of the convex hull; and O'Rourke and Rippel [12] proved it for segments where no segment is crossed by the supporting line of any other segment. (Two segments or lines cross, iff there is a common point in the relative interior of both.)

Hamiltonian polygons with special properties, however, do not necessarily exist: There are sets of line segments for which there is no *circumscribing* Hamiltonian polygon, that is, a Hamiltonian polygon whose closure contains all the segments [19]. Similarly, there is not always an *alternating* Hamiltonian polygon for a set S of segments, that is, a Hamiltonian polygon in which every line segment of S is a side. It is NP-complete to decide whether a set S admits an alternating Hamiltonian polygon, if the segments of S are allowed to intersect at endpoints [16], although it can be decided efficiently in some special cases [17].

1.2. Applications

An immediate consequence of Theorem 1 is a recent result of Bose, Houle and Toussaint [3]. They show that for every set S of disjoint line segments, the segment endpoint visibility graph contains an *encompassing tree*, which is defined as a planar embedding of a tree with maximal degree three that contains all segment edges. Indeed, a Hamiltonian polygon together with all segment edges forms a planar spanning subgraph H of $\mathrm{Vis}(S)$ with maximum degree three. Contracting the segment edges in H and finding a spanning tree of the resulting graph, gives an encompassing tree for S.

Using the existence of a Hamiltonian polygon, we could also show recently [8] that there is always an *alternating path* (segment edges and visibility edges in alternating order) of length $\Omega(\log n)$ in the segment endpoint visibility graph of n disjoint line segments.

1.3. Proof technique

We build a Hamiltonian polygon P algorithmically, starting from the convex hull $\mathrm{conv}(S)$ (Fig. 2(a)). The polygon P is then successively extended to pass through more segment endpoints. As a first phase, the second endpoints of those segments for which one endpoint is already on the convex hull, are included; this yields a new proof of Mirzaian's theorem for convexly independent segments [10] (Fig. 2(b)).

In a second phase, P is extended to some of the segments in its interior (Fig. 2(c)), and we create a convex subdivision of P. Once certain conditions (Lemma 3) are fulfilled, a simple induction completes the proof (Fig. 2(d) and 2(e)).

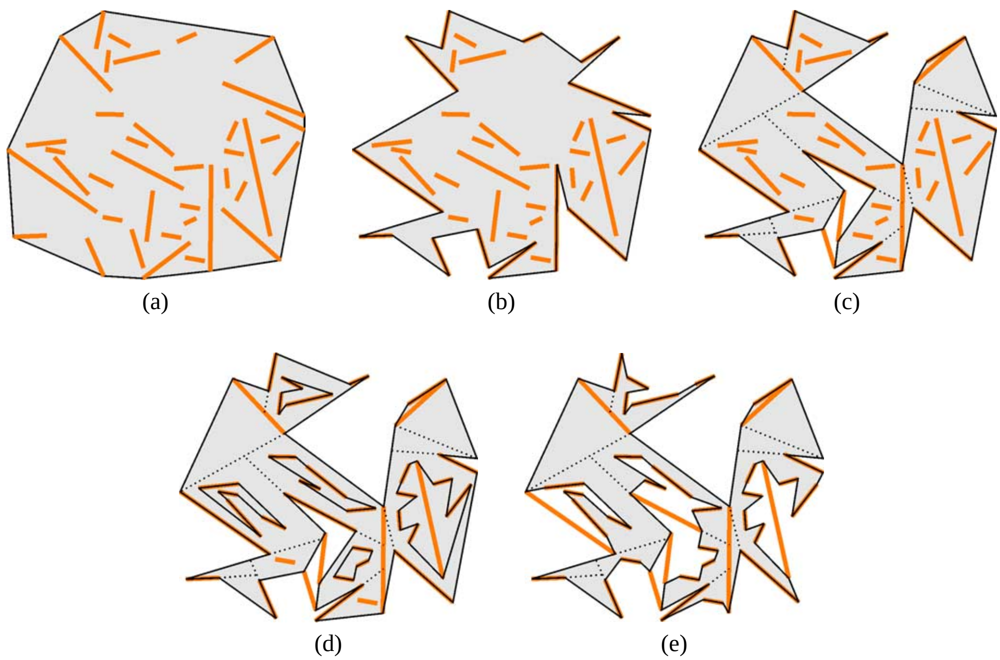

Fig. 2. Steps in the proof of Theorem 1.

Every step of the algorithm and every operation relies only on elementary geometry, like ray shooting, convex hull, or sorting angles. Based on our proof, it is straightforward to give an $O(n \log n)$ algorithm to find a Hamiltonian polygon for a given set of line segments. This running time is asymptotically optimal, as was shown by Bose et al. [3] for finding an encompassing tree; such a tree can be obtained from a Hamiltonian polygon in linear time, as explained above.

The rest of the paper is organized as follows. In Section 2, we prove Theorem 1 by induction. The key lemma of the proof, Lemma 3, is proved algorithmically in three phases. Section 3 gives some basic operations of our algorithm, Section 4 provides a new proof of the theorem of Mirzaian [10] and explains the first phase of our algorithm. The second phase and the complete algorithm are discussed in Sections 5 and 6.

2. Proof of Theorem 1

Given a set S of disjoint line segments in the plane, denote by $V(S)$ the set of segment endpoints from S. A *simple polygon* P is defined as a closed region in the plane enclosed by a simple closed polygonal curve ∂P consisting of a finite number of line segments. Let $V(P)$ denote the set of vertices of P.

Definition 2. A simple polygon P is a *Hamiltonian polygon* for S, if $V(P) = V(S)$ and the sides of P correspond to edges of $\mathrm{Vis}(S)$.

We say that a finite set $\mathcal{D}$ of pairwise non-overlapping simple polygons is a *dissection* of P, if $P = \bigcup_{D \in \mathcal{D}} D$. (Two polygons overlap, if there is a common point in the relative interior of both.) The following lemma is crucial in our argument, as it establishes Theorem 1 by a simple induction.

Lemma 3. *For a set S of disjoint line segments, not all in a line, and a side yz of* $\mathrm{conv}(S)$*, there is a simple polygon P whose sides correspond to edges of* $\mathrm{Vis}(S)$ *and a dissection $\mathcal{D}$ of P satisfying the following properties.*

(L1) *yz is a side of P;*
(L2) *for every $s = pq \in S$, either $s \subset \mathrm{int}(P)$ or $\{p, q\} \subset V(P)$;*
(L3) *for every $s \in S$, if $s \subset \mathrm{int}(P)$ then there is a $D \in \mathcal{D}$ such that $s \subset \mathrm{int}(D)$, otherwise $s \cap \mathrm{int}(D) = \emptyset$ for all $D \in \mathcal{D}$;*
(L4) *every polygon $D \in \mathcal{D}$ is convex;*
(L5) *every polygon $D \in \mathcal{D}$ has a common side with P which is different from yz.*

We prove Lemma 3 in the remaining sections assuming that the line segments are in *general position*, i.e., there are no three collinear segment endpoints. The extension for the case where some, but not all, segment endpoints are collinear will be indicated in Remark 10.

The outline of the proof is as follows. We start with $P := \mathrm{conv}(S)$ and $\mathcal{D} := \{P\}$ which together satisfy already (L1) and (L5). In the following, the polygon P and the set D are modified such that these properties are maintained and $V(P)$ never decreases. In a first phase, property (L2) is established by including the second endpoints of those segments for which one endpoint is already in $V(P)$. Then a simple dissection by diagonal segments assures (L3). Finally, during a second phase the dissection $\mathcal{D}$ is refined until all sets in $\mathcal{D}$ are convex, as demanded in (L4).

Proof of Theorem 1. We prove by induction the following statement. For a set S of disjoint line segments, not all in one line, and for any fixed side yz of the polygon conv(S), there is a Hamiltonian polygon H for S such that yz is a side of H.

The statement holds for $|S| = 2$. Suppose it holds for all S' with $1 < |S'| < |S|$.

Consider the simple polygon P and the set $\mathcal{D}$ of polygons described in Lemma 3. If both endpoints of every segment are in $V(P)$, then the statement holds. If there is a segment s whose neither endpoint is in $V(P)$, then by properties (L2) and (L3), s is in the interior of some $D \in \mathcal{D}$. By property (L5), D has a common side $ab \neq yz$ with P. By (L3) and (L4), $C(D) := \text{conv}(S \cap \text{int}(D)) \subset \text{int}(D)$. Moreover, $C(D)$ has a side cd such that both ac and bd are visibility edges. If $c_1d_1, c_2d_2, \ldots, c_md_m$, $m \geqslant 1$, are the segments in $\text{int}(D)$ and they are all collinear in this order, then replace the side ab of P by the path $ac_1d_1c_2d_2 \ldots c_md_mb$. Otherwise there is, by induction, a Hamiltonian polygon $H(D)$ for $S \cap \text{int}(D)$ such that cd is a side of $H(D)$. Replace the side ab of P by the path $(a, c) \oplus (\partial H(D) \setminus cd) \oplus (d, b)$. Doing so for each $D \in \mathcal{D}$ that contains segments from S results in a Hamiltonian polygon (see Fig. 2(e)). (For two polygonal arcs $A = (a_1, \ldots, a_k)$ and $B = (b_1, \ldots, b_\ell)$ with $a_k = b_1$, we denote by $A \oplus B$ the concatenation $(a_1, \ldots, a_k, b_2, \ldots, b_\ell)$ of A and B.) $\square$

3. Basic definitions and operations

Our goal is to find a simple polygon satisfying the conditions of Lemma 3. In order to construct such a polygon, we run an algorithm which, in each step, makes local changes to our polygon, that is, replaces one edge by a path or two consecutive edges by one edge.

This algorithm, however, leads out from the family of simple polygons. Therefore, we will use a slightly more general definition for polygons, such that the boundary of a polygon may have self-intersections but no self-crossings.

Definition 4. Consider a simply-connected closed region P in the plane which is the image of the unit disc under a continuous mapping ϱ. P is a *polygon*, if its boundary ∂P is the image of the unit circle under ϱ and consists of finitely many pairwise non-crossing line segments.

The endpoints of the segments on ∂P are called *vertices* of P. Let $P_\circlearrowleft$ denote the cyclic sequence of vertices of P along ∂P in counterclockwise order. The *sides* of the polygon are the segments connecting two consecutive vertices of $P_\circlearrowleft$ along ∂P.

The image of any arc A of the unit circle under ϱ is called *polygonal arc* of ∂P. A polygonal arc is *simple*, if ϱ is injective on A.

Observe that a vertex from $V(P)$ can appear several times in $P_\circlearrowleft$. We define the *multiplicity* $m_P(U)$ for a set $U \subset V(P)$ of vertices to be the number of occurrences of vertices from U in $P_\circlearrowleft$.

Definition 5. We say that an angle α is convex, strictly convex, reflex or flat, if $\alpha \leqslant \pi$, $\alpha < \pi$, $\alpha > \pi$ or $\alpha = \pi$, respectively. For three points a, b and c, denote by $\angle abc$ the angle between the rays $\overrightarrow{ba}$ and $\overrightarrow{bc}$, measured counterclockwise.

For $a \in P_{\circlearrowleft}$, denote by a^+ (respectively a^-) the next vertex of $P_{\circlearrowleft}$ in counterclockwise (respectively clockwise) direction. We call an occurrence of a vertex a in $P_{\circlearrowleft}$ convex (reflex), if $\angle_P a := \angle a^+ a a^-$ is convex (reflex). Similarly to $m_P(U)$, for a set $U \subset V(P)$ of vertices define $r_P(U)$ to be the number of reflex occurrences of vertices from U in $P_{\circlearrowleft}$. For a single vertex $v \in V(P)$ we simply write $m_P(v)$ for $m_P(\{v\})$ and $r_P(v)$ for $r_P(\{v\})$.

In order to be sure that we can apply certain operations to a polygon, a few additional properties are required; we summarize them under the concept of *frame* polygons defined below. All through our algorithm, we make sure that the intermediate polygons belong to this class.

Definition 6. A polygon P is called *frame* for a set S of disjoint line segments, if

(F1) $V(S) \subset P$ and $V(P) \subset V(S)$;
(F2) ∂P does not cross any segment from S;
(F3) $m_P(v) \leqslant 2$ for every vertex $v \in V(P)$;
(F4) if $m_P(v) = 2$ for $v \in V(P)$, then the angular domain around v intersects $\mathrm{int}(P)$ in two convex angles (that is, if $P_{\circlearrowleft} = (\ldots avb \ldots cvd \ldots)$, then both $\angle dva$ and $\angle bvc$ are convex, with possibly $a = d$ or $b = c$);
(F5) if $v \in V(P)$, and $u \in \mathrm{int}(P)$ for some $uv \in S$, then $m_P(v) = 1$ but $r_P(v) = 0$.

For example, Fig. 3(a) shows a frame, while the polygons in Fig. 3(b) ($\alpha > \pi$), 3(c) (crosses a segment), and 3(d) (violates (F5)) are not frames. The convex hull $\mathrm{conv}(S)$ is always a frame for S.

The idea behind allowing $P_{\circlearrowleft}$ to visit a vertex v twice is that we hope to eliminate one occurrence at the end of our algorithm. This can actually be done easily, if v appears in $P_{\circlearrowleft}$ once as a *cap* defined below.

Definition 7. Let $k \in \mathbb{N}$ and $(a, b_1, b_2, \ldots, b_k, c)$ be a sequence of consecutive vertices in $P_{\circlearrowleft}$ such that b_i, $i = 1, \ldots, k$, are reflex vertices and $\mathrm{int}(\mathrm{conv}(a, b_1, \ldots, b_k, c)) \cap S = \emptyset$. Then the sequence $(b_1, b_2, \ldots, b_k)$ is called *cap*. If $k = 1$, we usually omit the parentheses.

A reflex vertex of $P_{\circlearrowleft}$ that is not a cap is called *anti-cap*.

A sequence $(a, b_1, b_2, \ldots, b_k, c)$ of consecutive vertices in $P_{\circlearrowleft}$ is called *wedge*, if $m_P(b_i) = 2$, for all $i = 1, 2, \ldots, k$, and $(b_1, b_2, \ldots, b_k)$ is a cap.

Assuming that every sequence of double occurrences in $P_{\circlearrowleft}$ corresponds to a wedge, it is easy to create a simple polygon from a frame P by the following operation.

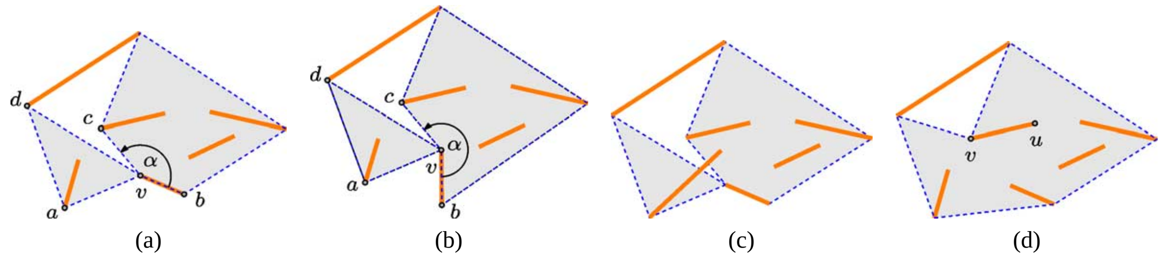

Fig. 3. Examples for (non-)frames.

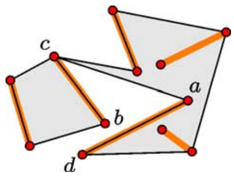

Fig. 4. One occurrence of vertex $c \in P_{\circlearrowleft}$ forms a cap and (b, c, a) is a wedge; a is an anti-cap, since segment bc intersects triangle $\triangle(cda)$.

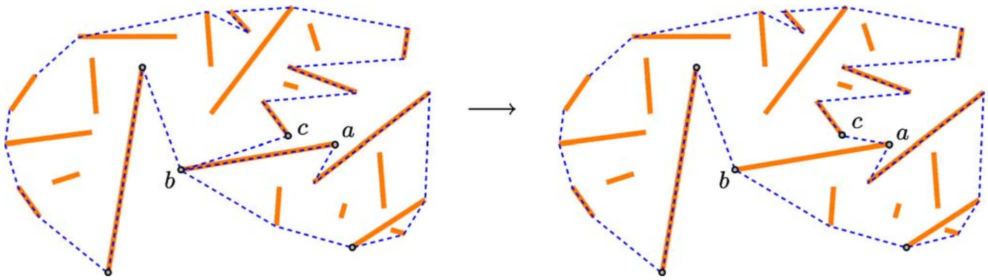

Fig. 5. Chopping the wedge (a, b, c).

Operation 1 (*Chop_wedges*(P)) (Fig. 5).
Input: a frame P.
Operation: As long as there is a wedge $(a, b_1, b_2, \ldots, b_k, c)$,

Replace the path $(a, b_1, b_2, \ldots, b_k, c)$ in $P_{\circlearrowleft}$ by the single edge ac.

Output: P.

Proposition 8. *The output of Chop_wedges is a frame.*

In order to create a simple polygon from a frame P, it is crucial to have a hold on the vertices with multiplicity two in $P_{\circlearrowleft}$. It is easy to see that a polygon cannot have two strictly convex angles at a vertex of multiplicity two. The following proposition states a stronger property for frames assuming that the segment endpoints are in general position.

Proposition 9. *Let S be a set of line segments in general position. Any frame P for S has the following property*:

(F6) *If $v \in V(P)$ is a vertex with $m_P(v) = 2$, then $r_P(v) \geqslant 1$.*

Proof. Let $P_{\circlearrowleft} = (\ldots, a, b, c \ldots, d, b, e, \ldots)$ such that $\angle cba$ is convex. The general position assumption assures that $\angle cba$ is strictly convex. As P is a polygon, i.e., it is simply connected, the edges bd and be must lie in the angular domain $\angle cba$, therefore $\angle ebd$ is reflex, as drawn in Fig. 6. □

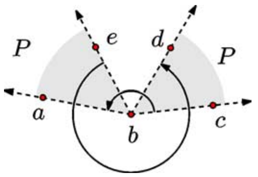

Fig. 6. Illustration for Proposition 9.

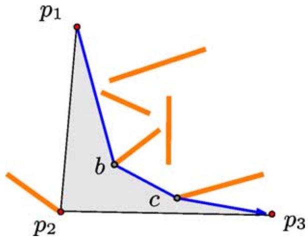

Fig. 7. Example: $\mathrm{carc}(p_1, p_2, p_3) = (p_1, b, c, p_3)$.

Remark 10. In the rest of this paper we assume that the segment endpoints are in general position. A complete proof of Lemma 3, of course, cannot use Proposition 9. We may state instead another property:

(F6′) If $v \in V(P)$ is a vertex with $m_P(v) = 2$, then there is a sequence $s = (b_1, b_2, \ldots, b_m)$, $m \geqslant 1$ containing v such that both s and $s^R = (b_m, b_{m-1}, \ldots, b_1)$ are sequences of consecutive vertices in $P_{\circlearrowleft}$; moreover, $\angle_P b_1$ and $\angle_P b_m$ are reflex in the same sequence (s or s^R), and $b_2, b_3, \ldots, b_{m-1}$ are flat in both s and s^R.

It can be shown that property (F6′) is maintained during our algorithm, even if there are collinearities. Using this property and checking all possible degenerate cases throughout the argument, the proof can be extended to establish Lemma 3 in its general form.

3.1. Including second segment endpoints

Our first objective is to ensure property (L2). The method is really simple: We start with the convex hull of S; whenever there is a line segment s whose one endpoint is in $V(P)$ but the other is not, we extend the polygon locally to visit the other endpoint as well. This extension can be done in two different ways, which will be determined by an orientation defined as follows.

Definition 11. Consider a simple polygonal arc $A = (p_1, p_2, p_3)$ that does not cross any segment from S. Define the *convex arc* $\mathrm{carc}(p_1, p_2, p_3)$ of A to be the shortest polygonal arc from p_1 to p_3 such that there is no segment endpoint in the interior of the closed polygonal curve $\mathrm{carc}(p_1, p_2, p_3) \oplus (p_3, p_2, p_1)$. (See Fig. 7.)

If p_1, p_2 and p_3 are not collinear, then $\mathrm{carc}(p_1, p_2, p_3) \oplus (p_3, p_2, p_1)$ is a *pseudo-triangle* where all internal vertices of $\mathrm{carc}(p_1, p_2, p_3)$ are reflex.

Definition 12. For a polygon P, an *orientation* $u(P)$ is a function $u : P_{\circlearrowleft} \to \{-, +\}$.

Operation 2 (*Build_cap*(P, u, a)) (Fig. 8).
Input: a frame P, an orientation $u(P)$, and a convex vertex $a \in P_{\circlearrowleft}$ such that $b \notin V(P)$, for the vertex $b \in V(S)$ with $ab \in S$.
Operation: Let $c := a^{u(a)}$.
Obtain P' from P by replacing the edge ac by the path $ab \oplus \mathrm{carc}(b, a, c)$.

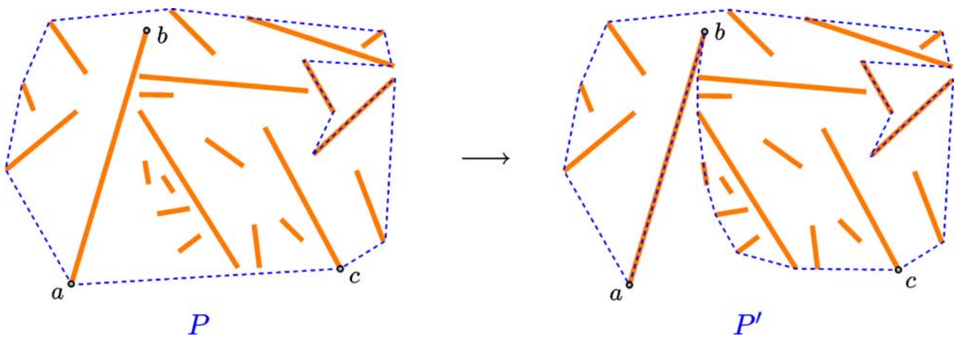

Fig. 8. Build_cap(P, u, a) with $u(a) = +$.

Set $u(p) := u(a)$ for all p on carc(b, a, c).
Output: (P', u).

Observe that $r_{P'}(V(P')) = 1 + r_P(V(P))$, since *Build_cap* produces exactly one new reflex vertex: at b. Note also that P' is not necessarily simple, since some of the vertices from carc(b, a, c) might already have been in $V(P)$.

Proposition 13. *The output P' of Build_cap is a frame.*

Proof. We have to check properties (F1)–(F5). (F1) and (F2) follow directly from the definition of carc and from the fact that the input polygon P is a frame.

Let carc$(b, a, c) = (b = p_0, \dots, p_k = c)$ for some $k \in \mathbb{N}$. Build_cap inserts vertices $p_0, \dots, p_{k-1}$ into $P_{\circlearrowleft}$. Obviously, $m_{P'}(b) = 1$ and $m_{P'}(a) = m_P(a) = 1$ by property (F5); also, the vertices $p_1, \dots, p_{k-1}$ are inserted as convex vertices, that is, $r_{P'}(p_i) = r_P(p_i)$ for any p_i, $i = 1, 2, \dots, k$. This immediately implies that P' has properties (F4) and (F5).

For (F3), we argue by contradiction. Suppose that $m_P(p_i) = 2$ for some $i \in \{1, \dots, k-1\}$. By (F4), the angular domain around p_i intersects int(P) in two convex angles. So by definition of carc, p_i cannot be on carc(b, a, c). □

Operation 3 (*Both_endpoints*(P, u)).
Input: a frame P and an orientation $u(P)$.
Operation: As long as there exists an $a \in P_{\circlearrowleft}$ such that $ab \in S$ and $b \notin V(P)$,
let $(P, u) \leftarrow$ *Build_cap*(P, u, a).
$P' \leftarrow P$.
Output: (P', u).

Proposition 14. *Both_endpoints does not create any anti-cap (that is, every anti-cap in $P'_{\circlearrowleft}$ is already an anti-cap in $P_{\circlearrowleft}$). Sequences of consecutive caps in $P'_{\circlearrowleft}$ form one cap, if the same was true for $P_{\circlearrowleft}$.*

Proof. Let carc$(b, a, c) = (b = p_0, \dots, p_k = c)$ for some $k \in \mathbb{N}$. *Build_cap* produces exactly one new reflex vertex: b. Vertex b is a cap, because int$(\triangle(abp_1)) \cap S = \emptyset$ by construction.

By property (F5), $r_P(a) = r_{P'}(a) = 0$. In fact, all the other new vertices are convex as well, i.e., $r_{P'}(\{a, p_1, \dots, p_{k-1}\}) = r_P(\{a, p_1, \dots, p_{k-1}\})$. Hence, there is nothing more to show, if $k > 1$. So let us consider the case $k=1$, that is, $\mathrm{carc}(b,a,c)=bc$. Suppose that c is a reflex vertex of $P_{\circlearrowleft}$, which is part of a cap $(c=c_1,c_2,\ldots,c_r)$; in particular, this implies $\mathrm{int}(\mathrm{conv}(\{a,c_1,c_2,\ldots,c_r,d\}))\cap S=\emptyset$, where d is the other ($\neq c_{r-1}$) neighbor of c_r in $P_{\circlearrowleft}$. If $\angle bcd>\pi$, then c appears as a convex vertex in $P'_{\circlearrowleft}$. Otherwise, we have $\mathrm{int}(\mathrm{conv}(\{a,b,c_1,c_2,\ldots,c_r,d\}))\cap S=\emptyset$ and $(b,c_1,c_2,\ldots,c_r)$ is a cap in $P'_{\circlearrowleft}$. □

4. Convexly independent segments and more

In this section, we describe a simple algorithmic proof for the case where S is a set of convexly independent segments. The procedure then serves as a base step to our main algorithm (Algorithm 2) for arbitrary S.

Algorithm 1.
Input: a set S of disjoint line segments and an orientation u for the vertices of $\mathrm{conv}(S)$.

(1) $P\leftarrow\mathrm{conv}(S)$.
(2) $(P',u)\leftarrow\mathrm{Both\_endpoints}(P,u)$.
(3) $P''\leftarrow\mathrm{Chop\_wedges}(P')$.

Output: P''.

Proposition 15. *The output P'' of Algorithm* 1 *is a simple frame with property* (L2).

Proof. Property (L2) follows from the loop condition in *Both_endpoints*, Proposition 13, and the fact that *Chop_wedges* does not alter the set of visited vertices. P'' is simple because, by Proposition 9, for every vertex v with $m_{P'}(v)>1$, we have $r_{P'}(v)\geqslant 1$. Proposition 14 tells us that every sequence of consecutive reflex vertices in $P'_{\circlearrowleft}$ forms a cap, and thus all repetitions in $P'_{\circlearrowleft}$ are deleted by *Chop_wedges*. □

Corollary 16 [10]. *If the line segments of S are convexly independent and in general position, then Algorithm* 1 *outputs a Hamiltonian polygon for any orientation u of the vertices of* $\mathrm{conv}(S)$.

Note that we did not make any use of the orientation u for the proof of Corollary 16. We could simply run Algorithm 1 with a uniform orientation $u\equiv +$. But in this case we cannot guarantee that a prescribed side yz of $\mathrm{conv}(S)$ is a side of the output polygon, as required in (L1).

Suppose that y precedes z in $\mathrm{conv}(S)_{\circlearrowleft}$. Define the orientation u_{yz} of $\mathrm{conv}(S)$ by $u_{yz}(y)=-$, and $u_{yz}(v)=+$ for any other vertex $v\in\mathrm{conv}(S)_{\circlearrowleft}$.

Proposition 17. *If Algorithm* 1 *is applied to S with the orientation u_{yz}, then the output P'' is a simple frame satisfying properties* (L1) *and* (L2).

Proof. Segment yz is a side of $P=\mathrm{conv}(S)$, and none of the *Build_cap* operations replaces yz by something else. Moreover, both y and z remain convex vertices throughout *Both_endpoints*. Since *Chop_wedges* does only cut off edges adjacent to reflex vertices, the edge yz remains part of P'' as well. □

Proposition 18. *If Algorithm 1 is applied to S with orientation u_{yz}, then the output P'' has at most one cap with exactly two reflex vertices* (*double-cap*); *all other caps consist of exactly one reflex vertex.*

Proof. An operation *Build_cap*(P, u, a) creates exactly one new reflex vertex, namely at b where $ab \in S$. Let $c := a^{u(a)}$. As in Proposition 14, we can have two consecutive reflex vertices only if $\mathrm{carc}(b, a, c) = bc$, and if c is a reflex vertex of P. Assuming this scenario, the reflex vertex c is created in a previous operation *Build_cap*$(\widetilde{P}, \tilde{u}, d)$ such that in $\widetilde{P}$ we had $a^{\tilde{u}(a)} = d$, $d^{\tilde{u}(d)} = a$ and $\mathrm{carc}(c, d, a) = ca$. This already implies that there is no cap of three consecutive vertices in $P'_{\circlearrowleft}$.

A pair $a^{u(a)} = d$, $d^{u(d)} = a$ corresponds to a subsequence $(+, -)$ in an orientation u along $P_{\circlearrowleft}$. The orientation u_{yz} has exactly one subsequence $(+, -)$ throughout Algorithm 1, since *Build_cap* does not induce alternations in the orientation. Thus, there is at most one *double cap* in $P'_{\circlearrowleft}$. □

5. Dissecting P

Consider the frame P produced by *Both_endpoints*. Recall that P is not necessarily simple, since it may have multiple vertices at *wedges*. We call a diagonal ab of P *segment diagonal*, if $ab \in S$. By cutting P at wedges and along segment diagonals, we obtain a dissection $\mathrm{Diss}(P)$ into simple polygons (Fig. 9). Observe that $\mathrm{Diss}(P)$ satisfies property (L3).

Unfortunately, the polygons of $\mathrm{Diss}(P)$ are not necessarily convex. A first idea to obtain a dissection into convex polygons from $\mathrm{Diss}(P)$ is the following: for every $D \in \mathrm{Diss}(P)$ draw consecutively rays from every reflex vertex b of D dissecting $\angle_D b$ into two convex angles, until the ray hits the boundary of D or a previously drawn ray. If no ray crosses a segment of $S \cap \mathrm{int}(D)$, then they dissect D into non-overlapping convex regions satisfying properties (L2), (L3) and (L4). The resulting partition depends on the order in which the rays are drawn, but any order would do at this point. But if any of the rays crosses a segment $s \in S$, such a partitioning would not grant (L3). In this case, we extend P to incorporate s by means of two new basic operations that are introduced below.

5.1. Extension to interior segments

Definition 19. Consider a simple polygonal arc (a, b, c, d) that does not cross any segment from S. Denote by $\mathrm{marc}(a, b, c, d) = (a = p_0, \ldots, p_k = d)$, for some $k \in \mathbb{N}$, the shortest polygonal arc from a to d such that there is no segment endpoint in the interior of the closed polygonal curve $M = \mathrm{marc}(a, b, c, d) \oplus (d, c, b, a)$.

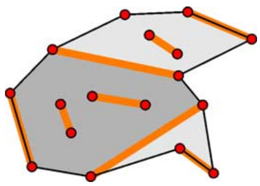

Fig. 9. This frame is dissected into three polygons by $\mathrm{Diss}(P)$.

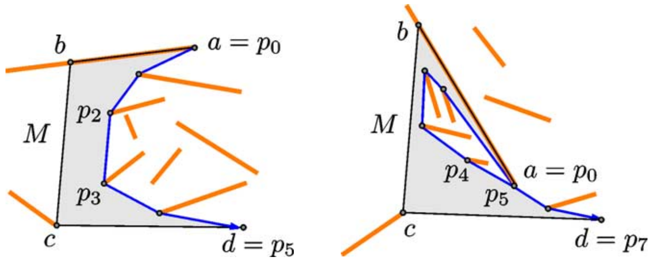

Fig. 10. marc(a, b, c, d) for convex and concave quadrilaterals $abcd$.

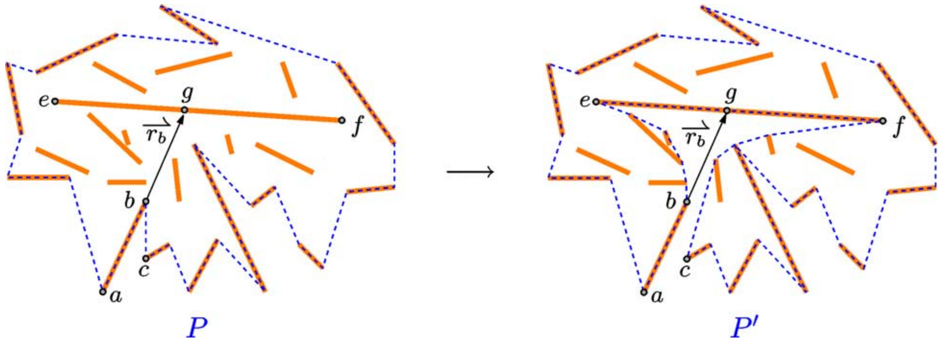

Fig. 11. *Extend_reflex*$(P, u, \mathcal{D}, b, c, \overrightarrow{r_b})$ to a segment ef.

M has reflex vertices at $p_1, \ldots, p_{k-1}$, but—in contrast to carc—it is not necessarily simple: a or d may occur twice on the arc, see Fig. 10(b).

Operation 4 (*Extend_reflex*$(P, u, \mathcal{D}, b, c, \overrightarrow{r_b})$ (Fig. 11).
Input: a frame P along with an orientation $u(P)$, a dissection $\mathcal{D}$ of P, a reflex vertex b of some $D \in \mathcal{D}$, a vertex c, and a ray $\overrightarrow{r_b}$ emanating from b.
Preconditions: bc is a common side of D and P, $\overrightarrow{r_b}$ cuts $\angle_D b$ into two convex angles, $r_D(c) = 0$, and $\overrightarrow{r_b}$ hits[2] the segment $ef \subset \mathrm{int}(D)$ at a point g. We may suppose that c and f are on the same side of the supporting line of $\overrightarrow{r_b}$.
Operation: Obtain P' from P and D' from D by replacing the edge bc by the path carc$(b, g, e) \oplus (e, f) \oplus$ marc(f, g, b, c). Split D' into simple polygons in $\mathcal{D}$ if necessary. Set $u(\cdot) := -$ for all interior vertices of carc(b, g, e), and $u(\cdot) := +$ for all interior vertices of marc(f, g, b, c).
Output: $(P', u, \mathcal{D})$.

There are two variants of *Extend_reflex*, depending on whether c follows or precedes b in $P_{\circlearrowleft}$. We have described only the first above, and refer to this variant in the notation of Fig. 11 and Propositions 21–24. The other variant is completely symmetric.

[2] More precisely, the intersection of the open segment bg with $(S \cup \partial D)$ is empty.

Proposition 20. *Given a frame P for S, a dissection $\mathcal{D}$ of P, and a polygon $D \in \mathcal{D}$, we have $m_P(b) = 1$ for every $b \in V(P)$ with $r_D(b) = 1$.*

Proof. If $m_P(b) = 2$, then b cannot be a reflex vertex of any $D \in \mathcal{D}$ by property (F4). □

Proposition 21. *The output P' of* Extend_reflex *is a frame.*

Proof. Properties (F1) and (F2) follow directly from the definition of carc and marc and from the fact that P is a frame. For internal vertices of $\mathrm{carc}(b, g, e)$ and $\mathrm{marc}(f, g, b, c)$, one can argue as in Proposition 13. Hence, we have to consider the vertices b, c, e and f, only.

Since c is a convex vertex of D by assumption, it cannot appear twice on $\mathrm{marc}(f, g, b, c)$, even if it is a reflex vertex of the quadrilateral $fgbc$. Thus, f is the only vertex possibly visited twice by $\mathrm{marc}(f, g, b, c)$. Since $m_P(e) = m_P(f) = 0$, (F3) follows.

For (F4) note that $m_P(b) = m_{P'}(b) = 1$ (Proposition 20); if $m_{P'}(c) = 2$, then the convex angles at c described in (F4) cannot increase. Also f fulfills (F4), even if it appears twice on $\mathrm{marc}(f, g, b, c)$, since $\mathrm{marc}(f, g, b, c)$ is locally convex and the second (reflex) occurrence of f is inside this convex angle (look at vertex a in Fig. 10(b)). Finally, (F5) follows from the fact that the line segment adjacent to the two new reflex vertices, e and f, is $ef \subset \partial P'$. □

Next, we would like to prove an analog to Proposition 14 for *Extend_reflex*. Unfortunately, *Extend_reflex* can create anti-caps, but—fortunately—at most one. Recall that the problem with anti-caps is that they cannot be chopped off; hence, we have to make sure that $P_\circlearrowleft$ does not visit this anti-cap in a later step, for instance, along a convex arc constructed by a *Build_cap* operation. Therefore, whenever an anti-cap is created, we draw the next ray from this anti-cap, immediately reverting it into a convex vertex of two non-overlapping polygons in $\mathcal{D}$. For this purpose, we have to control carefully the number of anti-caps appearing in the course of our algorithm.

Proposition 22. Extend_reflex *creates at most one new anti-cap (that is, there is at most one more anti-cap in ${P'}_\circlearrowleft$ than in $P_\circlearrowleft$).*

Proof. Both b and c are convex vertices of D'. Compared to P, there are at most two new reflex vertices in P': e and f. We will show that at least one of e or f is a cap in ${P'}_\circlearrowleft$.

Let d be the second vertex of $\mathrm{carc}(e, g, b)$, and let h be the second vertex of $\mathrm{marc}(f, g, b, c)$ (possibly $d = b$ or $h = c$). If $\mathrm{int}(\triangle(fgb)) \cap S = \emptyset$, then by definition of carc also $\mathrm{int}(\triangle(fed)) \cap S = \emptyset$, and e is a cap. Otherwise, the rays $\overrightarrow{ed}$ and $\overrightarrow{fh}$ intersect in a point $v \in \triangle(feb)$ (Fig. 12). Since the edges ed and fh do not cross by definition, we have $d \in ve$ or $h \in vf$. In the first case df is a visibility edge and e is a cap, and in the second case he is a visibility edge and f is a cap. □

Corollary 23. *If $g = e$ in* Extend_reflex, *then f is a cap in ${P'}_\circlearrowleft$.*

If f appears twice on $\mathrm{marc}(f, g, b, c)$, we have to make sure that the reflex occurrence of f is a cap of ${P'}_\circlearrowleft$ that can be chopped off later. Fortunately, this is not hard to achieve: before applying *Extend_reflex*, we apply the following rotation to $\overrightarrow{r_b}$.

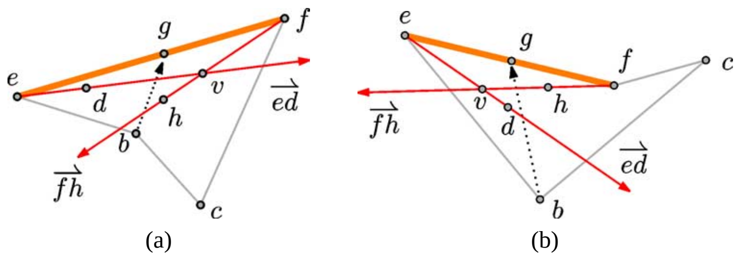

Fig. 12. Illustration for Proposition 22. (a) $fgbc$ is convex. (b) $fgbc$ is concave.

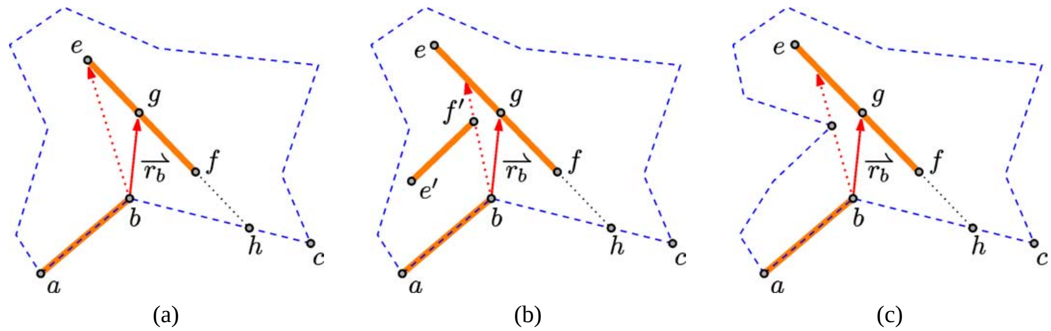

Fig. 13. The three possible outcomes of Rotate($\overrightarrow{r_b}, b, ef, D$).

Operation 5 (*Rotate*($\overrightarrow{r_b}, b, ef, D$)).
Input: a ray $\overrightarrow{r_b}$ emanating from b, a segment $ef \subset \operatorname{int} D$, and a polygon $D \in \mathcal{D}$.
Preconditions: b is a reflex vertex of D, $\overrightarrow{r_b}$ dissects $\angle_D b$ into two convex angles, $\overrightarrow{r_b}$ hits ef and ray $\overrightarrow{ef}$ hits a side of D incident to b.
Operation: Obtain $\overrightarrow{r_b}'$ by rotating $\overrightarrow{r_b}$ around b towards e, until it hits

- either e (Fig. 13(a))—Corollary 23 assures that f is a cap in this case;
- or the right endpoint f' of another segment $e'f' \subset \operatorname{int}(P)$ (Fig. 13(b))—Then we have $\operatorname{marc}(f', g' = f', b, c) = \operatorname{carc}(f', b, c)$, and e' is a cap in $P'_{\circlearrowleft}$;
- or a reflex vertex of D (Fig. 13(c))—We do not apply *Extend_reflex* here.
  Output: $\overrightarrow{r_b}'$.

Proposition 24. *The ray* $\overrightarrow{r_b}' = \operatorname{Rotate}(\overrightarrow{r_b}, b, ef, D)$ *cuts* $\angle_D b$ *into two convex angles.*

Proof. Let a and c denote the vertices of $D_{\circlearrowleft}$ adjacent to b. The ray $\overrightarrow{be}$ lies in the convex angle formed by the rays $\overrightarrow{ab}$ and $\overrightarrow{cb}$. Since reaching e is one of the stop conditions for the rotation of $\overrightarrow{r_b}$, therefore $\overrightarrow{r_b}$ stays in the convex angle formed by $\overrightarrow{ab}$ and $\overrightarrow{cb}$. □

5.2. *Common side for each $D \in \mathcal{D}$ and P*

If we just proceed to shoot rays from a reflex vertex of some $D \in \mathcal{D}$ and call *Extend_reflex* when applicable, we obtain a frame P and a dissection $\mathcal{D}$ of P fulfilling properties (L1)–(L4). Unfortunately,

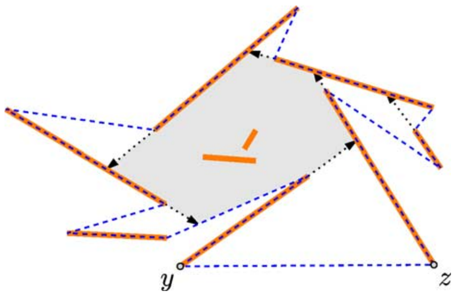

Fig. 14. The shaded polygon does not have a common side with the frame.

P and $\mathcal{D}$ do not necessarily have property (L5), as can be seen in Fig. 14. The problem is that all sides that a dissection polygon originally had in common with P might have been hit by rays. We have to take into account that, whenever a ray hits the boundary of the current region, and thus the region is split along this ray, the side hit might have been the last common side of P and one of the newly created regions.

Operation 6 (*Mend_cap*$(P, u, \mathcal{D}, b, \overrightarrow{r_b}, cd)$) (Fig. 15).
Input: a frame P with an orientation $u(P)$, a dissection $\mathcal{D}$ of P, a reflex vertex b of some $D \in \mathcal{D}$ which is a cap in $P_{\circlearrowleft}$, a ray $\overrightarrow{r_b}$ emanating from b, and a side cd of ∂D hit by $\overrightarrow{r_b}$.
Preconditions: cd is a common side of P and D, $\overrightarrow{r_b}$ cuts the reflex $\angle_P b$ into two convex angles, $r_D(c) = 0$.
Operation: Let q denote the point where $\overrightarrow{r_b}$ hits cd. Obtain P' from P and D' from D by replacing the edge cd by the path $\mathrm{carc}(c, q, b) \oplus \mathrm{carc}(b, q, d)$. Split D' into simple polygons in $\mathcal{D}$. Set $u(\cdot) := -$ for all interior vertices of $\mathrm{carc}(c, q, b)$ and $u(\cdot) := +$ for all interior vertices of $\mathrm{carc}(b, q, d)$.
Output: $(P', u, \mathcal{D})$.

Proposition 25. *The output P' of* Mend_cap *is a frame.*

Proof. We have to check properties (F1)–(F5). (F1) and (F2) are obvious from the definition of carc. For internal vertices of convex arcs, one can argue as in Proposition 13. Hence, we have to consider vertices b, c and d only.

By Proposition 20, $m_P(b) = 1$, and, thus, $m_{P'}(b) = 2$. Since $m_P(c) = m_{P'}(c)$ and $m_P(d) = m_{P'}(d)$, (F3) follows. (F4) is clearly true for b, since for both visiting paths, the adjacent vertices are on different sides of the line through b and q. For both c and d, the angles mentioned in (F4) cannot increase. Hence, (F4) holds for all vertices in $V(P')$. Finally, for (F5) note that *Mend_cap* does not create any new reflex

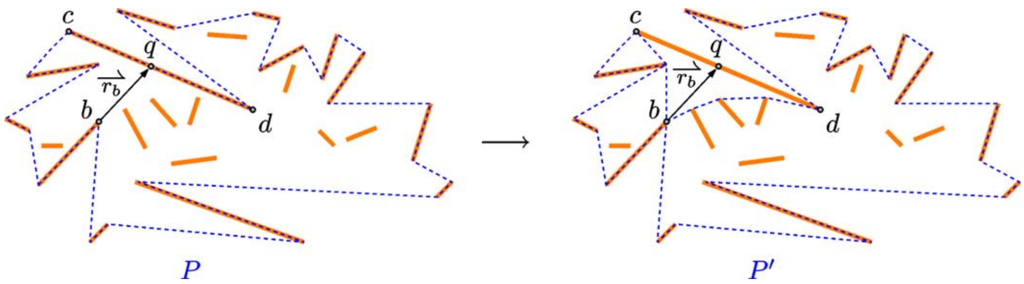

Fig. 15. Mending a cap.

vertex, except for the fact that $r_{P'}(b) = r_P(b) + 1 = 2$. Let $p_b, p_d \in V(S)$ such that $bp_b \in S$ and $dp_d \in S$. Since P is a frame and $r_P(b) = r_P(d) = 1$, we can conclude by (F5) that $p_b, p_d \in V(P) \subset V(P')$. □

Proposition 26. Mend_cap *creates at most one new anti-cap* (*that is, there is at most one more anti-cap in* $P'_{\circlearrowleft}$ *than in* $P_{\circlearrowleft}$).

Proof. The operation does not create any new reflex vertices, so only the existing reflex vertices b and (possibly) d might become anti-caps. But by the definition of carc, the new occurrence of b in $P'_{\circlearrowleft}$ is a cap. □

Remark 27. If vertex b appears twice as a cap in $P'_{\circlearrowleft}$, there is some choice which one to chop off as a wedge by *Chop_wedges*. For reasons that will become apparent later (cf. Lemma 37), we decide to consider the original cap as a wedge.

6. Algorithm and its analysis

Algorithm 2.
Input: a set S of disjoint line segments and a side yz of conv(S).

P	$\leftarrow$	conv(S).	(*frame*)
$\mathcal{D}$	$\leftarrow$	$\{P\}$.	(*dissection*)
(a, b, c)	$\leftarrow$	$\emptyset$.	(*vertex + adjacent reflex vertex + adjacent vertex*)
u	$\leftarrow$	u_{yz}.	(*orientation*)

Repeat until every $D \in \mathcal{D}$ is convex in step c below.
- (a) $(P, u) \leftarrow$ *Both_endpoints*(P, u).
- (b) Update $\mathcal{D}$ by replacing each $D \in \mathcal{D}$ by Diss(D).
- (c) If every $D \in \mathcal{D}$ is convex, then $P \leftarrow$ *Chop_wedges*(P) and exit.
- (d) If $(a, b, c) = \emptyset$, then
  - (1) If there is a double-cap (k, l) in some $D_b \in \mathcal{D}$, then $(a, b, c) \leftarrow (k, l, m)$, where m is the other ($\neq k$) neighbor of l in ∂D_b.
  - (2) Else let b be a reflex vertex of some $D_b \in \mathcal{D}$, and let a and c be the adjacent (in ∂D_b) convex vertices, such that c is also adjacent to b in $P_{\circlearrowleft}$ (see Proposition 30).
- (e) If $\overrightarrow{r_b} := \overrightarrow{ab}$ hits a segment $ef \subset \text{int}(D_b)$ whose supporting line crosses the side bc, then $\overrightarrow{r_b} \leftarrow$ *Rotate*$(\overrightarrow{r_b}, b, ef, D_b)$.
- (f) If $\overrightarrow{r_b}$ hits a segment $ef \subset \text{int}(D_b)$, then
  - (1) $(P, u, \mathcal{D}) \leftarrow$ *Extend_reflex*$(P, u, \mathcal{D}, b, c, \overrightarrow{r_b})$.
  - (2) If *Extend_reflex* created an anti-cap h in $P_{\circlearrowleft}$, then $b \leftarrow h$; $c \leftarrow$ one convex neighbor, and $a \leftarrow$ the other neighbor of b in $P_{\circlearrowleft}$;
  - (3) else $(a, b, c) \leftarrow \emptyset$.
- (g) If $\overrightarrow{r_b}$ hits ∂D_b at a point g on side de (w.l.o.g., $r_{D_b}(d) \leqslant r_{D_b}(e)$), then
  - (1) Dissect D_b by bg and update $\mathcal{D}$ accordingly.
  - (2) If $de \neq yz$, and de is a common side of D_b and P which is not part of a wedge, then
    - (i) If not both ab and bc are common sides of D_b and P, then

$(P, u, \mathcal{D}) \leftarrow Mend\_cap(P, u, \mathcal{D}, b, \overrightarrow{r_b}, de)$.

(ii) If $r_{D_e}(e) = 1$ for some region $D_e \in \mathcal{D}$, then
$a, c \leftarrow$ neighbors of e in ∂D_e, such that b and c are in different open halfplanes w.r.t. the line de; and $b \leftarrow e$.

(iii) Else $(a, b, c) \leftarrow \emptyset$.

(3) Else $(a, b, c) \leftarrow \emptyset$.

Output: $(P, \mathcal{D})$.

An example illustrating the different steps of Algorithm 2 is provided in Fig. 16.

Proposition 28. *Algorithm* 2 *terminates.*

Proof. If P is changed in step (a), at least one segment endpoint is added to $P_{\circlearrowleft}$ that was not visited before. As no vertex ever leaves $P_{\circlearrowleft}$, these changes can only occur in a finite number of steps. Apart from this, either step (f) or step (g) is executed in every iteration. Either $P_{\circlearrowleft}$ is augmented by a segment that was in the interior of P before (step (f)); or a reflex angle of a region $D_b \in \mathcal{D}$ is destroyed (step (g)), while no new reflex angle is added. Hence, after a finite number of iterations, every $D \in \mathcal{D}$ is convex and the algorithm terminates. □

To ensure that Algorithm 2 works correctly and P is a frame all the time, it is enough to check that the preconditions of our operations are satisfied.

Proposition 29. *Whenever* Mend_cap$(P, u, \mathcal{D}, b, \overrightarrow{r_b}, de)$ *is called in Algorithm* 2, *then* b *is a cap in* $P_{\circlearrowleft}$.

Proof. Whenever an anti-cap h is created during Algorithm 2, the next ray is shot from h. At that point, the edges incident to h are common edges of both P and the corresponding dissection polygon $D_h \in \mathcal{D}$. □

Proposition 30. *If* a, b, c *are three consecutive vertices in* $P_{\circlearrowleft}$, *during Algorithm* 2, *where* b *is a reflex vertex of some* $D_b \in \mathcal{D}$, *then either* ab *or* bc *is a side of* D_b.

Proof. The side ab (or bc) is not a side of D_b if and only if the ray drawn from a previous reflex vertex hit it. Algorithm 2 is organized so that right after a ray hits, say, side ab (step g(2)), it shoots a ray from b in the next step, such that from there on, b is no longer a reflex vertex of any set in $\mathcal{D}$. □

The following lemmata show three invariants of Algorithm 2, finally establishing the conditions of Lemma 3.

Lemma 31. *In each step of Algorithm* 2, *the total number of pairs of adjacent reflex vertices over all* $D \in \mathcal{D}$ *is at most one.*

Proof. The statement holds after the first execution of step (a) by Proposition 18. It suffices to check that each operation maintains this property.

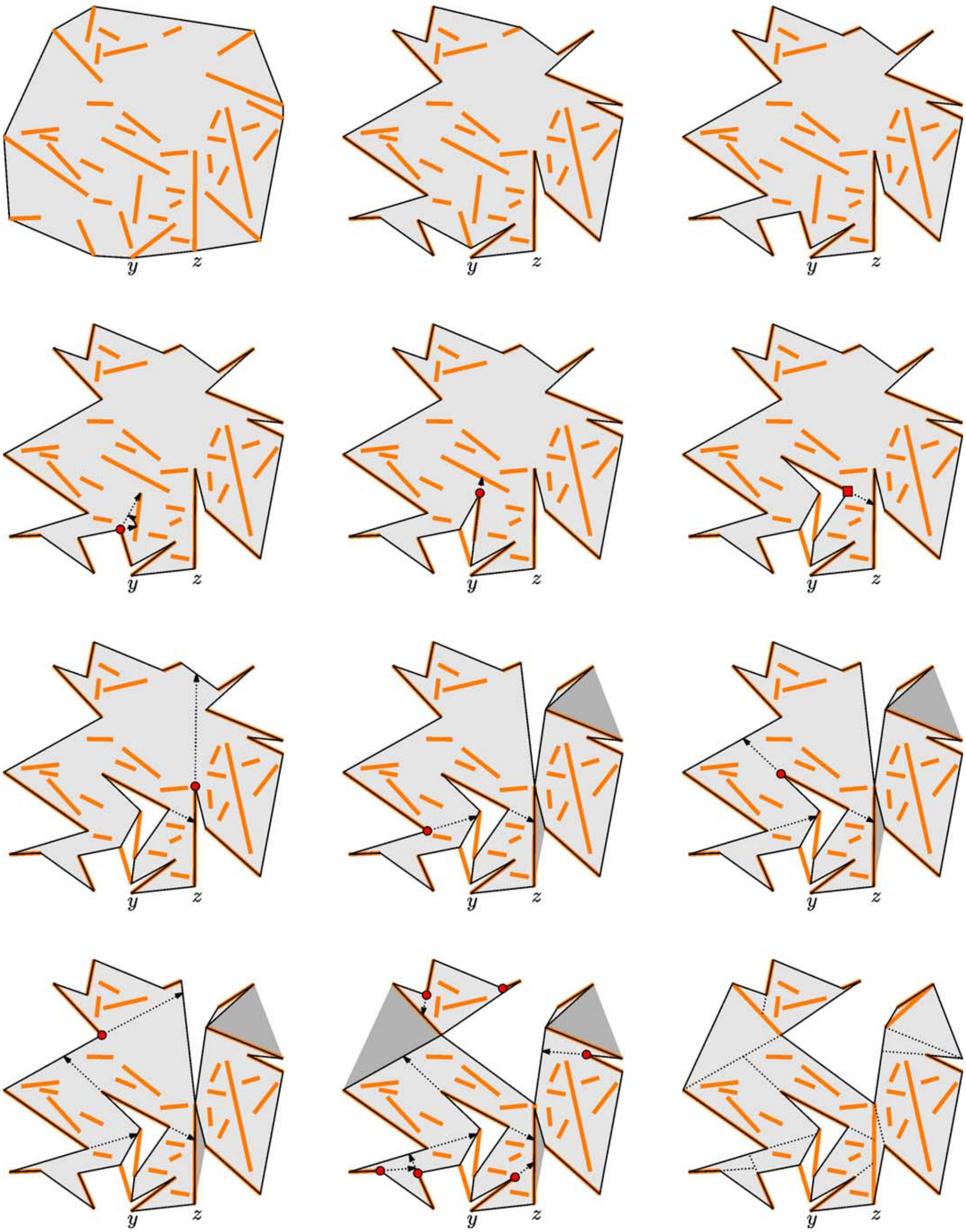

Fig. 16. Running Algorithm 2 on an example; wedges are shaded dark, and the points from which a ray is shot are marked: a circle denotes a cap, while a square stands for an anti-cap. In the last step, the wedges are chopped off, and we obtain a dissection of P into convex polygons.

Mend_cap does not create any reflex vertex of any $D \in \mathcal{D}$. *Extend_reflex* creates at most two adjacent reflex vertices; if it does so, one of these reflex vertices is chosen (step d(1) or step f(2)) as the vertex b to shoot the next ray from, thereby reverting b to a convex vertex of the resulting regions in $\mathcal{D}$.

It rests to consider the call to *Both_endpoints* in step (a). Recall that every interior vertex v of every single carc and marc is always oriented such that $v^{u(v)}$ is a convex vertex of the corresponding $D \in \mathcal{D}$. As in Proposition 18, the fact that all interior vertices of any single carc or marc get the same orientation assures that no two consecutive reflex vertices are created during *Both_endpoints*. □

Corollary 32. *Whenever* Extend_reflex$(P, u, \mathcal{D}, b, c, \overrightarrow{r_b})$ *is called in Algorithm* 2*, we have* $r_D(c) = 0$*, where* D *is the region from* $\mathcal{D}$ *of which* b *is a reflex vertex.*

Corollary 33. *Whenever* Mend_cap$(P, u, \mathcal{D}, b, \overrightarrow{r_b}, de)$ *is called in Algorithm* 2*, we have* $r_D(d) = 0$*, where* D *is the region from* $\mathcal{D}$ *of which* b *is a reflex vertex.*

Now we have shown that all the preconditions of both *Extend_reflex*$(P, u, \mathcal{D}, b, c, \overrightarrow{r_b})$ and *Mend_cap* $(P, u, \mathcal{D}, b, \overrightarrow{r_b}, de)$ are satisfied whenever these operations are called. It remains to show that the preconditions of *Chop_wedges* in step (c) of Algorithm 2 are satisfied, too.

Proposition 34. *During Algorithm* 2*, there is always at most one anti-cap which is a common reflex vertex of* $P_{\circlearrowleft}$ *and some* $D \in \mathcal{D}$*.*

Proof. An anti-cap can be created in two places only: in *Extend_reflex* (step f(1)), or in *Mend_cap* (step g(2)(i)). In both cases, at most one anti-cap is created (Propositions 22 and 26). Assume that a vertex e is inserted into $P_{\circlearrowleft}$ as an anti-cap by *Extend_reflex* or *Mend_cap*. At this point, $m_P(e) = 1$ by Proposition 20. In the next iteration, Algorithm 2 dissects the region $D \in \mathcal{D}$ containing e along a ray emanating from e. From there on, e is not a common reflex vertex of P and any $D \in \mathcal{D}$ anymore. □

Lemma 35. *For every anti-cap* e *in* $P_{\circlearrowleft}$*, we have* $m_P(e) = 1$ *during Algorithm* 2*.*

Proof. The only point where an anti-cap e could possibly be revisited by $P_{\circlearrowleft}$ is in the call to *Both_endpoints* (step (a)) immediately following the step where e became an anti-cap. We argue that the orientation u along carc and marc is set such that $P_{\circlearrowleft}$ cannot revisit e in any of the resulting *Build_cap* operations:

We consider only the variant of *Extend_reflex* described in Operation 4 and we use the same notation as there; the argument is similar for the symmetric variant of *Extend_reflex* and for *Mend_cap*.

First we show that *Both_endpoints* applied to vertices of carc(b, g, e) does not revisit e. Recall that $u(k) = -$, for all $k \in$ carc(b, g, e), and that *Build_cap* preserves this orientation for all new vertices. In particular, for every interior vertex k of a carc, $k^{u(k)}$ is convex in $P_{\circlearrowleft}$.

Denote by P' the frame resulting from *Both_endpoints*(P, u). For every vertex k inserted by *Both_endpoints* into $P_{\circlearrowleft}$, we define recursively a polygonal arc $\varepsilon(k)$ connecting k to b. If k is inserted as part of a carc(p, q, q^-) in a step *Build_cap*$(\tilde{P}, u, q)$, then let $\varepsilon(k)$ follow carc(p, q, q^-) from p to q^-, and then continue along $\varepsilon(q^-)$ to b (an example is given in Fig. 17). For any such k, the arc $\varepsilon(k) = (p = p_0, \ldots, p_j = k, \ldots, p_m = b)$ is a simple locally convex polygonal arc within P. Moreover, $\varepsilon(k)$ forms a *right-turn*, that is, for every $i = 1, \ldots, (m-1)$, p_{i+1} as well as all the neighbors of $p_i \in P'_{\circlearrowleft}$ lie to the right of the oriented line $\overrightarrow{p_{i-1}p_i}$.

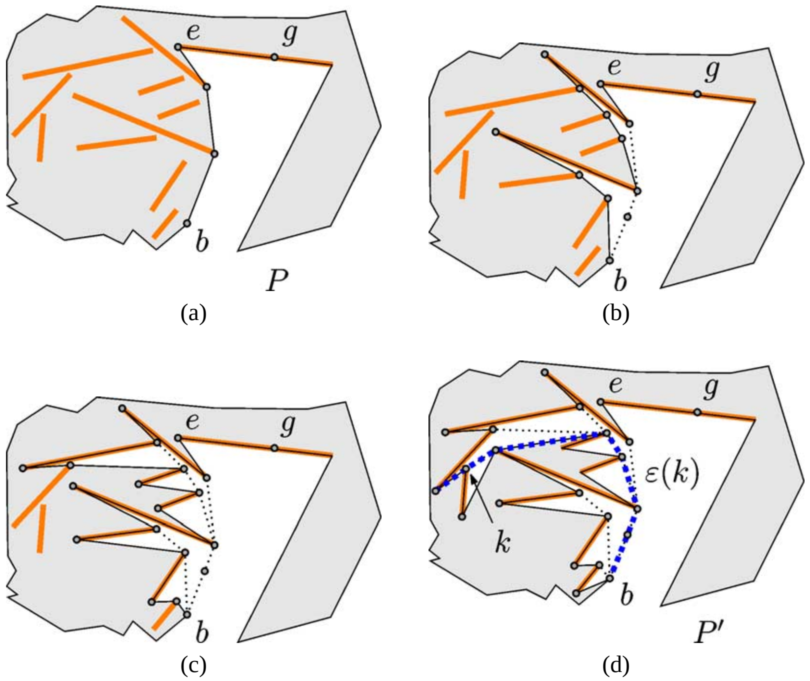

Fig. 17. Illustration for Lemma 35.

Suppose that $m_{P'}(e) = 2$. Notice that e is inserted into $P'_{\circlearrowleft}$ as a convex vertex by *Both_endpoints*, since the other endpoint f of the segment edge ef is already in $P_{\circlearrowleft}$. Therefore, there is a vertex k_0, $k_0 \neq e$, such that $e \in \varepsilon(k_0)$. Since $\varepsilon(k_0)$ is a simple *right-turn* path from e to b within P, it has to stay within $P \cap \triangle(bge)$, with g lying on its reflex side at vertex e. On the other hand, by property (F4) of the frame P', g must lie on the convex side of $\varepsilon(k_0)$ at e, giving a contradiction.

For the case of $\mathrm{marc}(f, g, b, c)$, observe that if $\overrightarrow{ef}$ hits bc, then $g = e$ by the rotation of $\overrightarrow{r_b}$; and by Corollary 23, f is a cap in $P'_{\circlearrowleft}$. If $\overrightarrow{ef}$ does not hit bc, then the argument from above shows that *Both_endpoints* applied to vertices of $\mathrm{marc}(f, g, b, c)$ does not revisit f. □

Corollary 36. *During Algorithm* 2*, every* $v \in V(P)$ *with* $m_P(v) = 2$ *appears at least once as a cap in* $P_{\circlearrowleft}$.

Proof. A vertex $b \in V(P)$ can be revisited in two different ways (we may assume that a, b, c are consecutive vertices in $P_{\circlearrowleft}$):

(i) If b is a cap and *Mend_cap*$(P, u, \mathcal{D}, b, \overrightarrow{r_b}, cd)$ is applied.
(ii) If a cap b is a reflex vertex of some $D \in \mathcal{D}$ and carc or marc contain b.

In both cases, the first occurrence of b remains a cap, and both ab and bc remain sides of P. □

At the last step of Algorithm 2, *Chop_wedges* is applied. Lemma 35 assures that any vertex v, for which $m_P(v) = 2$, is adjacent to a wedge that can be chopped off. Thus, the output P of Algorithm 2 is a simple frame. To show that P and the partition $\mathcal{D}$ satisfy the properties of Lemma 3, it rests to prove the following.

Lemma 37. *All through Algorithm* 2, *every* $D \in \mathcal{D}$ *has a common side with* P *which is different from wedge edges and the special side* yz *of* P.

Proof. The statement holds for $\text{conv}(S)$. It is enough to check that it remains true after each iteration.

Build_cap, *Mend_cap* or *Extend_reflex* may dissect a region $D \in \mathcal{D}$ into several regions: either directly (*Mend_cap* dissects the current region at the mended cap), or because carc or marc

- pass through both endpoints of a segment (thus forming a segment diagonal),
- pass through an endpoint of a segment whose other endpoint is already in $P_{\circlearrowleft}$ (again creating a segment diagonal),
- or revisit a cap (thereby reverting sides of D to wedge-edges).

Still, in each new region $D' \subset D$, carc and marc have a side which is common with both D' and P. For *Mend_cap* we have to note that both occurrences of the mended cap are caps in $P_{\circlearrowleft}$ (cf. Proposition 26). We need to be a bit careful which of them is supposed to be chopped off in *Chop_wedges*, in order for the above argument to go through: one side adjacent to the original cap might have been hit by a ray; hence, we have to mark this original cap as wedge.

In step g(1) of Algorithm 2, the region $D_b \in \mathcal{D}$ is dissected into regions D_e and D_d by the ray $\overrightarrow{r_b}$, where b is a reflex vertex of both D_b and $P_{\circlearrowleft}$. We have to check that our statement still holds for both D_e and D_d. According to Proposition 30, we may assume that bc is a common side of D_e and P. Denote the other neighbor of b in $P_{\circlearrowleft}$ and D_d by a and α, respectively.

If b is an anti-cap, then $a = \alpha$, since Algorithm 2 draws the ray $\overrightarrow{r_b}$ right after the path $\alpha b \gamma$ is created. Hence, αb is a common side of D_d and P that is clearly neither a wedge edge nor equal to yz.

So suppose that b is a cap and, αb is not a side of D_d. This means that a previously drawn ray $\overrightarrow{r_{b'}}$ from a reflex vertex b' hits ab at α. Let γ be the neighbor of b' in D_b. Then $b'\gamma$ must be a common side of D_b and P, since otherwise *Mend_cap* would have been applied to b', $\overrightarrow{r_{b'}}$ and ab, and ab would not be a side of P anymore. Note that the dissection by $\overrightarrow{r_b}$ immediately follows the dissection by $\overrightarrow{r_{b'}}$ (no operation is applied, hence the call to *Both_endpoints* does not change anything).

We claim $b' \in D_d$. Since $\overrightarrow{r_{b'}}$ and side αb are adjacent along ∂D_d, the only way to exclude b' from D_d is that $\overrightarrow{r_b}$ hits back to $\overrightarrow{r_{b'}}$. But this is impossible by the choice of $\overrightarrow{r_b}$, which always shoots along the edge that was hit by the previous ray (step g(2)(ii)), in this case ab. Thus, b' lies on the boundary of D_d, as claimed.

If side $b'\gamma$ does not belong to ∂D_d, it must be hit by $\overrightarrow{r_b}$. But in this case, *Mend_cap* is applied to b ($b'\gamma$ is not a wedge edge and side ab is not part of ∂D_b), and there is a common side of D_d and P along the constructed carc. Otherwise, $b'\gamma$ is a common side of D_d and P which is neither wedge edge nor equal to yz. □

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