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Superconnectivity of graphs with odd girth g and even girth h

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ABSTRACT

A maximally connected graph of minimum degree δ is said to be superconnected (for short super-κ) if all disconnecting sets of cardinality δ are the neighborhood of some vertex of degree δ. Sufficient conditions on the diameter to guarantee that a graph of odd girth g and even girth $h \geq g+3$ is super-κ are stated. Also polarity graphs are shown to be super-κ.

1. Introduction

The topology of a multiprocessor system can be modelled as an undirected graph $G = (V(G), E(G))$, where $V(G)$ represents the set of all processors and $E(G)$ represents the set of all connecting links between the processors. Among all fundamental properties for interconnection networks, the connectivity κ is a major parameter widely used for measures of functionality of the system. A basic definition of the connectivity of a graph G is defined as the minimum number of vertices whose removal from G produces a disconnected graph. The parameter κ of connectivity gives the minimum cost to disrupt the network, but they do not take into account what remains after destruction. One attribute which leads us to define a more reliable network is the notion of *superconnectivity* proposed for the first time in [7,8]. A graph G is *superconnected*, for short *super-*κ, if all minimum cut sets isolate one vertex. Therefore if a graph G is non-super-κ, there exists a cut set $X \subset V(G)$ of cardinality $|X| = \delta$ such that every connected component of $G - X$ has at least two vertices.

The main objective of this paper is to give sufficient conditions for a graph to be super-κ, in terms of the girth pair for odd g and even $h \geq g+3$. The *odd girth (even girth)* of G is the length of a shortest odd (even) cycle in G. If there is no odd (even) cycle in G then the odd (even) girth of G is taken as ∞. Let $g = g(G)$ denote the smaller of the odd and even girths, and let $h = h(G)$ denote the larger. Then g is called *girth* of G, and (g, h) is called the *girth pair* of G. Girth pairs were introduced by Harary and Kovács [15] and several interesting questions concerning girth pairs were posed in that paper. Campbell [10] studied the size of smallest cubic graphs with girth pairs $(6, 7)$, $(6, 9)$ and $(6, 11)$. And a lower bound on the order of a regular graph with girth pair (g, h), for odd g and even $h \geq g+3$ was found in [4].

1.1. Main results

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [11] for terminology and definitions.

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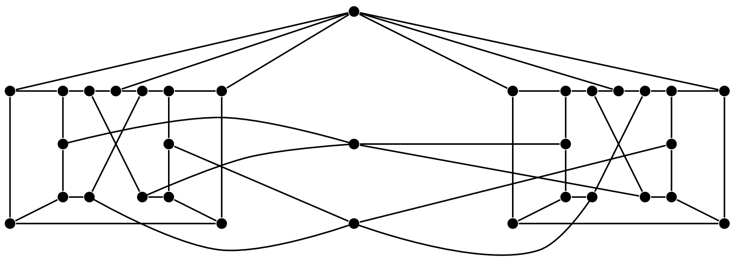

Fig. 1. A maximally connected graph with $g = 5, h = 8, \delta = 3$ of diameter 5 which is non-super-κ.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For any $S \subset V$, the subgraph *induced by S* is denoted by $G[S]$. For $u, v \in V$, $d(u, v) = d_G(u, v)$ denotes the *distance* between u and v; that is, the length of a shortest (u, v)-path. For $S, F \subset V$, $d(S, F) = d_G(S, F) = \min\{d(s, f) : s \in S, f \in F\}$ denotes the *distance* between S and F. For every $v \in V$ and every integer $r \geq 0$, $N_r(v) = \{w \in V : d(w, v) = r\}$ denotes the *neighborhood of v at distance r*. If $S \subset V$, then $N_r(S) = \{w \in V : d(w, S) = r\}$. When $r = 0$, $N_0(S) = S$ for every subset S of vertices, and when $r = 1$ we put simply $N(v)$ and $N(S)$ instead of $N_1(v)$ and $N_1(S)$. The *degree* of a vertex v is $d(v) = |N(v)|$, whereas the (minimum) degree $\delta = \delta(G)$ of G is the minimum degree over all vertices of G. The *diameter* denoted by *diam*(G) is the maximum distance over all pairs of vertices in G and G is connected if *diam*$(G) < \infty$.

A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V$ and $G - S$ is not connected, then S is said to be a *cut set*. Certainly, every connected graph different from a complete graph has a cut set. The graphs G considered in this paper are different from a complete graph. A *component* of a graph G is a maximal connected subgraph of G. A connected graph is called *k-connected* if every cut set has cardinality at least k. The *connectivity* $\kappa = \kappa(G)$ of a connected graph G is defined as the maximum integer k such that G is k-connected. A classic result due to Whitney is that for every graph G, $\kappa \leq \delta$. A graph is *maximally connected* if $\kappa = \delta$. Observe that the situation $\kappa < \delta$ is precisely a situation where no minimum cut set isolates a vertex. A graph G is said to be *super-κ* if $\kappa = \delta$ and the minimum cut sets of δ vertices are the neighboring of one vertex of degree δ.

Some known sufficient conditions on the diameter of a graph in terms of its girth to guarantee lower bounds on κ or super-κ graphs are listed in the following theorem.

Theorem 1. *Let G be a graph with minimum degree $\delta \geq 2$, diameter diam(G), girth g, and connectivity κ. Then,*

(i) [12,17,18] $\kappa = \delta$ *if diam*$(G) \leq 2\lfloor (g-1)/2 \rfloor - 1$.
(ii) [2] *The graph G is super-κ if diam*$(G) \leq g - 3$.
(iii) [6,5] *The graph G is super-κ if g is odd, diam*$(G) \leq g - 2$ *and the maximum degree* $\Delta \leq 3\delta/2 - 1$.

Hellwig and Volkmann [16] provide a comprehensive survey of sufficient conditions for a graph to achieve lower bounds on κ and other index of connectivities.

Item (i) of Theorem 1 was improved for graphs with girth pair (g, h), odd g and even $h \geq g + 3$ in [3].

Theorem 2 *([3])*. *Let G be a graph of minimum degree $\delta \geq 3$, girth pair (g, h), odd g and even h with $g + 3 \leq h < \infty$ and connectivity κ. Then $\kappa = \delta$ if diam$(G) \leq h - 4$.*

The main result of this paper is the following theorem in which we improve Theorem 2.

Theorem 3. *Let G be a graph with girth pair (g, h), odd $g \geq 5$ and even h with $g + 3 \leq h < \infty$, and minimum degree δ. Then G is super-κ if some of the following conditions hold.*

(i) *diam*$(G) \leq h - 5$ *and* $\delta = 3$.
(ii) *diam*$(G) \leq h - 4$ *and* $\delta \geq 4$ *or* $\delta = 3$ *and vertices of degree* 3 *are not on odd cycles of length less than* $h - 3$.

The hypothesis on the diameter and on the minimum degree of Theorem 3 are necessary because for instance Fig. 1 depicts a maximally connected graph with $g = 5$, $h = 8$, non-super-κ of diameter 5 and minimum degree $\delta = 3$. To prove Theorem 3 we need to use the following two results, which roughly speaking shows us that the even girth h is a suitable index to measure how far away a vertex of a non-super-κ graph with girth pair (g, h) can be from a minimum cut set X.

Proposition 1. *Let G be a connected graph with minimum degree $\delta \geq 3$ and girth pair (g, h), odd $g \geq 5$ and even h with $g + 3 \leq h < \infty$. Let X be a cut set of cardinality $|X| \leq \delta$ such that every component C of $G - X$ has $|V(C)| \geq 2$. Let denote $\mu(C) = \max\{d(u, X) : u \in V(C)\}$. Then $\mu(C) \geq (h-4)/2$.*

Proposition 2. *Let G be a connected graph with girth pair (g, h), odd $g \geq 5$ and even h with $g + 3 \leq h < \infty$. Suppose that minimum degree $\delta \geq 4$ or $\delta = 3$ and vertices of degree 3 are not on odd cycles of length less than $h - 3$. Let X be a cut set of cardinality $|X| \leq \delta$ such that every component C of $G - X$ has $|V(C)| \geq 2$. Then for all component C of $G - X$ there exists some vertex $u_0 \in V(C)$ at distance $d(u_0, X) \geq (h-4)/2$ such that $|N_{(h-4)/2}(u_0) \cap X| \leq 2$ and*

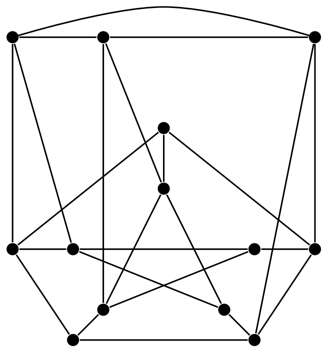

Fig. 2. A polarity graph with girth pair $g = 3$ and $h = 6$ on 13 vertices.

(i) *for every* $x \in X$ *if* $|N_{(h-4)/2}(u_0) \cap X| = 1$ *then* $x \notin N_{(h-4)/2}(u_0) \cap X$;
(ii) *for every* 2*-set* $\{x, x'\} \subset X$ *if* $|N_{(h-4)/2}(u_0) \cap X| = 2$ *then* $\delta = 4$ *and* $\{x, x'\} \cap (N_{(h-4)/2}(u_0) \cap X) = \emptyset$.

One important family of graphs with girth pair $g = 3$ and $h = 6$ are polarity graphs defined as follows. Let $\mathcal{P}$ be a finite projective plane, and let π be a polarity of $\mathcal{P}$ (a one-to-one mapping of points onto lines such that $p' \in \pi(p)$ whenever $p \in \pi(p')$). The *polarity graph* $G(\mathcal{P}, \pi)$ is the graph whose vertex set is the set of points of $\mathcal{P}$ and whose edge set is $\{pp' : p \in \pi(p')\}$. A polarity graph has diameter 2, $g = 3$ and no 4-circuits, then they are maximally connected according to Theorem 2. Moreover, they are the unique graphs satisfying these requirements [9]. Fig. 2 shows a polarity graph on 13 vertices. Polarity graphs are extremal graphs for the extremal problem of finding graphs with maximum number of edges with no 4-circuits of order n when $n = q^2 + q + 1$, q being a prime power; see [1,13,14]. Moreover, these graphs have order $\delta^2 + \delta + 1$, the vertices have degrees δ or $\delta + 1$, and the vertices of degree δ do not belong to any triangle. Using these properties we finish by proving the following result.

Theorem 4. *Polarity graphs of minimum degree* $\delta \geq 3$ *are super-*κ.

2. Proofs

In what follows the goal is to prove Theorem 3. To do that we use the following notation introduced in [2]. Let $G = (V, E)$ be a graph and let $X \subset V$, $v \in V \setminus X$ and $u \in N(v)$. Let us define the sets

$$\begin{aligned} S_u^+(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) + 1\}; \\ S_u^=(v) &= \{z \in N(v) - u : d(z, X) = d(v, X)\}; \\ S_u^-(v) &= \{z \in N(v) - u : d(z, X) = d(v, X) - 1\}. \end{aligned} \tag{1}$$

Clearly, $S_u^+(v)$, $S_u^=(v)$ and $S_u^-(v)$ form a partition of $N(v) - u$.

Proof of Proposition 1. Let C be any component of $G - X$ and denote $\mu = \mu(C) = \max\{d(u, X) : u \in V(C)\}$ and $\mathcal{F}(C) = \{u \in V(C) : d(u, X) = \mu\}$. □

Claim 1. Every vertex $u \in \mathcal{F}(C)$ satisfies that

$$\sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^=(z)| \leq \delta - |N(u) \setminus \mathcal{F}(C)|.$$

Proof. Observe that the sets $N_\mu(S_u^=(z)) \cap X$ are pairwise disjoint for all $z \in N(u) \cap \mathcal{F}(C)$, because otherwise even cycles of length at most $2\mu + 4 \leq h - 2$ would be created. By the same reason $(N_\mu(S_u^=(z)) \cap X) \cap (N_\mu(u) \cap X) = \emptyset$ and $|N_\mu(S_u^=(z)) \cap X| \geq |S_u^=(z)|$ for all $z \in N(u) \cap \mathcal{F}(C)$; see Fig. 3. Then

$$\begin{aligned} \delta \geq |X| &\geq \sum_{z \in N(u) \cap \mathcal{F}(C)} |N_\mu(S_u^=(z)) \cap X| + |N_\mu(u) \cap X| \\ &\geq \sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^=(z)| + |N(u) \setminus \mathcal{F}(C)| \end{aligned}$$

which implies the desired result. □

Claim 2. $\mu \geq 2$.

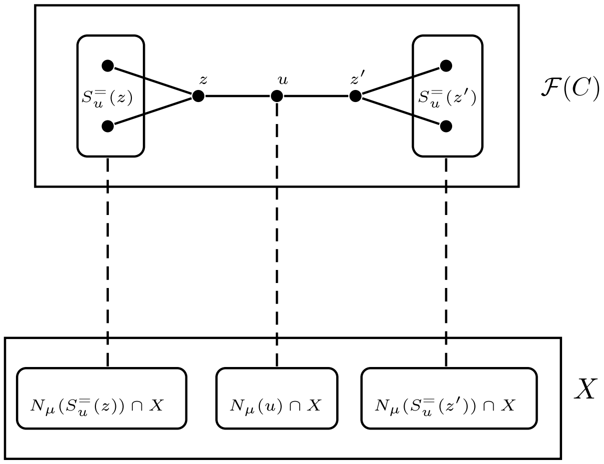

Fig. 3. Detail of the proof of Claim 1.

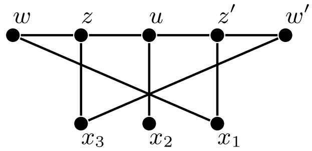

Fig. 4. Detail of the proof of Claim 2 for $\delta = 3$.

Proof. On the contrary suppose that $\mu = 1$ which means $V(C) = \mathcal{F}(C)$ and $|N(u) \setminus \mathcal{F}(C)| = |N(u) \cap X| \geq 1$ for all $u \in V(C)$. Suppose that there exist $u_0 \in V(C)$ and $z_0 \in N(u_0) \cap V(C)$ such that $S_{u_0}^{=}(z_0) = \emptyset$, then $|N(z_0) \cap X| \geq \delta - 1$. Since $g \geq 5$, $N(u_0) \cap X$ and $N(z_0) \cap X$ are two disjoint sets, hence $|N(u_0) \cap X| = 1$, $|N(z_0) \cap X| = \delta - 1$ and $|X| = \delta$ and X is partitioned into these two sets. As $\delta \geq 3$ there is other vertex $z_1 \in (N(u_0) - z_0) \cap V(C)$ which must have a common vertex with $N(u_0) \cap X$ or with $N(z_0) \cap X$, in both cases a contradiction because $g \geq 5$. Therefore $|S_u^{=}(z)| \geq 1$ for all $u \in V(C)$ and for all $z \in N(u) \cap V(C)$. This fact together with Claim 1 gives for all $u \in V(C)$ that

$$d(u) = |N(u) \cap \mathcal{F}(C)| + |N(u) \cap X| \leq \sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^{=}(z)| + |N(u) \cap X| \leq \delta.$$

Therefore $d(u) = \delta$ and $|S_u^{=}(z)| = 1$ for all $u \in V(C)$ and for all $z \in N(u) \cap V(C)$. Hence for all $uz \in E(C)$ it follows that $|N(z) \cap X| = \delta - 2$ and $|N(u) \cap X| = \delta - 2$. As $g \geq 5$ a path z, u, z' of length 2 can be considered in C and the sets $N(z) \cap X$, $N(u) \cap X$ and $N(z') \cap X$ are pairwise disjoint because $g \geq 5$, thus $\delta \geq |X| \geq |N(u) \cap X| + |N(z) \cap X| + |N(z') \cap X| \geq 3\delta - 6$ which is a contradiction for all $\delta \geq 4$. Therefore $\mu \geq 2$ and Claim 2 is valid if $\delta \geq 4$. If $\delta = 3$ the above inequalities become equalities, thus $X = \{x_1, x_2, x_3\}$ and assume that $z'x_1, ux_2, zx_3 \in E(G)$. A path w, z, u, z', w' of length 4 in C can be considered. As $|N(w) \cap X| = 1$ and $|N(w') \cap X| = 1$ the only possibility is that $w'x_3, wx_1 \in E(G)$ because $g \geq 5$; see Fig. 4. However the cycle $z, x_3, w', z', x_1, w, z$ has length 6 which is a contradiction. Therefore $\mu \geq 2$ and Claim 2 is also valid if $\delta = 3$. □

To continue the proof assume that $2 \leq \mu \leq (h-6)/2$ and observe that for any given arbitrary vertex $u \in \mathcal{F}(C)$, the sets $S_u^{=}(v)$, for all $v \in N(u) \cap \mathcal{F}(C)$, are pairwise disjoint because $g \geq 5$.

Claim 3. Every vertex $u \in \mathcal{F}(C)$ satisfies that $|N(u) \cap \mathcal{F}(C)| \leq 2$ and hence $|N(u) \setminus \mathcal{F}(C)| \geq \delta - 2$.

Proof. Note that $N(z) - u = S_u^{=}(z) \cup S_u^{-}(z)$ for all $u \in \mathcal{F}(C)$ and for all $z \in N(u) \cap \mathcal{F}(C)$. Then by Claim 1, the set $U_1 = \bigcup_{z \in N(u) \cap \mathcal{F}(C)} S_u^{-}(z)$ has cardinality

$$\begin{aligned}
|U_1| &\geq |N(u) \cap \mathcal{F}(C)|(\delta - 1) - \sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^{=}(z)| \\
&\geq |N(u) \cap \mathcal{F}(C)|(\delta - 1) - (\delta - |N(u) \setminus \mathcal{F}(C)|) \\
&\geq |N(u) \cap \mathcal{F}(C)|(\delta - 2).
\end{aligned}$$

As $N_{\mu-1}(v) \cap N_{\mu-1}(v') \cap X = \emptyset$ for all $v, v' \in U_1$ because otherwise an even cycle of length $2(\mu - 1) + 4 = 2\mu + 2$ is formed, it follows that

$$\delta \geq |X| \geq |N_{\mu-1}(U_1)| \geq |U_1| \geq |N(u) \cap \mathcal{F}(C)|(\delta - 2).$$

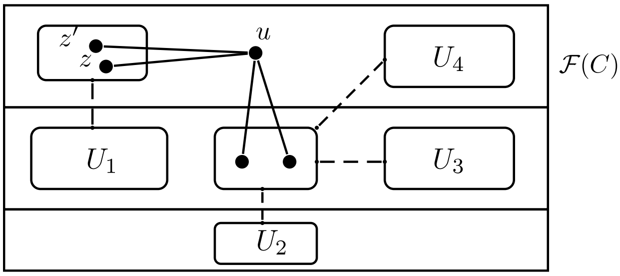

Fig. 5. Detail of the sets $U_1,\ U_2,\ U_3$ and U_4.

Consequently if $\delta \geq 4$ then $|N(u) \cap \mathcal{F}(C)| \leq 2$ and the claim is valid; and if $\delta = 3$ then $|N(u) \cap \mathcal{F}(C)| \leq 3$. Suppose that $N(u) \cap \mathcal{F}(C) = \{z_1, z_2, z_3\}$, then by the above inequality $\delta = 3 \geq |X| \geq |U_1| \geq 3$, i.e., $|U_1| = 3$ yielding $|S_u^-(z_i)| = 1$ and $|S_u^=(z_i)| \geq 1$ for all $z_i \in N(u) \cap \mathcal{F}(C)$. Let $w_i \in (N(z_i) - u) \cap \mathcal{F}(C)$, $i = 1, 2, 3$. Since $g \geq 5$, w_1, w_2, w_3 are three distinct vertices such that $N_\mu(w_i) \cap X$, $i = 1, 2, 3$, and $N_\mu(u) \cap X$ are four pairwise disjoint sets because otherwise an even cycle of length $2\mu + 4 \leq h - 2$ is formed. Hence $3 = |X| \geq \sum_{i=1}^{3} |N_\mu(w_i) \cap X| + |N_\mu(u) \cap X| \geq 4$ which is a contradiction. Therefore $|N(u) \cap \mathcal{F}(C)| \leq 2$ and the claim is also valid for $\delta = 3$. □

To finish the proof we consider the following sets

$$U_1 = \bigcup_{z \in N(u) \cap \mathcal{F}(C)} S_u^-(z)$$

$$U_2 = \bigcup_{v \in N(u) \setminus \mathcal{F}(C)} S_u^-(v)$$

$$U_3 = \bigcup_{v \in N(u) \setminus \mathcal{F}(C)} S_u^=(v)$$

$$U_4 = \bigcup_{v \in N(u) \setminus \mathcal{F}(C)} S_u^+(v)$$

which are pairwise disjoint because $g \geq 5$ (see Fig. 5).

As $N_2(u) \setminus \mathcal{F}(C) = U_1 \cup U_2 \cup U_3$ then

$$|N_2(u) \setminus \mathcal{F}(C)| = |U_1| + |U_2| + |U_3|. \tag{2}$$

Furthermore, by Claims 1 and 3 it follows that $\sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^=(z)| \leq 2$, hence

$$\begin{aligned} |N_2(u) \setminus \mathcal{F}(C)| &= |N_2(u)| - |N_2(u) \cap \mathcal{F}(C)| \\ &= |N_2(u)| - \sum_{z \in N(u) \cap \mathcal{F}(C)} |S_u^=(z)| - \sum_{v \in N(u) \setminus \mathcal{F}(C)} |S_u^+(v)| \\ &\geq \delta(\delta - 1) - 2 - |U_4|. \end{aligned} \tag{3}$$

From (2) and (3) it follows that

$$|U_1| + |U_2| + |U_3| \geq \delta(\delta - 1) - 2 - |U_4|. \tag{4}$$

Observe that $|N_{\mu-1}(U_1 \cup U_3) \cap X| \geq |U_1| + |U_3|$, because otherwise by the pigeonhole principle, even cycles of length at most $2(\mu - 1) + 4 \leq h - 4$ would be created. First suppose that $|U_4| = 0$. As $\delta \geq |N_{\mu-2}(U_2) \cap X| \geq |U_2|$, then by (4)

$$\delta \geq |X| \geq |N_{\mu-1}(U_1 \cup U_3) \cap X| \geq |U_1| + |U_3| \geq \delta(\delta - 1) - 2 - |U_2| \geq \delta(\delta - 2) - 2$$

yielding a contradiction. Therefore $\mu \geq (h - 4)/2$ in this case. So suppose that $|U_4| \geq 1$ and let $d \in U_4$, then $d \in S_u^+(v)$ for some $v \in N(u) \setminus \mathcal{F}(C)$. By Claim 3, $|(N(d) - v) \setminus \mathcal{F}(C)| \geq \delta - 3$ for all $d \in U_4$. First, suppose that $|(N(d) - v) \setminus \mathcal{F}(C)| \geq 1$ for all $d \in U_4$ (which always happen if $\delta \geq 4$). Then $|(N(U_4) \setminus \mathcal{F}(C)) \setminus N(u)| \geq |U_4|$. Further as $N_{\mu-2}(U_2) \cap X$ and $N_\mu(U_4) \cap X$ are pairwise disjoint then

$$\delta \geq |X| \geq |N_{\mu-1}((N(U_4) \setminus \mathcal{F}(C)) \setminus N(u)) \cap X| + |N_{\mu-2}(U_2) \cap X| \geq |U_4| + |U_2|$$

because otherwise by the pigeonhole principle even cycles of length at most $2\mu + 2 \leq h - 4$ would be created. Then, by (4)

$$\delta \geq |X| \geq |N_{\mu-1}(U_1 \cup U_3) \cap X| \geq |U_1| + |U_3| \geq \delta(\delta - 1) - 2 - |U_4| - |U_2| \geq \delta(\delta - 2) - 2$$

yielding a contradiction. Therefore $\mu \geq (h - 4)/2$ also in this case.

Suppose that there exists a vertex $d \in S_u^+(v)$ for some $v \in N(u) \setminus \mathcal{F}(C)$ such that $|(N(d) - v) \setminus \mathcal{F}(C)| = 0$. Therefore $\delta = 3$ and by Claim 3 it follows that $N(d) \cap \mathcal{F}(C) = \{d_1, d_2\}$. As $N_{\mu-1}(U_1) \cap X$ and $N_\mu(d_i) \cap X$, $i = 1, 2$ are pairwise disjoint sets, otherwise cycles of length $2\mu + 4$ would appear, then $|U_1| = 1$ and $|X| = 3$ yielding that $N(u) \cap \mathcal{F}(C) = \{z\}$ and

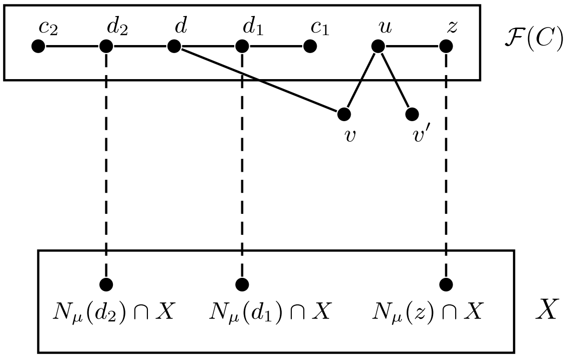

Fig. 6. Detail of the proof of Proposition 1 when $\delta = 3$.

$|N(u) \setminus \mathcal{F}(C)| \geq 2$. That is, X is partitioned into three disjoint sets $N_\mu(z) \cap X$, $N_\mu(d_1) \cap X$ and $N_\mu(d_2) \cap X$ meaning that $|N_\mu(d_i) \cap X| = 1, i = 1, 2$. By the pigeonhole principle $1 \leq |N(d_i) \setminus \mathcal{F}(C)| \leq |N_\mu(d_i) \cap X| = 1$, hence $|(N(d_i) - d) \cap \mathcal{F}(C)| \geq 1$, $i = 1, 2$; see Fig. 6. Take $c_i \in N(d_i) - d$, $i = 1, 2$, and noticing that $N_\mu(u) \cap X$, $N_\mu(c_1) \cap X$ and $N_\mu(c_2) \cap X$ are pairwise disjoint sets (otherwise cycles of length $2\mu + 4$ would appear) we obtain

$$|X| = 3 \geq |N_\mu(u) \cap X| + |N_\mu(c_1) \cap X| + |N_\mu(c_2) \cap X| \geq |N(u) \setminus \mathcal{F}(C)| + 2 \geq 4$$

which is a contradiction. Therefore $\mu \geq (h-4)/2$, thus the proposition is valid. □

Lemma 1. *Let G be a graph with minimum degree $\delta \geq 3$ and girth pair (g, h), odd $g \geq 5$ and even h with $g + 3 \leq h < \infty$. Let X be a cut set of cardinality $|X| \leq \delta$ such that every component C of $G - X$ has $|V(C)| \geq 2$. Let C be a component of $G - X$ such that* $\max\{d(u, X) : u \in V(C)\} = (h-4)/2$ *and denote by $\mathcal{F}(C) = \{u \in V(C) : d(u, X) = (h-4)/2\}$. Then*

(i) $|N(u) \setminus \mathcal{F}(C)| \leq \delta - 1$ *for all $u \in \mathcal{F}(C)$ if either $\delta \geq 4$ or $\delta = 3$ and vertices of degree* 3 *are not on odd cycles of length less than $h - 3$.*
(ii) *There exists a vertex $u \in \mathcal{F}(C)$ such that $|N(u) \setminus \mathcal{F}(C)| \leq \delta - 2$ if $\delta \geq 4$.*

Proof. Notice that $(h-4)/2 \geq 2$ because $h \geq g + 3 \geq 8$. First let us see that $|N(u) \setminus \mathcal{F}(C)| \leq \delta - 1$ for all $u \in \mathcal{F}(C)$. Suppose that the result does not hold and take any $u \in \mathcal{F}(C)$. Then $\delta \geq |X| \geq |N_{(h-4)/2}(u) \cap X| \geq |N(u) \setminus \mathcal{F}(C)| \geq \delta$ which means that $\delta = |X|$, $|N_{(h-6)/2}(v) \cap X| = 1$ for all $v \in N(u) \setminus \mathcal{F}(C)$, that is, $|S_u^-(v)| = 1$. Let us denote by $N(u) \setminus \mathcal{F}(C) = \{u_1, \ldots, u_\delta\}$ and $X = \{x_1, \ldots, x_\delta\}$ and suppose that $N_{(h-6)/2}(u_i) \cap X = \{x_i\}$ for $i = 1, \ldots, \delta$. Let us see that $S_u^+(u_i) = \emptyset$ for every $i = 1, \ldots, \delta$. Otherwise, if $z_i \in S_u^+(u_i)$ then $d(z_i, x_j) \geq (h-4)/2 + 1$ for all $i \neq j$, for if not, the shortest (z_i, x_j)-path (of length $(h-4)/2$) together with the shortest (u_j, x_j)-path (of length $(h-4)/2 - 1$) and the path z_i, u_i, u, u_j forms an even cycle of length $h - 2$, which is a contradiction. Since $z_i \in \mathcal{F}(C)$ and $|N(z_i) \setminus \mathcal{F}(C)| = \delta$, taking a vertex $y \in (N(z_i) - u_i) \setminus \mathcal{F}(C)$ the shortest path $z_i, y, \ldots, x_j$, the shortest path $u, u_j, \ldots, x_j$, both of length $(h-4)/2$, and the path z_i, u_i, u of length 2 produces an even cycle of length at most $h - 2$ which is a contradiction. Hence, $S_u^+(u_i) = \emptyset$ for every $i = 1, \ldots, \delta$, implying that $|S_u^=(u_i)| = d(u_i) - 1 - |S_u^-(u_i)| = d(u_i) - 2$. Note that the sets $N_{(h-6)/2}(S_u^=(u_i)) \cap X$, for $i = 1, \ldots, \delta$, are pairwise disjoint for if not, an even cycle of length at most $h - 2$ is formed. Furthermore, $|N_{(h-6)/2}(S_u^=(u_i)) \cap X| \geq |S_u^=(u_i)| = d(u_i) - 2$, since otherwise, an even cycle of length $h - 4$ is formed and this is not possible. Hence,

$$\begin{aligned} \delta = |X| &\geq \sum_{i=1}^{\delta} |N_{(h-6)/2}(S_u^=(u_i)) \cap X| \\ &\geq \sum_{i=1}^{\delta} |S_u^=(u_i)| \\ &= \sum_{i=1}^{\delta} (d(u_i) - 2), \end{aligned}$$

a contradiction if $\delta \geq 4$, thus $|N(u) \setminus \mathcal{F}(C)| \leq \delta - 1$ for all $u \in \mathcal{F}(C)$ if $\delta \geq 4$. Suppose that $\delta = 3$ in which case all the above inequalities are equalities, i.e., $|N_{(h-6)/2}(S_u^=(u_i)) \cap X| = |S_u^=(u_i)| = d(u_i) - 2 = 1$, hence $d(u_i) = 3, i = 1, 2, 3$. Let $S_u^=(u_i) = \{a_i\}, i = 1, 2, 3$. If $d(a_i, x_i) = (h-6)/2$ then the edge $a_i u_i$, the shortest (u_i, x_i)-path and the shortest (a_i, x_i)-path both of length $(h-6)/2$, form an odd cycle of length at most $h - 5$ going through u_i which is a contradiction with the hypothesis. Thus $d(a_i, x_j) = (h-6)/2, i \neq j, i, j = 1, 2, 3$. Then the shortest (a_i, x_j)-path of length $(h-6)/2$, (u, x_j)-path of length $(h-4)/2$ and the path u, u_i, a_i of length 2 forms an odd cycle of length at most $h - 3$ which is a contradiction with the hypothesis. Hence (i) holds.

(ii) Now suppose that $\delta \geq 4$ and let us see that $|N(u) \setminus \mathcal{F}(C)| \leq \delta - 2$ for some $u \in \mathcal{F}(C)$. Take any $u \in \mathcal{F}(C)$ and assume to the contrary that the vertices $u_1, \ldots, u_{\delta-1} \in N(u) \setminus \mathcal{F}(C)$ can be considered.

Claim 1. $S_u^+(u_i) \neq \emptyset$ for some $i \in \{1, \ldots, \delta - 1\}$.

Suppose $S_u^+(u_i) = \emptyset$ for all $i \in \{1, \ldots, \delta - 1\}$. Note that from

$$\delta \geq |X| \geq |N_{(h-4)/2}(u) \cap X| \geq \sum_{i=1}^{\delta-1} |S_u^-(u_i)| \geq \delta - 1$$

it follows that we may suppose that $|S_u^-(u_1)| \leq 2$ and $|S_u^-(u_i)| = 1$ for $i \neq 1$. Therefore, $|S_u^=(u_1)| \geq \delta - 1 - 2 \geq 1$ and $|S_u^=(u_i)| \geq \delta - 1 - 1 \geq 2$ for $i = 2, \ldots, \delta - 1$, because $\delta \geq 4$. Since the sets $N_{(h-6)/2}(S_u^=(u_i)) \cap X$ are pairwise disjoint (because there is no even cycles of length at most $4 + 2(h-6)/2 = h - 2$) and $|N_{(h-6)/2}(S_u^=(u_i)) \cap X| \geq |S_u^=(u_i)|$ we obtain

$$\delta \geq |X| \geq \sum_{i=1}^{\delta-1} |N_{(h-6)/2}(S_u^=(u_i)) \cap X| \geq 1 + 2(\delta - 2) = 2\delta - 3 > \delta,$$

because $\delta \geq 4$, which is a contradiction. Thus, there exists $i \in \{1, \ldots, \delta - 1\}$ such that $S_u^+(u_i) \neq \emptyset$. □

Now take $w \in S_u^+(u_i)$ and observe that the sets $N_{(h-4)/2}(u) \cap X$ and $N_{(h-6)/2}(S_{u_i}^-(w)) \cap X$ are disjoint for if not an even cycle of length at most $2 + 2(h-4)/2 = h - 2$ is created. Furthermore, $|N_{(h-4)/2}(u) \cap X| \geq |N(u) \setminus \mathcal{F}(C)| \geq \delta - 1$ and $|N_{(h-6)/2}(S_{u_i}^-(w)) \cap X| \geq |S_{u_i}^-(w)|$ because even cycles of length $h - 4$ do not exist. Hence,

$$\delta \geq |X| \geq |N_{(h-4)/2}(u) \cap X| + |N_{(h-6)/2}(S_{u_i}^-(w)) \cap X| \geq \delta - 1 + |S_{u_i}^-(w)|$$

and therefore $|S_{u_i}^-(w)| \leq 1$, hence $|N(w) \setminus \mathcal{F}(C)| = |S_{u_i}^-(w)| + 1 \leq 2 \leq \delta - 2$, as $\delta \geq 4$. Then (ii) holds. □

Proof of Proposition 2. Let us denote $\mu(C) = \max\{d(u, X) : u \in V(C)\}$. Obviously the proposition is valid for $\mu(C) > (h-4)/2$. So assume that $\mu(C) = (h-4)/2$ and denote by $\mathcal{F}(C) = \{u \in V(C) : d(u, X) = (h-4)/2\}$. For all $u \in \mathcal{F}(C)$ the sets $N_{(h-4)/2}(v) \cap X$ are pairwise disjoint for all $v \in N(u) \cap \mathcal{F}(C)$, otherwise an even cycle of length $2 + 2(h-4)/2 = h - 2$ is created. By the same reason, the sets $N_{(h-4)/2}(v) \cap X$ for all $v \in N(u) \setminus \mathcal{F}(C)$ are pairwise disjoint. Then $|X| \geq 2$. Furthermore if $|X| = 2$, $|N(u) \cap \mathcal{F}(C)| \leq 2$ and $|N(u) \setminus \mathcal{F}(C)| \leq 2$ for all $u \in \mathcal{F}(C)$, thus $\delta \leq 4$. If $\delta = 4$, $N_{(h-4)/2}(z) \cap X = N_{(h-4)/2}(z') \cap X = \{x_1, x_2\}$ for all $z, z' \in N(u)$, where $u \in \mathcal{F}(C)$ and $\{x_1, x_2\} \subset X$. Then, an even cycle of length at most $h - 2$ is produced by the shortest (z, x_1)-path, (z', x_1)-path and the z', u, z path of length 2 which is a contradiction. Hence we conclude that $|X| \geq 2$ if $\delta \geq 3$ and $|X| \geq 3$ if $\delta \geq 4$.

First let us prove the proposition for $\delta = 3$. Assume that $|N(u) \cap \mathcal{F}(C)| \leq 1$ for all $u \in \mathcal{F}(C)$. From Lemma 1(i), it follows that $|N(u) \cap \mathcal{F}(C)| = 1$ and $|N(u) \setminus \mathcal{F}(C)| = 2$, that is, $d(u) = 3$ for all $u \in \mathcal{F}(C)$. Therefore considering $z \in N(u) \cap \mathcal{F}(C)$ it is clear that $N_{(h-4)/2}(u) \cap X$ and $N_{(h-4)/2}(z) \cap X$ must have a common vertex. Thus an odd cycle of length at most $h - 3$ going through vertices of degree 3 is produced which is a contradiction. Consequently there exists some $u_0 \in \mathcal{F}(C)$ such that two distinct vertices $z, z' \in N(u_0) \cap \mathcal{F}(C)$ can be considered, then $3 \geq |X| \geq |N_{(h-4)/2}(z) \cap X| + |N_{(h-4)/2}(z') \cap X|$. Thus one of z, z' for instance z satisfies that $|N_{(h-4)/2}(z) \cap X| = 1$ because otherwise, an even cycle of length $2 + 2(h-4)/2 = h - 2$ would be created. Then $|N(z) \cap \mathcal{F}(C)| = d(z) - 1$. If $d(z) = 3$ then the sets $N_{(h-4)/2}(z) \cap X$, $N_{(h-4)/2}(u_0) \cap X$ and $N_{(h-4)/2}(w) \cap X$, where $w \in N(z) \cap \mathcal{F}(C)$, are pairwise disjoint because otherwise an odd cycle of length at most $h - 3$ passing through z would be created, which is a contradiction. Thus $|N_{(h-4)/2}(z) \cap X| = |N_{(h-4)/2}(u_0) \cap X| = |N_{(h-4)/2}(w) \cap X| = 1$ because $|X| \leq 3$ and the result follows. If $d(z) \geq 4$ then the sets $N_{(h-4)/2}(u_0) \cap X$, $N_{(h-4)/2}(w_1) \cap X$ and $N_{(h-4)/2}(w_2) \cap X$ where $w_1, w_2 \in N(z) \cap \mathcal{F}(C)$ are pairwise disjoint. Thus $|N_{(h-4)/2}(u_0) \cap X| = |N_{(h-4)/2}(w_1) \cap X| = |N_{(h-4)/2}(w_2) \cap X| = 1$ and again the result is valid.

Next suppose $\delta \geq 4$ and let us show the following claim.

Claim 1. Suppose that $\delta \geq 4$. There exists a vertex $u \in \mathcal{F}(C)$ such that $|N_{(h-4)/2}(u) \cap X| = 1$ or $\delta = 4$ and $|N(u) \cap \mathcal{F}(C)| = 2$; in this case it follows that $|N_{(h-4)/2}(v) \cap X| = 2$ for all $v \in N(u) \cap \mathcal{F}(C)$.

Proof. Let us denote by $r = \min\{|N_{(h-4)/2}(w) \cap X| : w \in \mathcal{F}(C)\}$. If $r = 1$ the claim holds, so assume that $r \geq 2$. First assume that $r \geq \delta - 1$. By Lemma 1(ii), there exists some vertex $u_0 \in \mathcal{F}(C)$ such that $|N(u_0) \cap \mathcal{F}(C)| \geq 2$. Since $|N_{(h-4)/2}(v) \cap X| \geq r \geq \delta - 1$ for all $v \in N(u_0) \cap \mathcal{F}(C)$,

$$\delta \geq |X| \geq \sum_{v \in N(u_0) \cap \mathcal{F}(C)} |N_{(h-4)/2}(v) \cap X| \geq |N(u_0) \cap \mathcal{F}(C)|\, r \geq 2(\delta - 1)$$

which is a contradiction because $\delta \geq 4$. Therefore $2 \leq r \leq \delta - 2$ and let $u \in \mathcal{F}(C)$ be such that $|N_{(h-4)/2}(u) \cap X| = r$, then from the inequalities

$$r = |N_{(h-4)/2}(u) \cap X| \geq |N(u) \setminus \mathcal{F}(C)| \geq \delta - |N(u) \cap \mathcal{F}(C)|$$

it follows that $|N(u) \cap \mathcal{F}(C)| \geq \delta - r$. Moreover, the sets $N_{(h-4)/2}(v) \cap X$, for $v \in N(u) \cap \mathcal{F}(C)$ are pairwise disjoint because otherwise, an even cycle of length $2 + 2(h-4)/2 = h - 2$ is created. Hence,

$$\delta \geq |X| \geq \sum_{v \in N(u) \cap \mathcal{F}(C)} |N_{(h-4)/2}(v) \cap X| \geq r|N(u) \cap \mathcal{F}(C)| \geq r(\delta - r) \tag{5}$$

which is a contradiction unless $r = 2$ and $\delta = 4$. Furthermore, if $r = 2$ and $\delta = 4$ then all the inequalities of (5) become equalities, that is, $|N(u) \cap \mathcal{F}(C)| = 2$ and $|N_{(h-4)/2}(v) \cap X| = 2$ for all $v \in N(u) \cap \mathcal{F}(C)$, which also proves the claim for $r = 2$ and $\delta = 4$. □

By Claim 1 two cases need to be studied.

Case 1. There exists a vertex $u \in \mathcal{F}(C)$ such that $|N_{(h-4)/2}(u) \cap X| = 1$.

This implies that $|N(u) \setminus \mathcal{F}(C)| = 1$. Let $N_{(h-4)/2}(u) \cap X = \{x_u\}$; obviously the proposition is true for every $x \in X - x_u$. As that $|N(u) \cap \mathcal{F}(C)| \geq \delta - |N(u) \setminus \mathcal{F}(C)| = \delta - 1$ and the sets $N_{(h-4)/2}(w) \cap X$ are pairwise disjoint for all $w \in N(u) \cap \mathcal{F}(C)$, then there exists a set $R \subset N(u) \cap \mathcal{F}(C)$ of $|R| \geq \delta - 2$ such that $x_u \notin N_{(h-4)/2}(z) \cap X$ for all $z \in R$. Suppose that every $z \in R$ is such that $|N_{(h-4)/2}(z) \cap X| \geq 2$. Then

$$\delta - 1 \geq |X - x_u| \geq \sum_{z \in R} |N_{(h-4)/2}(z) \cap X| \geq 2(\delta - 2),$$

which is a contradiction for $\delta \geq 4$. Thus, a vertex $z \in R$ such that $|N_{(h-4)/2}(z) \cap X| = 1$ and $x_u \notin N_{(h-4)/2}(z) \cap X$ can be selected and hence the proposition holds in this case. This finishes the proof of item (i).

Case 2. There exists a vertex $u_0 \in \mathcal{F}(C)$ such that $|N_{(h-4)/2}(u_0) \cap X| = 2$ which implies that $\delta = 4$ by Claim 1. Let us prove that there exists a vertex $z \in \mathcal{F}(C)$ such that $|N_{(h-4)/2}(z) \cap X| = 2$ and $\{x_1, x_2\} \cap (N_{(h-4)/2}(z) \cap X) = \emptyset$, where x_1, x_2 are two prescribed vertices of X.

Suppose there exists a vertex $w \in \mathcal{F}(C)$ such that $|N(w) \setminus \mathcal{F}(C)| = 1$, then $|N(w) \cap \mathcal{F}(C)| \geq \delta - 1 = 3$. Take $\{w_1, w_2, w_3\} \subset N(w) \cap \mathcal{F}(C)$. Since $(N_{(h-4)/2}(w_i) \cap X) \cap (N_{(h-4)/2}(w_j) \cap X) = \emptyset$ for all pair $i \neq j$, there exists some vertex $w_i \in N(w) \cap \mathcal{F}(C)$ such that $\{x_1, x_2\} \cap (N_{(h-4)/2}(w_i) \cap X) = \emptyset$ hence the result holds. Therefore we may suppose that $|N(w) \setminus \mathcal{F}(C)| \geq 2$ for every vertex $w \in \mathcal{F}(C)$ yielding $|N_{(h-4)/2}(w) \cap X| \geq 2$ for every $w \in \mathcal{F}(C)$. By Lemma 1(ii), it follows that $|N(w) \setminus \mathcal{F}(C)| = 2$ for every $w \in \mathcal{F}(C)$.

Take $u_1, u_2 \in N(u_0) \cap \mathcal{F}(C)$ and $|N_{(h-4)/2}(u_1) \cap X| = |N_{(h-4)/2}(u_2) \cap X| = 2$. Let us prove that this vertex u_0 satisfies the following assertion.

$$|S^{+}_{u_0}(v)| \geq 2 \quad \text{for all } v \in N(u_0) \setminus \mathcal{F}(C). \tag{6}$$

First, observe that $|S^{=}_{u_0}(v)| = 0$ for all $v \in N(u_0) \setminus \mathcal{F}(C)$; otherwise, since the sets $N_{(h-4)/2}(u_1) \cap X$, $N_{(h-4)/2}(u_2) \cap X$ and $N_{(h-6)/2}(S^{=}_{u_0}(v)) \cap X$ are pairwise disjoint we get

$$\begin{aligned} 4 \geq |X| &\geq |N_{(h-4)/2}(u_1) \cap X| + |N_{(h-4)/2}(u_2) \cap X| + |N_{(h-6)/2}(S^{=}_{u_0}(v)) \cap X| \\ &\geq 2 + 2 + 1 = 5, \end{aligned}$$

which is a contradiction. Therefore $|S^{=}_{u_0}(v)| = 0$ for all $v \in N(u_0) \setminus \mathcal{F}(C)$. Second let us show that $|S^{-}_{u_0}(v)| = 1$ for all $v \in N(u_0) \setminus \mathcal{F}(C)$. On the contrary we would have $|N_{(h-4)/2}(u_0) \cap X| \geq |S^{-}_{u_0}(v_1)| + |S^{-}_{u}(v_2)| \geq 3$ where $v_1, v_2 \in N(u) \setminus \mathcal{F}(C)$. Taking $z \in (N(u_1) - u_0) \cap \mathcal{F}(C)$, which exists because $|N(u_1) \setminus \mathcal{F}(C)| = 2$, and noting that $N_{(h-4)/2}(u_0) \cap X$ and $N_{(h-4)/2}(z) \cap X$ are pairwise disjoint then

$$4 \geq |X| \geq |N_{(h-4)/2}(u_0) \cap X| + |N_{(h-4)/2}(z) \cap X| \geq 3 + 2 = 5,$$

which is a contradiction, hence $|S^{-}_{u_0}(v)| = 1$ for all $v \in N(u_0) \setminus \mathcal{F}(C)$. Since $\delta = 4$, $|S^{=}_{u_0}(v)| = 0$ and $|S^{-}_{u_0}(v)| = 1$, $|S^{+}_{u}(v)| \geq 2$ for all $v \in N(u_0) \setminus \mathcal{F}(C)$ as claimed.

Now note that if $\{x_1, x_2\} \cap (N_{(h-4)/2}(u_0) \cap X) = \emptyset$ then we are done. Thus, we may assume that $x_1 \in N_{(h-4)/2}(u_0) \cap X$. If $x_2 \in N_{(h-4)/2}(u_0) \cap X$, then take $u_1 \in N(u_0) \cap \mathcal{F}(C)$ and $z \in (N(u_1) - u_0) \cap \mathcal{F}(C)$, and observe that the sets $N_{(h-4)/2}(z) \cap X$ and $N_{(h-4)/2}(u_0) \cap X$ are pairwise disjoint. Then $\{x_1, x_2\} \cap (N_{(h-4)/2}(z) \cap X) = \emptyset$, and we are done. If $x_2 \notin N_{(h-4)/2}(u_0) \cap X$, let us consider a vertex $v \in N(u_0) \setminus \mathcal{F}(C)$ such that $x_1 \notin N_{(h-6)/2}(v) \cap X$. Clearly, $x_2 \notin N_{(h-6)/2}(v) \cap X$. From (6), it follows that two vertices $z_1, z_2 \in S^{+}_{u_0}(v)$ can be selected. Since $x_1 \notin N_{(h-4)/2}(z_i) \cap X$, $|N_{(h-4)/2}(z_i) \cap X| \geq 2$ for $i = 1, 2$, and $(N_{(h-4)/2}(z_1) \cap X) \cap (N_{(h-4)/2}(z_2) \cap X) = N_{(h-6)/2}(v) \cap X$, one of the z_1 or z_2, for instance, z_1, satisfies that $x_2 \notin N_{(h-4)/2}(z_1) \cap X$. As $x_1 \notin N_{(h-4)/2}(z_1) \cap X$ and $|X| = 4$, we deduce that $|N_{(h-4)/2}(z_1) \cap X| = 2$ and $\{x_1, x_2\} \cap (N_{(h-4)/2}(z_1) \cap X) = \emptyset$. This finishes the proof of item (ii). □

Proof of Theorem 3. Suppose that G is not super-k and let X be a cut set with cardinality $|X| \leq \delta$ then any component C of $G - X$ has $|V(C)| \geq 2$. Let C and C' denote two components of $G - X$ and $\mu(C) = \max\{d(u, X) : u \in V(C)\}$ and $\mu(C') = \max\{d(X, u') : u' \in V(C')\}$. Then, from Proposition 1, it follows that $\mu(C) \geq (h-4)/2$ and $\mu(C') \geq (h-4)/2$. Hence, if $u \in V(C)$ and $u' \in V(C')$ are such that $d(u, X) = \mu(C)$ and $d(X, u') = \mu(C')$, then

$$diam(G) \geq d(u, u') \geq d(u, X) + d(X, u') \geq \mu(C) + \mu(C') = h - 4. \tag{7}$$

If $diam(G) \leq h - 5$ we arrive at a contradiction and then G is super-κ, thus item (i) is proved. So assume that $\delta \geq 4$ or $\delta = 3$ and vertices of degree 3 are not on odd cycles of length less than $h - 3$. The hypothesis on the diameter means that all the inequalities of (7) become equalities, that is, $\mu(C) = (h-4)/2$ and $\mu(C') = (h-4)/2$. Then by applying Proposition 2, a vertex $u \in V(C)$ can be selected such that $d(u, X) = (h-4)/2$ and $|N_{(h-4)/2}(u) \cap X| \leq 2$. Moreover, a vertex $u' \in V(C')$ can be selected such that $d(X, u') = (h-4)/2$, $|N_{(h-4)/2}(u') \cap X| \leq 2$ and $(N_{(h-4)/2}(u) \cap X) \cap (N_{(h-4)/2}(u') \cap X) = \emptyset$. Hence,

$$diam(G) \geq d(u, u') \geq d(u, X) + 1 + d(X, u') \geq (h-4)/2 + 1 + (h-4)/2 = h-3,$$

contradicting the hypothesis. So G is super-κ and the result holds. □

Proof of Theorem 4. Let G be a polarity graph. Since $g = 3$, $h = 6$ and $diam(G) = 2$ it follows from Theorem 2 point (i) that G is maximally connected. Suppose that G is non-super-κ, thus there exists a cut set X of G of $|X| = \delta$ such that every connected component of $G - X$ has at least two vertices. Observe that $|N(v) \cap X| \geq 1$ for all $v \in V(G) \setminus X$ because $diam(G) = 2$. First assume that $|N(v) \cap X| \geq 2$ for all $v \in V(G) \setminus X$. As $\delta \leq d(s) \leq \delta + 1$ for all $s \in X$ it follows that

$$2|V(G) \setminus X| \leq |[V(G) \setminus X, X]| \leq |X|(\delta + 1) = \delta(\delta + 1).$$

Since $|V(G)| = \delta^2 + \delta + 1$, $2(\delta^2 + 1) \leq \delta^2 + \delta$ which is a contradiction. Therefore there exists a vertex $v_0 \in V(G) \setminus X$ such that $N(v_0) \cap X = \{x_0\}$. Let C be a component of $G - X$ such that $v_0 \notin V(C)$. Since $d(v_0, w) = 2$ for all $w \in V(C)$ it is forced that $V(C) \subset N(x_0)$. Let $uv \in E(C)$. If there exists other vertex $z \in V(C)$ such that $z \in N(u) \cap V(C)$, then the 4-cycle x_0, z, u, w, x_0 is created which is a contradiction. Therefore $(N(u) - w) \cup (N(w) - u) \subset X$. Further as $N(u) \cap N(w) \cap (X - x_0) = \emptyset$ (otherwise a 4-cycle would appear) then

$$\delta - 1 = |X - x_0| \geq |N(u) \setminus \{w, x_0\}| + |N(w) \setminus \{u, x_0\}| = d(u) + d(w) - 4,$$

which is only possible if $\delta = d(u) = d(w) = 3$ because by hypothesis $\delta \geq 3$. Hence u, w are vertices of degree 3 and x_0, u, v, x_0 is a triangle. This is a contradiction in polarity graphs, because vertices lying on triangles must have degree $\delta + 1 = 4$. Therefore G has no κ_1-cut X of δ vertices meaning that G is super-κ. □

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References

[1] M. Abreu, C. Balbuena, D. Labbate, Adjacency matrices of polarity graphs and of other C_4-free graphs of large size, Des. Codes Cryptogr. 55 (2010) 221–231.
[2] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez, X. Marcote, On the restricted connectivity and superconnectivity in graphs with given girth, Discrete Math. 307 (6) (2007) 659–667.
[3] C. Balbuena, M. Cera, A. Diánez, P. García-Vázquez, X. Marcote, Connectivity of graphs with given girth pair, Discrete Math. 307 (2007) 155–162.
[4] C. Balbuena, T. Jiang, Y. Lin, X. Marcote, M. Miller, Order of regular graphs with given girth pair, J. Graph Theory 55 (2007) 153–163.
[5] C. Balbuena, K. Marshall, L.P. Montejano, Connectivity and superconnectivity of graphs with small diameter, Discrete Appl. Math. 158 (5) (2010) 397–403.
[6] C. Balbuena, K. Marshall, J. Tang, Y. Lin, Superconnectivity of regular graphs with small diameter, Discrete Appl. Math. 157 (7) (2009) 1349–1353.
[7] F.T. Boesch, Synthesis of reliable networks—a survey, IEEE Trans. Reliab. 35 (1986) 240–246.
[8] F.T. Boesch, R. Tindell, Circulants and their connectivities, J. Graph Theory 8 (1984) 487–499.
[9] J.A. Bondy, Erdős, S. Fajtlowicz, Graphs of diameter two with no 4-circuits, Discrete Math. 200 (1999) 21–25.
[10] C.M. Campbell, On cages for girth pair (6, b), Discrete Math. 177 (1997) 259–266.
[11] G. Chartrand, L. Lesniak, Graphs and Digraphs, 4th edition, Chapman and Hall, London, UK, 2005.
[12] J. Fàbrega, M.A. Fiol, Maximally connected digraphs, J. Graph Theory 13 (3) (1989) 657–668.
[13] Z. Füredi, Graphs without quadrilaterals, J. Combin. Theory, Ser. B 34 (1983) 187–190.
[14] Z. Füredi, On the number of edges of quadrilateral-free graphs, J. Combin. Theory, Ser. B 68 (1996) 1–6.
[15] F. Harary, P. Kovács, Regular graphs with given girth pair, J. Graph Theory 7 (1983) 209–218.
[16] A. Hellwig, L. Volkmann, Maximally edge-connected and vertex-connected graphs and digraphs: a survey, Discrete Math. 308 (15) (2008) 3265–3296.
[17] M. Imase, T. Soneoka, K. Okada, Connectivity of regular directed graphs with small diameter, IEEE Trans. Comput. C-34 (1985) 267–273.
[18] T. Soneoka, H. Nakada, M. Imase, C. Peyrat, Sufficient conditions for maximally connected dense graphs, Discrete Math. 63 (1987) 53–66.