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Time-Optimal Control of Disturbance-Rejection Tracking Systems for Discrete-Time Time-Delayed Systems by State Feedback

S. M. KARBASSI
Faculty of Mathematics, Yazd University
Yazd, 89195–741 Iran
smkarbassi@yazduni.ac.ir

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Abstract—In this paper, we solve the disturbance-rejection and tracking problem for linear multivariable discrete-time systems with time-delayed controlled inputs. A set of necessary and sufficient conditions under which the proposed problem is controllable is defined. Also, the nilpotency properties of such systems is established and used as the basis of a comprehensive design procedure. This general procedure is illustrated by designing a time-optimal disturbance rejection tracking system for a stirred-tank with time-delayed control inputs. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Disturbance-rejection, Tracking, Time-optimal, Discrete-time Systems, Control Systems.

1. INTRODUCTION

In this paper, the design of time-optimal disturbance-rejection tracking controllers for discrete-time systems with multiple delays is investigated. The problem of disturbance-rejection has been solved by Porter [1] with a polynomial command input vector in the absence of time-delay. Schmitendorf studied the robustness of tracking systems [2] and Wang and Daley [3] and Tao, Joshi and Ma [4] studied the tracking control of systems with actuator failures. The results of Porter are extended to establish controllability and nilpotency properties of disturbance-rejection tracking systems in the presence of time-delays. By employing the method of augmentation of the state vector and applying the results of Klein and Ramirez [5] and Modarres and Karbassi [6], an explicit expression for the feedback matrix for time-optimal control of such systems is obtained. For the sake of simplicity, only the tracking of step command vectors and the rejection of constant disturbances is considered. Clearly, the design technique can be readily extended to the tracking of a polynomial input vector in the manner of Porter. The well known problem of stirred-tank, presented by Kwakernaak and Sivan [7] is chosen as an illustrative example and time-optimal disturbance rejection of the stirred-tank with input time-delays is simulated for design purposes.

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The problem of tracking is to design a controller such that the resulting closed-loop system is asymptotically stable and the controlled output tracks a given reference signal in the presence of any initial condition and external disturbances. The purpose of this paper is to present a comprehensive procedure for the design of time-optimal disturbance-rejection tracking systems for a class of linear multivariable discrete-time systems with time-delayed inputs.

Such tracking systems consist of a controllable system governed in the discrete-time domain by state and output equations of the respective forms,

$$x_0(k+1) = A_0 x_0(k) + \sum_{j=0}^{r} B_j u(k-j) + E_0 d(k) \tag{1}$$

and

$$y(k) = C_0 x_0(k), \tag{2}$$

together with a controller which is required to cause the output vector $y(k)$ of the system to track the piecewise constant command input vector $v(k)$ whilst simultaneously rejecting the piecewise constant disturbance vector $d(k)$ in the sense that

$$y(k) = v(k) \qquad (k \geq k_{\min}), \tag{3}$$

where $k_{\min}$ is to be as small as possible.

In equations (1)–(3), $x_0(k) \in \Re^n$, is the state vector and $u(k-j) \in \Re^m$, $j = 0, 1, \dots, r$, are the control input vectors delayed by j sampling intervals of duration T, $y(k) \in \Re^p$ is the output vector, $d(k) \in \Re^q$ is the disturbance vector and $v(k) \in \Re^p$ is a command input vector. The matrices A_0, B_j $(j = 0, 1, \dots, r)$, E_0 are known matrices of appropriate sizes which are functions of the sampling period T but C_0 is in general independent of T.

2. THEORY

The first stage in the design of the controller involves the introduction of a vector comparator and a discrete-time vector integrator in order to generate the $n \times 1$ vector $z(k)$ defined by the equation,

$$z(k+1) = z(k) + T(v(k) - y(k)). \tag{4}$$

Then, it is evident from equations (1), (2), and (4) that the open-loop tracking system is governed by a state equation of the form,

$$x(k+1) = Ax(k) + Bu(k) + Dv(k) + Ed(k), \tag{5}$$

where

$$x(k) = \begin{bmatrix} x_0(k) \\ u(k-r) \\ u(k-r+1) \\ \vdots \\ u(k-2) \\ u(k-1) \\ z(k) \end{bmatrix} \tag{6}$$

is an $(n+rm+p)\times 1$ augmented state vector [5] obtained by storing the r previous $m\times 1$ control input vectors $u(k-j)\in\Re^m$, $j=0,1,\ldots,r$, together with the $p\times 1$ integrator state vector $z(k)$,

$$A=\begin{bmatrix} A_0 & B_r & B_{r-1} & \cdots & B_1 & 0 \\ 0 & 0 & I_m & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ -TC_0 & 0 & 0 & \cdots & 0 & I_p \end{bmatrix} \tag{7}$$

is an $(n+rm+p)\times(n+rm+p)$ augmented open-loop system matrix,

$$B=[B_0 \quad 0 \quad 0 \quad \cdots \quad 0 \quad I_m \quad 0]^{\mathsf{T}} \tag{8}$$

is an $(n+rm+p)\times m$ augmented input matrix,

$$D=[0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad TI_p]^{\mathsf{T}} \tag{9}$$

is an $(n+rm+p)\times p$ augmented command input matrix, and

$$E=[E_0 \quad 0 \quad 0 \quad \ldots \quad 0 \quad 0 \quad 0]^{\mathsf{T}} \tag{10}$$

is an $(n+rm+p)\times q$ augmented disturbance matrix.

The second stage in the design of the controller involves the introduction of a feedback controller in order to generate the present $m\times 1$ control input vector $u(k)$ according to the control-law equation,

$$u(k)=K_0x_0(k)+K_ru(k-r)+\cdots+K_2u(k-2)+K_1u(k-1)+Kz(k)=Fx(k), \tag{11}$$

where K_0 is a $m\times n$ matrix, the K_j $(j=0,1,\ldots,r)$ are $m\times m$ matrices, and K is a $m\times p$ matrix. It is evident from equations (5) and (11) that the closed-loop tracking system is governed by a state equation of the form,

$$x(k+1)=(A+BF)x(k)+DV(k)+Ed(k), \tag{12}$$

and therefore, that the closed-loop system will behave so that (3) is satisfied with

$$k_{\min}=\sigma. \tag{13}$$

In case the feedback controller is designed so that the closed-loop system matrix $(A+BF)$ is nilpotent with minimal index σ.

However, it is obviously possible to synthesize a controller of this kind if and only if the open-loop tracking system governed by equation (5) is controllable. The controllability and nilpotency properties of this system are accordingly established in the following sequence of theorems.

THEOREM 1. *In the case of systems for which*

$$\left(A_0, \sum_{j=0}^{r} A_0^{r-j}B_j\right) \tag{14}$$

is a controllable pair, then the pair (A, B) is controllable if and only if

$$\text{rank} = \begin{bmatrix} A_0 - I_n & B_r & B_{r-1} & \cdots & B_1 & B_0 \\ 0 & -I_m & I_m & \cdots & 0 & 0 \\ 0 & 0 & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m & 0 \\ 0 & 0 & 0 & \cdots & -I_m & I_m \\ -TC_0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} = n + rm + p. \tag{15}$$

PROOF. It is evident that, in the case of systems for which $(\bar{A}_r, \bar{B}_r)$ is a controllable pair, the pair

$$(A, B) = \left(\begin{bmatrix} \bar{A}_r & 0 \\ -\bar{C}_r & I_p \end{bmatrix}, \begin{bmatrix} \bar{B}_r \\ 0 \end{bmatrix} \right) \tag{16}$$

is controllable if and only if

$$\text{rank} \begin{bmatrix} \bar{A}_r - I_{n+rm} & \bar{B}_r \\ -\bar{C}_r & 0 \end{bmatrix} = n + rm + p, \tag{17}$$

where

$$\bar{A}_r = \begin{bmatrix} A_0 & B_r & B_{r-1} & \cdots & B_1 \\ 0 & 0 & I_m & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & I_m \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{18}$$

is an $(n + rm) \times (n + rm)$ matrix,

$$\bar{B}_r = [B_0 \quad 0 \quad 0 \quad \dots \quad 0 \quad I_m]^{\mathsf{T}} \tag{19}$$

is an $(n + rm) \times m$ matrix, and

$$\bar{C}_r = [TC_0 \quad 0 \quad 0 \quad \dots \quad 0 \quad 0]^{\mathsf{T}} \tag{20}$$

is a $p \times (n+rm)$ matrix. However, $(\bar{A}_r, \bar{B}_r)$ is a controllable pair if and only if (14) is a controllable pair since clearly,

$$\text{rank}\,[\, sI_{n+rm} - \bar{A}_r, \quad \bar{B}_r\,] = \text{rank} \left[sI_n - A_0, \quad \sum_{j=0}^{r} A_0^{r-j} B_j \right] + rm. \tag{21}$$

Thus, the theorem is established in view of the obvious equivalence of conditions (15) and (17).

THEOREM 2. *In the case of systems for which (A, B) is a controllable pair, then $\{\kappa_1 + r, \kappa_2 + r, \dots, \kappa_m + r\}$ is the set of controllability indices associated with the pair,*

$$\left(\begin{bmatrix} A_0 & 0 \\ -TC_0 & I_p \end{bmatrix}, \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \right). \tag{22}$$

PROOF. Clearly, there are at most $(\kappa_j + r)$ linearly independent vectors in the chain,

$$b_j, Ab_j, A^2 b_j, A^3 b_j, \dots, \tag{23}$$

generated by the j^{th} column b_j of the $(n + rm + p) \times m$ matrix B. But, there are exactly $(n + rm + p)$ linearly independent vectors in the entire set formed from the m chains of the form (23) associated with the m columns of B because the pair (A, B) is controllable [1]. Thus, the theorem is established since, therefore, it follows that there are exactly $(\kappa_j + r)$ linearly independent vectors generated by b_j $(j = 1, 2, \dots, m)$ in the entire set of $(n + rm + p)$ vectors: for only then, is it the case that

$$\sum_{j=1}^{m} (\kappa_j + r) = rm + \sum_{j=1}^{m} \kappa_j = n + rm + p, \tag{24}$$

where

$$\sum_{j=1}^{m} \kappa_j = n + p, \tag{25}$$

in view of the controllability of the pair (22).

THEOREM 3. *In the case of systems for which (A, B) is a controllable pair, there exists a feedback control law such that tracking occurs in the sense that*

$$y(k) = v(k) \qquad (k \geq \kappa_{\max} + r), \tag{26}$$

in the presence of the piecewise constant disturbance $d(k)$ where $\kappa_{\max}$ is the largest member of the set of controllability indices $\{\kappa_1, \kappa_2, \dots, \kappa_m\}$ associated with the controllable pair,

$$\left(\begin{bmatrix} A_0 & 0 \\ -TC_0 & I_p \end{bmatrix}, \begin{bmatrix} B_0 \\ 0 \end{bmatrix} \right). \tag{27}$$

PROOF. It follows from Theorem 2 that the vector companion form [1] of the pair (A, B) is such that the largest block matrix in the canonical form of A is of dimension $(\kappa_{\max} + r) \times (\kappa_{\max} + r)$ which is, therefore, the minimal achievable index of nilpotency of the closed-loop matrix $(A + BF)$. Thus, the theorem is established since the algorithm of Karbassi and Tehrani [8] can clearly be used to synthesize a control law of the form (11), such that

$$(A + BF)^{\sigma} = 0, \tag{28}$$

where

$$\sigma = \kappa_{\max} + r, \tag{29}$$

thus, establishing the tracking condition (26).

REMARKS. It is evident that Theorem 1 establishes the conditions under which the pair (A, B) in the state equation (5) of the open-loop tracking system is controllable. In the case of systems for which the pair (A, B) is controllable, Theorem 2 indicates that each member of the set of controllability indices $\{\kappa_1, \kappa_2, \dots, \kappa_m\}$ associated with the nondelayed system is increased by r because of the presence of the time-delayed control inputs. Thus, it is finally evident that Theorem 3 establishes that tracking is achievable in the case of controllable time-delayed systems in a minimum time of $(\kappa_{\max} + r)$ sampling intervals where $\kappa_{\max}$ is the largest member of the set $\{\kappa_1, \kappa_2, \dots, \kappa_m\}$.

3. ILLUSTRATIVE EXAMPLE

These general results can be conveniently illustrated by designing a time-optimal disturbance-rejection tracking system for a stirred-tank with time-delayed control inputs under computer control. The stirred-tank [7] is fed with two incoming flows whose incremental flow rates $\mu_1(t)$ and $\mu_2(t)$ are controlled by valves commanded by a process control computer but is subject to an unmeasurable disturbance because of fluid loss due to evaporation which is equivalent to an incremental flow rate $\nu(t)$ of the outgoing flow. The desired values of the incremental flow rate $\eta_1(t)$ and the incremental concentration $\eta_2(t)$ of the outgoing flow from the stirred-tank are the command inputs to the tracking system. The two feeds are mixed before flowing through a single inlet pipe into the stirred-tank thus introducing a transport delay τ. The deviations $\xi_1(t)$ and $\xi_2(t)$ of the volume and the concentration of fluid from the steady-state tank conditions are convenient state variables for a linearised representation of the plant. It is assumed that the tank is well stirred so that $\xi_2(t) = \eta_2(t)$.

In case the steady state conditions correspond to incoming flow rates of F_{10} and F_{20}, incoming feed concentrations of c_{10} and c_{20}, a tank fluid volume of v_0, a tank fluid concentration of c_0, and an outgoing flow rate of F_0, then the stirred-tank is governed by linearised continuous-time state and output equations of the respective forms [7],

$$\dot{x}_0(t) = \begin{bmatrix} -\dfrac{F_0}{2v_0} & 0 \\ 0 & -\dfrac{F_0}{v_0} \end{bmatrix} x_0(t) + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ \dfrac{c_{10}-c_0}{v_0} & \dfrac{c_{20}-c_0}{v_0} \end{bmatrix} u(t-\tau) + \begin{bmatrix} -1 \\ 0 \end{bmatrix} d(t) \quad (30)$$

and

$$y(t) = \begin{bmatrix} \dfrac{F_0}{2v_0} & 0 \\ 0 & 1 \end{bmatrix} x_0(t), \quad (31)$$

where

$$x_0(t) = [\,\xi_1(t) \quad \xi_2(t)\,]^{\mathrm{T}}, \quad (32)$$

$$y(t) = [\,\eta_1(t) \quad \eta_2(t)\,]^{\mathrm{T}}, \quad (33)$$

$$u(t) = [\,\mu_1(t) \quad \mu_2(t)\,]^{\mathrm{T}}, \quad (34)$$

and

$$d(t) = \nu(t). \quad (35)$$

The discrete-time state and output equations of the unaugmented computer-controlled plant corresponding to equations (1) and (2) in which the valve settings are constant on the intervals $[\,kT, \quad (k+1)T\,]$ $(k = 0, 1, 2, \dots)$ accordingly assume the respective forms,

$$\begin{aligned} x_0(k+1) = {} & \begin{bmatrix} 0.9512 & 0 \\ 0 & 0.9048 \end{bmatrix} x_0(k) + \begin{bmatrix} 4.8770 & 4.8770 \\ 0 & 0 \end{bmatrix} u(k) \\ & + \begin{bmatrix} 0 & 0 \\ -1.1895 & 3.5890 \end{bmatrix} u(k-1) + \begin{bmatrix} -4.8770 \\ 0 \end{bmatrix} d(k) \end{aligned} \quad (36)$$

and

$$y(k) = \begin{bmatrix} 0.01 & 0 \\ 0 & 1 \end{bmatrix} x_0(k). \quad (37)$$

When the sampling interval T and the transport delay τ are both 5, and [7]

$$F_{10} = 0.015\, m^3/s, \quad (38a)$$

$$F_{20} = 0.005\, m^3/s, \quad (38b)$$

$$c_{10} = 1\, \mathrm{kmol}/m^3, \quad (38c)$$

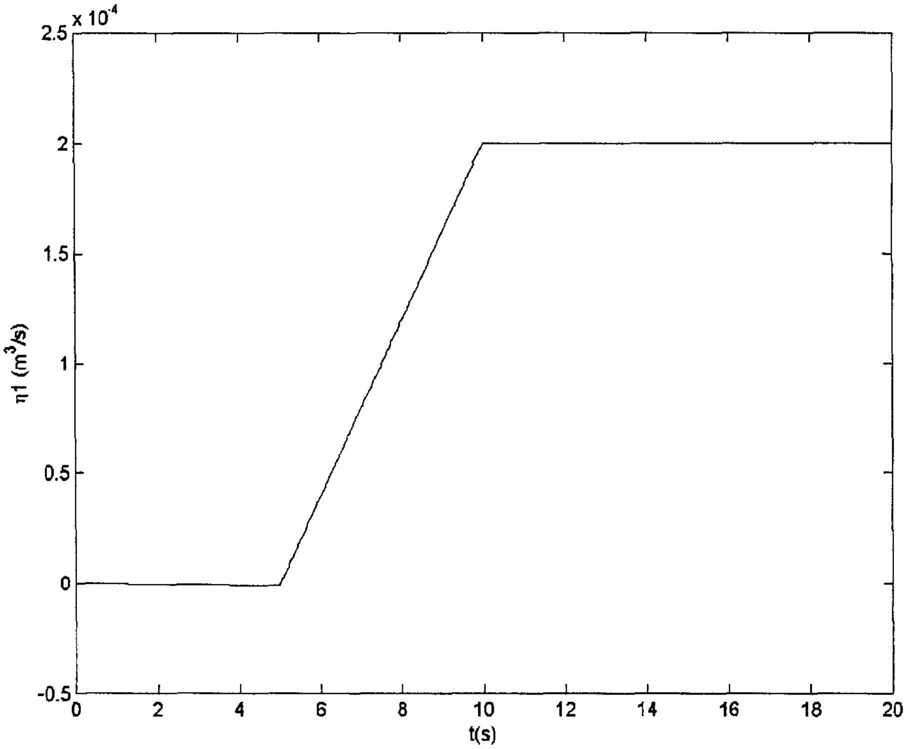

Figure 1. Rate of outgoing flow from tank.

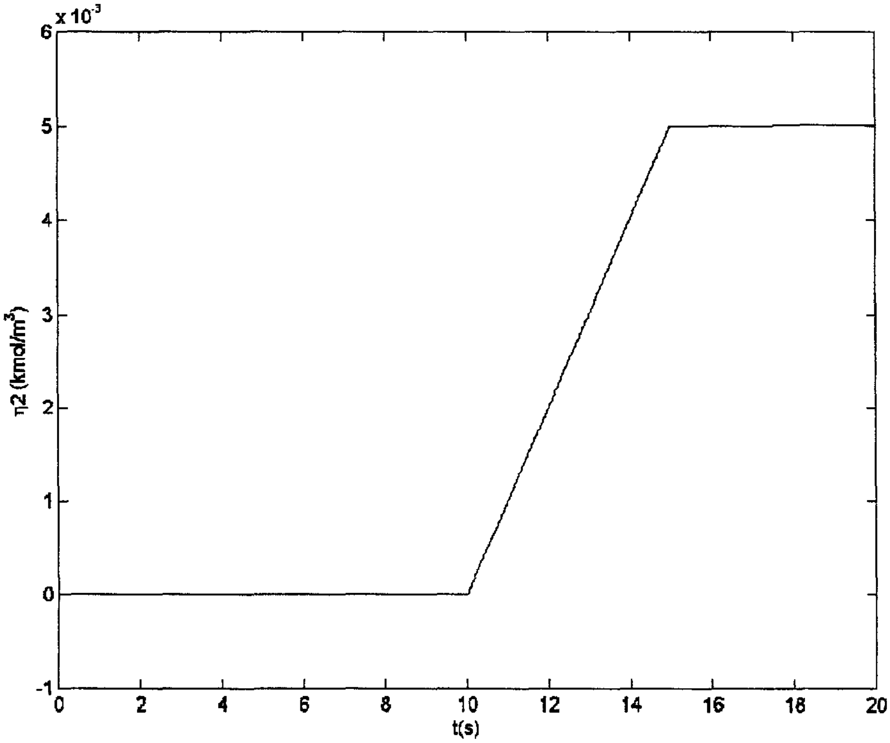

Figure 2. Concentration of outgoing flow from tank.

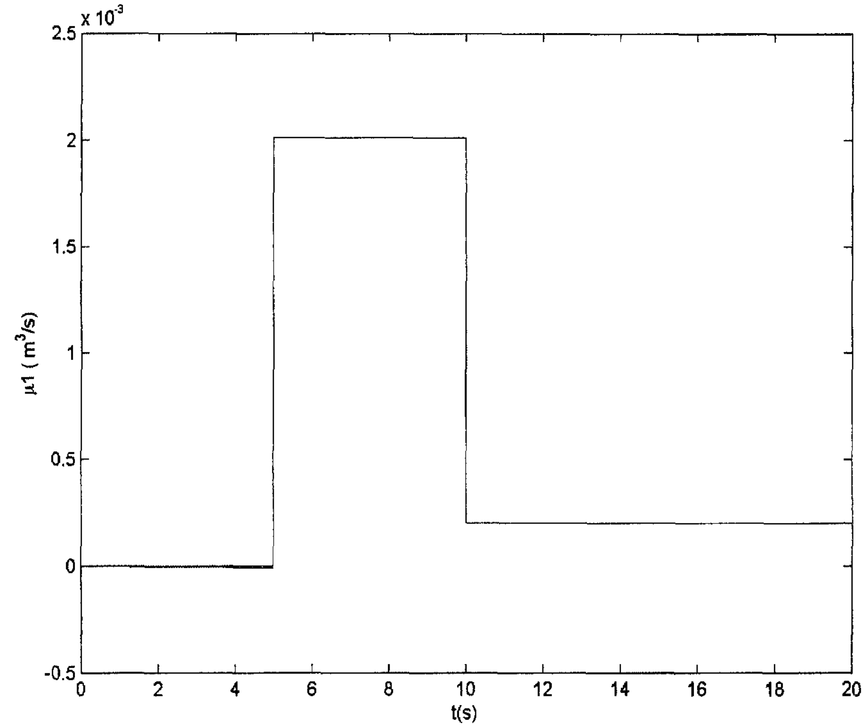

Figure 3. Commanded rate of incoming flow to tank from feed 1.

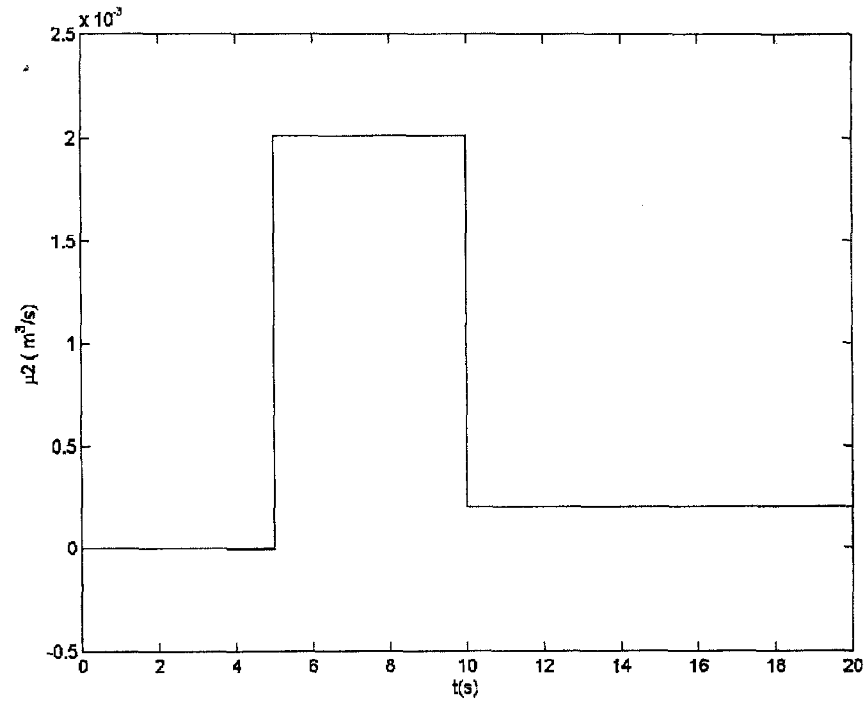

Figure 4. Commanded rate of incoming flow to tank from feed 2.

$$c_{20} = 2\,\text{kmol}/m^3, \tag{38d}$$

$$v_0 = 1\,m^3, \tag{38e}$$

$$c_0 = 1.25\,\text{kmol}/m^3, \tag{38f}$$

and

$$F_0 = 0.020\,m^3/s. \tag{38g}$$

The discrete-time state equation corresponding to equation (4) for the 2×1 integrator state vector $z(k)$ assumes the form,

$$z\,(k+1) = z\,(k) + 5\,(v\,(k) - y\,(k))\,, \tag{39}$$

and therefore, it is evident from Theorem 1 that the augmented plant is controllable and from Theorems 2 and 3 that tracking is achievable in three sampling intervals since $\{\,\kappa_1 \quad \kappa_2\,\} = \{\,2 \quad 2\,\}$ and $r = 1$. The algorithm of Karbassi and Bell [9] indicates that

$$\begin{aligned} u(k) = \begin{bmatrix} -0.3001 & 0.5724 \\ -0.1000 & -0.5724 \end{bmatrix} x_0(k) + \begin{bmatrix} -0.4762 & 1.4287 \\ 0.4762 & -1.4287 \end{bmatrix} u(k-1) \\ + \begin{bmatrix} 3.0757 & -0.0420 \\ 1.0251 & 0.0420 \end{bmatrix} z(k) \end{aligned} \tag{40}$$

is an appropriate time-optimal control law of the form (11).

Extensive simulation studies demonstrate that the resulting tracking system exhibits excellent finite settling-time characteristics as exemplified by the rejection of the step disturbance,

$$d(t) = \nu(t) = 0.0001 \qquad (t \geq 0), \tag{41}$$

whilst simultaneously tracking the step command input,

$$v(t) = [\,\omega_1(t) \quad \omega_2(t)\,]^{\mathsf{T}} = [\,0.0002 \quad 0.0005\,]^{\mathsf{T}} \qquad (t \geq 0), \tag{42}$$

where the units of $\nu(t)$ and $\omega_1(t)$ and of $\omega_2(t)$ are respectively m^3/s and kmol$/m^3$. In this case, tracking is completely achieved in $15s$ as shown in Figures 1 and 2 and the largest value of $\mu_1(t)$ and $\mu_2(t)$ are only $0.0022\,m^3/s$ and $0.0021\,m^3/s$, respectively, as shown in Figures 3 and 4. Excellent tracking and disturbance-rejection behavior is also found to occur when the unmeasurable disturbances arise from fluctuations in the incoming feed concentrations or from fluctuations in the incoming flow rates due to pressure fluctuations before the valves.

4. CONCLUSION

The design of time-optimal disturbance-rejection tracking systems for linear multivariable discrete-time plants with time-delayed control inputs has been considered. The controllability and nilpotency properties of such systems have been established and used as the basis of a comprehensive design procedure. The general procedure has been illustrated by designing a time-optimal disturbance-rejection tracking system for a stirred-tank with time-delayed control inputs under computer control. This design procedure can be readily extended so as to embrace plants with inaccessible states too.

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