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Regularizations of products of residue and principal value currents

Håkan Samuelsson

Matematik, Chalmers tekniska högskola och Göteborgs universitet, S-412 96 Göteborg, Sweden

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Abstract

Let f_1 and f_2 be two functions on some complex n-manifold and let φ be a test form of bidegree $(n, n-2)$. Assume that (f_1, f_2) defines a complete intersection. The integral of $\varphi/(f_1 f_2)$ on $\{|f_1|^2 = \epsilon_1, |f_2|^2 = \epsilon_2\}$ is the residue integral $I^{\varphi}_{f_1,f_2}(\epsilon_1, \epsilon_2)$. It is in general discontinuous at the origin. Let χ_1 and χ_2 be smooth functions on $[0,\infty]$ such that $\chi_j(0) = 0$ and $\chi_j(\infty) = 1$. We prove that the regularized residue integral defined as the integral of $\bar\partial\chi_1 \wedge \bar\partial\chi_2 \wedge \varphi/(f_1 f_2)$, where $\chi_j = \chi_j(|f_j|^2/\epsilon_j)$, is Hölder continuous on the closed first quarter and that the value at zero is the Coleff–Herrera residue current acting on φ. In fact, we prove that if φ is a test form of bidegree $(n, n-1)$ then the integral of $\chi_1\bar\partial\chi_2 \wedge \varphi/(f_1 f_2)$ is Hölder continuous and tends to the $\bar\partial$-potential $[(1/f_1) \wedge \bar\partial(1/f_2)]$ of the Coleff–Herrera current, acting on φ. More generally, let f_1 and f_2 be sections of some vector bundles and assume that $f_1 \oplus f_2$ defines a complete intersection. There are associated principal value currents U^f and U^g and residue currents R^f and R^g. The residue currents equal the Coleff–Herrera residue currents locally. One can give meaning to formal expressions such as e.g. $U^f \wedge R^g$ in such a way that formal Leibnitz rules hold. Our results generalize to products of these currents as well.
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Keywords: Multidimensional residue; Coleff–Herrera current; Cauchy–Fantappiè–Leray type current; Complete intersection

E-mail address: hasam@math.chalmers.se.

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1. Introduction

Consider a holomorphic function f defined on some complex n-manifold X and let $V_f = f^{-1}(0)$. Schwartz found that there is a distribution, or current, U on X such that $fU = 1$, [19,20]. The existence of the principal value current $[1/f]$ defined by

$$\mathscr{D}_{n,n}(X) \ni \varphi \mapsto \lim_{\epsilon \to 0} \int\limits_{|f|^2 > \epsilon} \varphi/f$$

was proved by Herrera and Lieberman in [9] using Hironaka's desingularization theorem, [10], and gives a realization of such a current U. The $\bar\partial$-image of the principal value current is the residue current associated to f. By Stokes' theorem its action on a test form of bidegree $(n, n-1)$ is given by the limit as $\epsilon \to 0$ (along regular values for $|f|^2$) of the residue integral

$$I_f^{\varphi}(\epsilon) = \int\limits_{|f|^2 = \epsilon} \varphi/f. \tag{1}$$

One main point discovered by Herrera and Lieberman is that if φ has bidegree $(n-1, n)$ then for each k, $I_{f^k}^{\varphi}(\epsilon) = \mathcal{O}(\epsilon^{\delta_k})$ for some positive δ_k. Using this, one can then smoothen the integration over $|f|^2 = \epsilon$ and regularize the residue current by using smooth functions χ defined on $[0, \infty)$ such that χ is 0 at zero and tends to 1 at infinity. In fact, we can make a Leray decomposition and write any (n,n)-test form φ as $\phi \wedge \partial f/f^k$ for some k, where ϕ is a test form of bidegree $(n-1, n)$ whose restriction to $|f|^2 = t$ is unique, for each $t > 0$. Then writing the integral of $\chi(|f|^2/\epsilon)\varphi/f$ as an integral over the level surfaces $|f|^2 = t$ and using Herrera's and Lieberman's result one sees that $\chi(|f|^2/\epsilon)/f$ is a regularization of the principal value current $[1/f]$. It follows that the residue current can be obtained as the weak limit of the smooth form $\bar\partial\chi(|f|^2/\epsilon)/f$. This is also a consequence of Corollary 5 below. A natural choice for χ is $\chi(t) = t/(t+1)$ and we see that we get the well-known result that the residue current can be obtained as the weak limit of $\bar\partial(\bar f/(|f|^2 + \epsilon))$. We also briefly mention the more general currents studied by Barlet, [2]. If we instead integrate over the fiber $f = s$ in (1) and let φ have bidegree $(n-1, n-1)$ then the integral has an asymptotic expansion in s with current coefficients. The constant term is Lelong's integration current on V_f and the residue current $\bar\partial[1/f]$ can be obtained from the coefficient of s^n.

We turn to the main focus of this paper which is the codimension two case. Let f and g be two holomorphic functions on X such that f and g define a complete intersection, that is, the common zero set $V_{f \oplus g}$ has codimension two. Consider the residue integral

$$I_{f,g}^{\varphi}(\epsilon_1, \epsilon_2) = \int\limits_{\substack{|f|^2 = \epsilon_1 \\ |g|^2 = \epsilon_2}} \frac{\varphi}{fg}. \tag{2}$$

The unrestricted limit of the residue integral as $\epsilon_1, \epsilon_2 \to 0$ does not exist in general. The first example of this phenomenon was discovered by Passare and Tsikh in [15], and Björk later found that this indeed is the typical case, [5]. See also [17]. After a localization, one may assume that the hypersurface $f \cdot g = 0$ has normal crossings thaks to Hironaka's theorem. This means that there is a (finite) atlas of charts such that $f(\zeta)=\tilde{f}(\zeta)\zeta^{\alpha}$ and $g(\zeta)=\tilde{g}(\zeta)\zeta^{\beta}$ where α and β are multi-indices (depending on the chart) and $\tilde{f}$ and $\tilde{g}$ are invertible holomorphic functions. It is actually the invertible factors which cause problems. One can always dispose of one of the factors, but in general not of both. However, if the matrix A, whose two rows are the integer vectors α and β, respectively, has rank two there is a change of variables $z=\tau(\zeta)$ such that $z^{\alpha}=\tilde{f}(\zeta)\zeta^{\alpha}$ and $z^{\beta}=\tilde{g}(\zeta)\zeta^{\beta}$, see e.g. [12]. Hence, when α and β are not linearly dependent we can make both the invertible factors disappear. Problems therefore arise in so-called charts of resonance where α and β are linearly dependent. Coleff and Herrera realized that if one demands that ϵ_1 and ϵ_2 tend to zero in such a way that $\epsilon_1/\epsilon_2^k\to 0$ for all $k\in\mathbb{Z}_+$, along a so-called admissible path, then one will get no contributions from the charts of resonance because one cannot have $|\tilde{f}(\zeta)\zeta^{\alpha}|\ll|\tilde{g}(\zeta)\zeta^{\beta}|$ if α and β are linearly dependent. They proved in [7] that the limit, along an admissible path, of the residue integral exists and defines the action of a $(0,2)$-current, the Coleff–Herrera residue current $[\bar\partial(1/f)\wedge\bar\partial(1/g)]$. In [12] Passare smoothened the integration over the set $\{|f|^2=\epsilon_1\}\cap\{|g|^2=\epsilon_2\}$ by introducing functions χ as described above, and he studied possible weak limits of forms

$$\frac{\bar\partial\chi_1(|f|^2/\epsilon_1)}{f}\wedge\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g}\tag{3}$$

along parabolic paths $(\epsilon_1,\epsilon_2)=(\epsilon^{s_1},\epsilon^{s_2})$ where $s=(s_1,s_2)$ belongs to the simplex $\Sigma_2(2)=\{(x,y)\in\mathbb{R}_+^2;\ s_1+s_2=2\}$. He found that it is enough to impose finitely many linear conditions $(n_j,s)\neq 0$ to assure that (3) has a weak limit along the corresponding parabolic path. The linear conditions partition $\Sigma_2(2)$ into finitely many open segments and the weak limit of (3) along a parabolic path corresponding to an s in such a segment only depends on the segment. We say that (ϵ_1,ϵ_2) tends to zero inside a Passare sector. Moreover, as we assume that f and g define a complete intersection, the limit is even independent of the choice of segment. In this case it also coincides with the Coleff–Herrera current. One can obtain a $\bar\partial$-potential to the Coleff–Herrera current e.g. by changing the integration set in (2) to $\{|f|^2>\epsilon_1\}\cap\{|g|^2=\epsilon_2\}$ and pass to the limit along an admissible path or by removing the first $\bar\partial$ in (3) and pass to the limit inside a Passare sector. This $\bar\partial$-potential is denoted $[(1/f)\bar\partial(1/g)]$. The main result in this paper implies that if $\chi_j\in C^{\infty}([0,\infty])$ satisfy $\chi_j(0)=0$ and $\chi_j(\infty)=1$ then, in the sense of currents,

$$\lim_{\epsilon_1,\epsilon_2\to 0}\frac{\chi_1(|f|^2/\epsilon_1)}{f}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{g}=\left[\frac{1}{f}\bar\partial\frac{1}{g}\right],\tag{4}$$

and the action of the smooth form on the left-hand side on a test form depends Hölder continuously on $(\epsilon_1,\epsilon_2)\in[0,\infty)^2$. For the particular case when $\chi_j(t)=t/(t+1)$ our result, apart from the Hölder continuity, was announced in [18]. Actually, it is possible to relax the smoothness assumption on one of the χ_j in (4). As mentioned above, one can always dispose of one of the invertible factors. Say that we always arrange so that $\tilde{f}\equiv 1$. Then, examining the proof, one finds that one may take χ_1 to be the characteristic function of $[1,\infty]$. Hence,

$$\int\limits_{|f|^2>\epsilon_1}\frac{\bar\partial\chi_2(|g|^2/\epsilon_2)}{fg}\wedge\varphi\to\left[\frac{1}{f}\bar\partial\frac{1}{g}\right].\varphi$$

with Hölder continuity. Note that if we let both χ_1 and χ_2 be the characteristic function of $[1,\infty]$ then this result is no longer true in view of the examples of Passare–Tsikh and Björk.

Our result also generalize to products of pairs of so-called Bochner–Martinelli blocks. Consider a tuple $f = (f_1, \dots, f_m)$ of holomorphic functions on X. The residue integral corresponding to f, $I_f^\varphi(\epsilon_1, \dots, \epsilon_m)$, is defined analogously to (2). If we take the mean value of the residue integral over $\epsilon = (\epsilon_1, \dots, \epsilon_m)$ in the simplex $\Sigma_m(\delta) = \{s \in \mathbb{R}_+^m;\ \sum s_j = \delta\}$ we obtain

$$c_m \int\limits_{|f|^2=\delta} \frac{\sum_{j=1}^m (-1)^{j+1} \bar{f}_j \bigwedge_{i\neq j} \bar\partial \bar{f}_i}{|f|^{2m}} \wedge \varphi, \tag{5}$$

where c_m is a constant only depending on m. It turns out, see [16], that the limit as δ tends to zero of (5) exists and defines the action of a $(0,m)$-current, which in the case f defines a complete intersection, coincides with the Coleff–Herrera current and also with the currents studied in [3,14]. Based on the work in [16], Andersson introduces more general currents of the Cauchy–Fantappiè–Leray type in [1]. We will briefly discuss Andersson's construction in Section 3. In short, he defines a singular form $u^f = \sum u^f_{k,k-1}$, where the terms $u^f_{k,k-1}$ are similar to the form in (5), and he shows that it is extendible to X as a current, U^f, either as principal values or by analytic continuation. The residue current, R^f, is derived from the current U^f and equals the Coleff–Herrera current locally if f defines a complete intersection. If g is also a tuple of functions there is a natural way of defining the product of the Cauchy–Fantappiè–Leray type currents corresponding to f and g so that formal Leibnitz rules hold, see [22]. If $f \oplus g$ defines a complete intersection and $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to high enough orders at zero and equals 1 at infinity then we prove that the smooth forms

$$\chi_1\big(|f|^2/\epsilon_1\big)u^f \wedge \bar\partial\chi_2\big(|g|^2/\epsilon_2\big) \wedge u^g \quad \text{and}$$
$$\bar\partial\chi_1\big(|f|^2/\epsilon_1\big) \wedge u^f \wedge \bar\partial\chi_2\big(|g|^2/\epsilon_2\big) \wedge u^g$$

are Hölder continuous as currents for $(\epsilon_1, \epsilon_2) \in [0,\infty)^2$ and tend to $U^f \wedge R^g$ and $R^f \wedge R^g$, respectively, as $\epsilon_1, \epsilon_2 \to 0$; see Theorem 21 and Corollary 23. If g is a function such that $f \oplus g$ defines a complete intersection, our techniques can also be used to prove that $\bar\partial\chi_1(|f|^2/\epsilon_1) \wedge u^f \chi_2(|g|^2/\epsilon_2) \to R^f$ when χ_2 equals the characteristic function of $[1,\infty]$. We use this to conclude that R^f has the standard extension property in the complete intersection case, Corollary 24. For more background we refer to the survey article [6] by Björk and the book [21] by Tsikh.

The disposition of the paper is as follows. In Section 2 we outline a proof of (4) since the proofs of the more general statements about Bochner–Martinelli or Cauchy–Fantappiè–Leray blocks are only more difficult to prove in the technical sense and to make it clear that it is not necessary to work through the constructions of Bochner–Martinelli or Cauchy–Fantappiè–Leray type currents in order to prove (4). In Section 3 we recall Andersson's construction and explain some useful notation. Section 4 contains some fairly well-known regularization results about Cauchy–Fantappiè–Leray type currents. As Andersson's formalism makes the arguments a little smoother we also supply the proofs. Section 5 contains the technical core of this paper. We study regularizations of products of monomial currents which we then use in Section 6 to prove our main results; Theorem 21 and its Corollaries 23, 25 and 26 and Theorem 27. In Section 7 we see by explicit computations that Corollary 26 holds for the example by Passare and Tsikh. This section is essentially self-contained.

2. Sketch of proof in the case of two functions

Let f and g be two holomorphic functions on X defining a complete intersection. We sketch how one can handle the difficulties arising in charts of resonance when proving (4). We study the integral

$$\int \frac{\chi_1(|f|^2/\epsilon_1)}{f} \frac{\bar{\partial}\chi_2(|g|^2/\epsilon_2)}{g} \wedge \varphi, \tag{6}$$

where φ is a test form of bidegree $(n, n-1)$. By Hironaka's theorem we may assume that $f = \zeta^\alpha \tilde{f}$ and $g = \zeta^\beta \tilde{g}$ are monomials times non-vanishing functions. One of the non-zero factors can be incorporated in a variable and so we assume that $\tilde{f} \equiv 1$. We assume also that we are in a chart of resonance, i.e. that α and β are linearly dependent. After resolving singularities, f and g no longer define a complete intersection in general, but on the other hand a degree argument shows that $d\bar{\zeta}_j/\bar{\zeta}_j \wedge \varphi$ becomes a test form for any ζ_j dividing both f and g. See the proof of Theorem 21 for more details. Since α and β are linearly dependent, $d\bar{\zeta}_j/\bar{\zeta}_j \wedge \varphi$ is a test form for all j such that $\alpha_j \neq 0$, or equivalently, $\beta_j \neq 0$. Now, (6) equals

$$\sum_j \beta_j \int \frac{\chi_1(|\zeta^\alpha|^2/\epsilon_1)}{\zeta^\alpha} \frac{\chi_2'(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^\beta} \frac{|\zeta^\beta|^2}{\epsilon_2} \wedge \frac{d\bar{\zeta}_j}{\bar{\zeta}_j} \wedge \varphi/\tilde{f},$$

where $\Psi = |\tilde{g}|^2$ is a strictly positive smooth function. It now follows from Corollary 15 that each term in this sum tends to zero as ϵ_1 and ϵ_2 tend to zero. Hence the charts of resonance do not give any contributions.

3. Preliminaries and notation

Assume that f is a section of the dual bundle E^* of a holomorphic m-bundle $E \to X$ over a complex n-manifold X. We will only deal with local problems and it is therefore no loss of generality in assuming that $E \to X$ is trivial. However, the formalism will run smoother with an invariant notation. As mentioned above, we will recall Andersson's construction in [1] and produce currents U^f and R^f and we emphasize that in the case $E \to X$ is the trivial line bundle then U^f and R^f are the currents $[1/f]$ and $\bar{\partial}[1/f]$ times some basis elements. On the exterior algebra ΛE of E, the section f induces mappings $\delta_f : \Lambda^{k+1}E \to \Lambda^k E$ of interior multiplication and $\delta_f^2 = 0$. We introduce the spaces $\mathcal{E}_{0,q}(X, \Lambda^k E)$ of the smooth sections of the exterior algebra of $E \oplus T^*_{0,1}X$ which are $(0,q)$-forms with values in $\Lambda^k E$. We also introduce the corresponding spaces of currents, $\mathscr{D}'_{0,q}(X, \Lambda^k E)$. The mappings δ_f extend to mappings $\delta_f : \mathscr{D}'_{0,q}(X, \Lambda^{k+1}E) \to \mathscr{D}'_{0,q}(X, \Lambda^k E)$ with $\delta_f^2 = 0$ and these mappings anti-commute with the $\bar{\partial}$-operator. Hence, $\mathscr{D}'_{0,q}(X, \Lambda^k E)$ is a double complex and the associated total complex is

$$\cdots \xrightarrow{\nabla_f} \mathcal{L}^{r-1}(X, E) \xrightarrow{\nabla_f} \mathcal{L}^r(X, E) \xrightarrow{\nabla_f} \cdots,$$

where $\mathcal{L}^r(X, E) = \bigoplus_{q-k=r} \mathscr{D}'_{0,q}(X, \Lambda^k E)$ and $\nabla_f = \delta_f - \bar{\partial}$. We will refer to the total complex as the Andersson complex. The exterior product, $\wedge$, induces mappings

$$\bigwedge : \mathcal{L}^r(X, E) \times \mathcal{L}^s(X, E) \to \mathcal{L}^{r+s}(X, E)$$

when possible, and ∇_f is an antiderivation, i.e. $\nabla_f(\tau \wedge \sigma) = \nabla_f \tau \wedge \sigma + (-1)^r \tau \wedge \nabla_f \sigma$ if $\tau \in \mathcal{L}^r(X,E)$ and $\sigma \in \mathcal{L}^s(X,E)$. If $\tau \in \mathcal{L}^r(X,E)$ we write $\tau_{k,k+r}$ for the component of τ belonging to $\mathscr{D}'_{0,k+r}(X, \Lambda^k E)$. Note that functions define elements of $\mathcal{L}^0(X,E)$ of degree $(0,0)$ and sections of E define elements of $\mathcal{L}^{-1}(X,E)$ of degree $(1,0)$. In the spirit of the duality theorem due, independently, to Dickenstein–Sessa, [8], and Passare, [13], one can show that if X is Stein and the zeroth cohomology group of the Andersson complex vanishes then for any holomorphic function h there is a holomorphic section ψ of E such that $\delta_f \psi = h$ [1]. This means that if $f = (f_1, \dots, f_m)$ in some local holomorphic frame for E^* then the division problem $\sum f_j \psi_j = h$ has a holomorphic solution. This cannot hold for all h if f has zeros and the Andersson complex can therefore not be exact in this case. Still, we try to look for an element $u^f \in \mathcal{L}^{-1}(X,E)$ such that $\nabla_f u^f = 1$. To this end we assume that E is equipped with some Hermitian metric $|\cdot|$ and we let s_f be the section of E with pointwise minimal norm such that $\delta_f s_f = |f|^2$. Outside $V_f = f^{-1}(0)$ we may put

$$u^f = \frac{s_f}{\nabla_f s_f} = \frac{s_f}{\delta_f s_f - \bar\partial s_f} = \sum_k \frac{s_f \wedge (\bar\partial s_f)^{k-1}}{|f|^{2k}}.$$

Observe that $\nabla_f s_f$ has even degree so the expression $s_f/\nabla_f s_f$ has meaning outside V_f and it follows immediately that $\nabla_f u^f = 1$ there. The following theorem is proved in [1].

Theorem 1. *Assume that f is locally nontrivial. The forms $|f|^{2\lambda} u^f$ and $\bar\partial |f|^{2\lambda} \wedge u^f$ are locally bounded if $\mathfrak{Re}\,\lambda$ is sufficiently large and they have analytic continuations as currents to $\mathfrak{Re}\,\lambda > -\epsilon$. Let U^f and R^f denote the values at $\lambda = 0$. Then U^f is a current extension of u^f, R^f has support on V_f and*

$$\nabla_f U^f = 1 - R^f.$$

Moreover, $R^f = R^f_{p,p} + \dots + R^f_{q,q}$ where $p = \mathrm{Codim}(V_f)$ and $q = \min(m,n)$.

Note that if $V_f = \emptyset$ then $\nabla_f U^f = 1$ on all of X, which implies that taking the exterior product with U^f is a homotopy operator for the Andersson complex. The current R^f is the Bochner–Martinelli, or more generally, the Cauchy–Fantappiè–Leray current associated to f, and if $f = (f_1, \dots, f_m)$ in some local holomorphic frame, $e_1, \dots, e_m$, of E then

$$R^f = \left[\bar\partial \frac{1}{f_1} \wedge \dots \wedge \bar\partial \frac{1}{f_m}\right] \wedge e_1 \wedge \dots \wedge e_m \tag{7}$$

if f defines a complete intersection, see [1].

Now if f_j, $j = 1,2$, are sections of the dual bundles E_j^* of holomorphic Hermitian m_j-bundles $E_j \to X$ we can apply the above construction to the section $f = f_1 \oplus f_2$ of the bundle $E_1^* \oplus E_2^*$ and obtain the currents U^f and R^f. We could also try to combine the individual currents U^{f_j} and R^{f_j}. It is shown in [22] that the forms

$$|f_1|^{2\lambda} u^{f_1} \wedge |f_2|^{2\lambda} u^{f_2}, \qquad |f_1|^{2\lambda} u^{f_1} \wedge \bar\partial |f_2|^{2\lambda} \wedge u^{f_2} \quad \text{and} \quad \bar\partial |f_1|^{2\lambda} \wedge u^{f_1} \wedge \bar\partial |f_2|^{2\lambda} \wedge u^{f_2},$$

which are locally bounded if $\mathfrak{Re}\,\lambda$ is large enough, have current extensions to $\mathfrak{Re}\,\lambda > -\epsilon$. The values at $\lambda = 0$ are denoted $U^{f_1} \wedge U^{f_2}$, $U^{f_1} \wedge R^{f_2}$, and $R^{f_1} \wedge R^{f_2}$, respectively, and formal computation rules such as e.g. $\nabla_f(U^{f_1}\wedge R^{f_2}) = (1-R^{f_1})\wedge R^{f_2} = R^{f_2} - R^{f_1}\wedge R^{f_2}$ hold. It is also shown in [22] that if f defines a complete intersection then $R^f = R^{f_1}\wedge R^{f_2}$.

We will use the names f and g, rather then f_1 and f_2, for the sections of the two bundles and the symbol ∇, without subscript, always denotes $\nabla_{f\oplus g}$. We will use multi-indices extensively in the sequel. Multi-indices will be denoted α and β or I and J and sometimes also r and ρ. The number of variables will always be n but it will be convenient to define multi-indices by expressions like $\alpha = (\alpha_j)_{j\in K}$ for $K\subseteq\{1,\dots,n\}$. By this we mean that $\alpha = (a_1,\dots,a_n)$, where $a_j = 0$ if $j\notin K$ and $a_j = \alpha_j$ if $j\in K$. Hence, if $z = (z_1,\dots,z_n)$ then $z^\alpha = \prod_{j\in K} z_j^{\alpha_j}$ and similarly for $\partial^\alpha/\partial z^\alpha$. Multi-indices are added and multiplied by numbers as elements in $\mathbb{Z}^n$ and $\alpha\pm 1 = (\alpha_1\pm 1,\dots,\alpha_n\pm 1)$. Also, $|\alpha|$ denotes the length of α as a vector in Euclidean space and $\#\alpha$ is the cardinality of the support of α.

Integration over domains in $\mathbb{C}^n$ will always be with respect to the volume form $(i/2)^n\,dz_1\wedge d\bar z_1\wedge\cdots\wedge dz_n\wedge d\bar z_n := (i/2)^n\,dz\wedge d\bar z$ if nothing else is said. If Δ is a Reinhardt domain in $\mathbb{C}^n$ and φ is a function which only depends on the moduli of the variables and such that $z^\alpha\varphi(z)$ is integrable on Δ then

$$\int\limits_{\Delta} z^\alpha\varphi(z) = 0$$

if α is a non-zero multi-index. This simple fact will play a fundamental role to us in what follows and we will refer to it as *anti-symmetry*.

Unless otherwise stated, the symbol χ with various subscripts will always denote a smooth function on $[0,\infty]$ which is zero to some order at 0 and such that $\chi(\infty) = 1$. By smooth at infinity we mean that $t\mapsto\chi(1/t)$ is smooth at zero.

4. Regularizations of Cauchy–Fantappiè–Leray type currents

Consider a function χ as above and let $\tilde\chi(s) = \chi(1/s)$. Then $\tilde\chi$ is differentiable at $s=0$ and $\tilde\chi'(s) = -\chi'(1/s)/s^2$. Letting $t = 1/s$ we see that $\chi'(t) = \mathcal{O}(1/t^2)$ as $t\to\infty$. This simple observation will be frequently used in the sequel. It follows that for any continuous function φ with compact support in $[0,\infty)$ we have $|\varphi(\epsilon t)\chi'(t)|\leqslant C(t+1)^{-2}$ for a constant independent of ϵ. Hence, by the dominated convergence theorem we see that

$$\int\limits_0^\infty \frac{d}{dt}\chi(t/\epsilon)\varphi(t)\,dt = \int\limits_0^\infty \frac{d}{d\tau}\chi(\tau)\varphi(\epsilon\tau)\,d\tau \to \varphi(0)\int\limits_0^\infty \frac{d}{d\tau}\chi(\tau)\,d\tau = \varphi(0),$$

and we have proved

Lemma 2. *Let $\chi\in C^1([0,\infty])$ satisfy $\chi(0)=0$ and $\chi(\infty)=1$. Then $(d/dt)\chi(t/\epsilon)\to\delta_0$ as measures on $[0,\infty)$.*

Proposition 3. *Assume $\chi\in C^\infty([0,\infty])$ vanishes to order ℓ at 0 and satisfies $\chi(\infty)=1$. Then*

$$\lim_{\epsilon\to 0^+}\int \chi\big(|f|^2/\epsilon\big)u^f_{\ell,\ell-1}\wedge\varphi = U^f_{\ell,\ell-1}.\varphi$$

for any test form φ.

Proof. On the set $\Omega = \{(z,t) \in \mathbb{C}^n \times (0,\infty); |f(z)|^2 > t\}$ we have, for all fixed $\epsilon > 0$, that

$$\left| u^f_{\ell,\ell-1} \frac{d}{dt}\chi(t/\epsilon) \wedge \varphi \right| \leqslant C \frac{1}{|f|^{2\ell-1}} \left| \frac{d}{dt}\chi(t/\epsilon) \right| \leqslant C \frac{t^{1/2}}{t^\ell} \left| \frac{d}{dt}\chi(t/\epsilon) \right| \leqslant C \frac{1}{t^{1/2}}$$

since $\frac{d}{dt}\chi(t/\epsilon) = \mathcal{O}(t^{\ell-1})$. Hence we have an integrable singularity on Ω and by Fubini's theorem we get

$$\int_0^\infty \frac{d}{dt}\chi(t/\epsilon) \int_{|f|^2>t} u^f_{\ell,\ell-1} \wedge \varphi \, dt = \int u^f_{\ell,\ell-1} \wedge \varphi \int_0^{|f|^2} \frac{d}{dt}\chi(t/\epsilon)\, dt$$

$$= \int u^f_{\ell,\ell-1} \chi\big(|f|^2/\epsilon\big) \wedge \varphi. \tag{8}$$

But $J(t) = \int_{|f|^2>t} u^f_{\ell,\ell-1} \wedge \varphi$ is a continuous function with compact support in $[0,\infty)$ with $J(0) = U^f_{\ell,\ell-1}.\varphi$, see [16] or [1]. Hence, by Lemma 2 the left-hand side of (8) tends to $U^f_{\ell,\ell-1}.\varphi$ and the proof is complete. □

If we take $\chi(t)$ equal to appropriate powers of $t/(t+1)$ we obtain the following natural ways to regularize the currents U^f and R^f.

Corollary 4. *For any test form φ we have*

$$\lim_{\epsilon\to 0^+} \int \sum_{\ell\geqslant 1} \frac{s_f \wedge (\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell} \wedge \varphi = U^f.\varphi \tag{9}$$

and

$$\lim_{\epsilon\to 0^+} \int \sum_{\ell\geqslant 1} \epsilon \frac{(\bar\partial s_f)^\ell}{(|f|^2+\epsilon)^{\ell+1}} \wedge \varphi = R^f.\varphi. \tag{10}$$

Proof. Letting $\chi_\ell(t) = t^\ell/(t+1)^\ell$ we see that

$$u^f_{\ell,\ell-1}\chi_\ell\big(|f|^2/\epsilon\big) = \frac{s_f \wedge (\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell}$$

and so (9) follows from Proposition 3. To show that (10) holds we first note that

$$\sum_{\ell\geqslant 1} \frac{s_f \wedge (\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell} = \frac{s_f}{\nabla_f s_f + \epsilon}.$$

Hence

$$\nabla_f \sum_{\ell\geqslant 1} \frac{s_f \wedge (\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell} = \nabla_f \frac{s_f}{\nabla_f s_f + \epsilon} = \frac{\nabla_f s_f}{\nabla_f s_f + \epsilon} = 1 - \sum_{\ell\geqslant 0} \epsilon \frac{(\bar\partial s_f)^\ell}{(|f|^2+\epsilon)^{\ell+1}}.$$

Since differentiation is a continuous operation on distributions it follows from (9) that

$$\lim_{\epsilon\to 0^+} 1-\sum_{\ell\geqslant 0}\epsilon\frac{(\bar\partial s_f)^\ell}{(|f|^2+\epsilon)^{\ell+1}} = \nabla_f \lim_{\epsilon\to 0^+}\sum_{\ell\geqslant 1}\frac{s_f\wedge(\bar\partial s_f)^{\ell-1}}{(|f|^2+\epsilon)^\ell} = \nabla_f U^f = 1-R^f$$

in the sense of currents. The term with $\ell=0$ in the sum on the left is easily seen to tend to zero in the sense of currents and hence (10) follows. □

Note that it is the difference

$$\bar\partial\big(\chi_\ell u^f_{\ell,\ell-1}\big)-\delta_f\big(\chi_{\ell+1}u^f_{\ell+1,\ell}\big)=\bar\partial\chi_\ell\wedge u^f_{\ell,\ell-1}+(\chi_\ell-\chi_{\ell+1})\delta_f u^f_{\ell+1,\ell} \tag{11}$$

which converges to the term of R^f of bidegree (ℓ,ℓ). It is only for the term of top degree, the last term in (11) is not present. This explains why the regularization result in [16, Theorem 2.1], coincides with our result for the top degree term but not for the terms of lower degree.

We can also take one χ which vanishes to high enough order at zero to regularize all terms of U^f and R^f.

Corollary 5. *Assume that* $\chi\in C^\infty([0,\infty])$, *vanishes to order* $\min(m,n)+1$ *at zero and satisfies* $\chi(\infty)=1$. *Then for any test form* φ *we have*

$$\lim_{\epsilon\to 0^+}\int\chi\big(|f|^2/\epsilon\big)u^f\wedge\varphi=U^f.\varphi, \tag{12}$$

$$\lim_{\epsilon\to 0^+}\int\bar\partial\chi\big(|f|^2/\epsilon\big)\wedge u^f\wedge\varphi=R^f.\varphi. \tag{13}$$

Proof. The first statement follows immediately from Proposition 3. For the second one we note that

$$\nabla\chi u^f=\nabla\chi\wedge u^f+\chi\nabla u^f=-\bar\partial\chi\wedge u^f+\chi\nabla u^f,$$

and since χ vanishes to high enough order at zero all terms are smooth. Outside $\{f=0\}$ we have $\nabla u^f=1$ and hence $\chi\nabla u^f=\chi$ everywhere. Moreover, $\chi(|f|^2/\epsilon)$ tends to 1 in the sense of currents and hence

$$\bar\partial\chi\wedge u^f=\chi\nabla u^f-\nabla\chi u^f\to 1-\big(1-R^f\big)=R^f$$

in the sense of currents. □

5. Regularizations of products of monomial currents

This section contains the technical result about the normal crossings case needed to prove our main theorems in the next section. Of particular importance is Proposition 11. First we need a generalization of Taylor's formula. Lemma 6 enables us to approximate a smooth function defined on $\mathbb{C}^n$ in a neighborhood of the union of the coordinate hyperplanes instead of in a neighborhood of their intersection as in the usual Taylor's formula. The approximating functions are in our case not polynomials in general but have enough similarities for our purposes. For tensor products of one-variable functions this corresponds to multiplying the individual Taylor expansions. Lemma 6 appears as Lemma 2.3 in [18] but the formulation there is unfortunately not completely correct. We also remark that Lemma 6 is very similar to Lemma 2.4 in [7] and that very general Taylor expansions are considered in [11, Chapter 1]. Define the linear operator $M_j^{r_j}$ on $C^\infty(\mathbb{C}^n)$ to be the operator that maps φ to the Taylor polynomial of degree r_j of the function $\zeta_j \mapsto \varphi(\zeta)$ (centered at $\zeta_j = 0$). We note that $M_j^{r_j}$ and $M_i^{r_i}$ commute. To see this we only need to observe that

$$\frac{\partial}{\partial\tilde{\zeta_i}}\left(\frac{\partial\varphi}{\partial\tilde{\zeta_j}}\bigg|_{\zeta_j=0}\right)\bigg|_{\zeta_i=0} = \frac{\partial^2\varphi}{\partial\tilde{\zeta_i}\partial\tilde{\zeta_j}}\bigg|_{\zeta_i=\zeta_j=0} = \frac{\partial}{\partial\tilde{\zeta_j}}\left(\frac{\partial\varphi}{\partial\tilde{\zeta_i}}\bigg|_{\zeta_i=0}\right)\bigg|_{\zeta_j=0},$$

where $\partial/\partial\tilde{\zeta_j}$ means that we do not specify whether we differentiate with respect to ζ_j or $\bar{\zeta_j}$.

Lemma 6. *Let $K \subseteq \{1, \ldots, n\}$ have cardinality κ and let $r = (r_j)_{j\in K}$. Define the linear operator M_K^r on $C^\infty(\mathbb{C}^n)$ by*

$$M_K^r = \sum_{j\in K} M_j^{r_j} - \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i} M_j^{r_j} + \cdots + (-1)^{\kappa+1} M_{j_1}^{r_{j_1}} \cdots M_{j_\kappa}^{r_{j_\kappa}}.$$

Then for any $\varphi \in C^\infty(\mathbb{C}^n)$ we have

$$\varphi(\zeta) = M_K^r\varphi(\zeta) + \int\limits_{[0,1]^\kappa} \frac{(1-t)^r}{r!}\frac{\partial^{r+1}}{\partial t^{r+1}}\varphi(t\zeta)\,dt, \tag{14}$$

where $t\zeta$ should be interpreted as $(\xi_1, \ldots, \xi_n)$, $\xi_j = t_j\zeta_j$ if $j \in K$ and $\xi_j = \zeta_j$ if $j \notin K$. In particular $\varphi - M_K^r\varphi = \mathcal{O}(|\zeta^{r+1}|)$. Moreover, $M_K^r\varphi$ can be written as a finite sum of terms, $\varphi_{IJ}(\zeta)\zeta^I\bar{\zeta}^J$, with the following properties:

(a) *$\varphi_{IJ}(\zeta)$ is independent of some variable and in particular of variable ζ_j if $I_j + J_j > 0$;*
(b) *$I_j + J_j \leqslant r_j$ for $j \in K$;*
(c) *if L is the set of indices $j \in K$ such that $\zeta_j \mapsto \varphi_{IJ}(\zeta)$ is non-constant then $\varphi_{IJ}(\zeta) = \mathcal{O}(\prod_{j\in L}|\zeta_j|^{r_j+1})$.*

Proof. It is enough to prove the lemma when $K = \{1, \ldots, n\}$. In case $n = 1$, (14) is Taylor's formula. For $n \geqslant 2$, we write the integral in (14) as an iterated integral. Formula (14) then follows by induction. One can also show (14) by repeated integrations by parts. The difference $\varphi - M_K^r\varphi$ is seen to be of the desired size after performing the differentiations of $\varphi(t\zeta)$ with respect to t inside the integral. To see that $M_K^r\varphi$ can be written as a sum of terms $\varphi_{IJ}(\zeta)\zeta^I\bar{\zeta}^J$ with the properties (a), (b), and (c), we let $r_{\tilde{K}}$, for any $\tilde{K} \subseteq K$, denote the multi-index $(r_{j_1}, \ldots, r_{j_{|\tilde{K}|}})$, $r_{j_i} \in \tilde{K}$. A straightforward computation now shows that

$$\begin{aligned} M_K^r\varphi &= \sum_{j\in K} M_j^{r_j}\big(\varphi - M_{K\setminus\{j\}}^{r_{K\setminus\{j\}}}\varphi\big) \\ &\quad + \sum_{\substack{i,j\in K\\ i<j}} M_i^{r_i} M_j^{r_j}\big(\varphi - M_{K\setminus\{i,j\}}^{r_{K\setminus\{i,j\}}}\varphi\big) + \cdots + M_{j_1}^{r_{j_1}} \cdots M_{j_\kappa}^{r_{j_\kappa}}\varphi. \end{aligned}$$

From the first part of the proof (and the definition of $M_j^{r_j}$) it follows that every term on the right-hand side is a finite sum of terms with the stated properties. $\square$

Lemma 7. *Let α be a multi-index and let $M = M_K^r$ be the operator defined in Lemma* 6 *with K the set of indices j such that $\alpha_j \geqslant 2$ and $r_j = \alpha_j - 2$, $j \in K$. Then for any $\varphi \in \mathscr{D}(\mathbb{C}^n)$ we have*

$$\int_\Delta \frac{1}{\zeta^\alpha}(\varphi - M\varphi) = \left[\frac{1}{\zeta^\alpha}\right].\varphi(i/2)^n\, d\zeta \wedge d\bar{\zeta}$$

if Δ is a polydisc containing the support of φ.

Proof. Note that by Lemma 6 we have $\varphi - M\varphi = \mathcal{O}(|\zeta^{\alpha-1}|)$ and so $(\varphi - M\varphi)/\zeta^\alpha$ is integrable on Δ. Hence if we let $\Delta_\delta = \Delta \cap \bigcap_j \{|\zeta_j| > \delta\}$ we get

$$\int_\Delta \frac{1}{\zeta^\alpha}(\varphi - M\varphi) = \lim_{\delta\to 0}\int_{\Delta_\delta} \frac{1}{\zeta^\alpha}(\varphi - M\varphi) = \lim_{\delta\to 0}\int_{\Delta_\delta} \frac{1}{\zeta^\alpha}\varphi - \lim_{\delta\to 0}\int_{\Delta_\delta} \frac{1}{\zeta^\alpha}M\varphi.$$

The first limit on the right-hand side is the tensor product of the principal value currents $[1/\zeta_j^{\alpha_j}]$ (acting on $\varphi\,(i/2)^n\, d\zeta \wedge d\bar{\zeta}$) and hence it equals $[1/\zeta^\alpha].\varphi\,(i/2)^n\, d\zeta \wedge d\bar{\zeta}$. It follows by anti-symmetry that actually

$$\int_{\Delta_\delta} \frac{1}{\zeta^\alpha} M\varphi = 0$$

for all $\delta > 0$. In fact, $M\varphi$ is a sum of terms $\varphi_{IJ}(\zeta)\zeta^I\bar{\zeta}^J$ where $I_j + J_j \leqslant \alpha_j - 2$ for all j and the coefficient $\varphi_{IJ}(\zeta)$ is at least independent of some variable. $\square$

Lemma 8. *Let $\chi_1, \chi_2 \in C^\infty([0,\infty])$ and let Φ and Ψ be smooth strictly positive functions on $\mathbb{C}^n$. Let also M_K^r be the operator defined in Lemma* 6 *with K and r arbitrary. Then*

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi) = M_K^r\big(\chi_1(t_1\Phi)\chi_2(t_2\Psi)\big) + \left|\zeta^{r+1}\right| B(t_1, t_2, \zeta),$$

where B is bounded on $(0,\infty)^2 \times D$ if $D \Subset \mathbb{C}^n$.

Proof. If $D \Subset \mathbb{C}^n$ both Φ and Ψ have strictly positive infima and finite suprema on D and so there is a neighborhood U of $[0,\infty]^2$ in $\widehat{\mathbb{R}} \times \widehat{\mathbb{R}}$ such that the function $(t_1, t_2, \zeta) \mapsto \chi_1(t_1\Phi)\chi_2(t_2\Psi)$ is smooth on $U \times D$. From Lemma 6 it follows that

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi) = M_K^r\big(\chi_1(t_1\Phi)\chi_2(t_2\Psi)\big) + \sum_{\substack{I,J\subseteq K\\ I_j+J_j=r_j+1}} G_{IJ}(t_1, t_2, \zeta)\zeta^I\bar{\zeta}^J$$

for some functions G_{IJ} which are smooth on $U \times D$, and the lemma readily follows. $\square$

To prove Proposition 11 we will need the estimates of the following two elementary lemmas.

Lemma 9. *Let Δ be the unit polydisc in $\mathbb{C}^n$ and put $\Delta_\epsilon^\alpha = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geqslant \epsilon\}$ and $\Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta} = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geqslant \epsilon_1, |\zeta^\beta|^2 \geqslant \epsilon_2\}$. Then for all $\epsilon, \epsilon_j \leqslant 1$ we have*

$$\int\limits_{\Delta \setminus \Delta_\epsilon^\alpha} \frac{1}{|\zeta_1| \cdots |\zeta_n|} \lesssim \epsilon^{1/(2|\alpha|)} |\log \epsilon|^{n-1}$$

and

$$\int\limits_{\Delta \setminus \Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta}} \frac{1}{|\zeta_1| \cdots |\zeta_n|} \lesssim \left|(\epsilon_1, \epsilon_2)\right|^\omega, \quad 2\omega < \min\left\{|\alpha|^{-1}, |\beta|^{-1}\right\}.$$

Proof. On the set $\Delta \setminus \Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta}$, either $|\zeta^\alpha|^2 < \epsilon_1$ or $|\zeta^\beta|^2 < \epsilon_2$ and so it follows from the first inequality that the integral in the second inequality is less then or equal to (a constant times)

$$\begin{aligned} \epsilon_1^{1/(2|\alpha|)} |\log \epsilon_1|^{n-1} + \epsilon_2^{1/(2|\beta|)} |\log \epsilon_2|^{n-1} &\lesssim \epsilon_1^{1/(2|\alpha|) - \nu} + \epsilon_2^{1/(2|\beta|) - \nu} \\ &\lesssim \left|(\epsilon_1, \epsilon_2)\right|^{\omega_\nu}, \end{aligned}$$

for any $\nu > 0$ and $\omega_\nu \leqslant \min\{|\alpha|^{-1}, |\beta|^{-1}\}/2 - \nu$. Hence the second inequality follows from the first one. To prove the first inequality we first integrate with respect to the angular variables and then we make the change of variables $x_j = \log|\zeta_j|$ to see that the integral in question equals

$$(4\pi)^n \int\limits_{Q_\epsilon} e^{\sum x_j}\, dx, \tag{15}$$

where

$$Q_\epsilon = \left\{x \in (-\infty, 0]^n;\ 2 \sum \alpha_j x_j < \log \epsilon\right\}.$$

Since all $x_j \leqslant 0$ on Q_ϵ we have $\exp(\sum x_j) \leqslant \exp(-|x|)$ here, and choosing $R = |\log \epsilon|/(2|\alpha|)$ we see that (15) is less then or equal to $\int_{\{|x| > R\}} \exp(-|x|)\, dx$. In polar coordinates this is easily seen to be of order $\epsilon^{1/(2|\alpha|)} |\log \epsilon|^{n-1}$. $\square$

Lemma 10. *Let Δ be the unit polydisc in $\mathbb{C}^n$ and put $\Delta_\epsilon^\alpha = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geqslant \epsilon\}$ and $\Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta} = \{\zeta \in \Delta; |\zeta^\alpha|^2 \geqslant \epsilon_1, |\zeta^\beta|^2 \geqslant \epsilon_2\}$. Then, for $\epsilon, \epsilon_j \leqslant 1$, we have*

$$\int\limits_{\Delta_\epsilon^\alpha} \frac{\epsilon}{|\zeta^\alpha|^2} \frac{1}{|\zeta_1| \cdots |\zeta_n|} \lesssim \epsilon^{1/(2|\alpha|)} |\log \epsilon|^{n-1},$$

$$\int\limits_{\Delta_{\epsilon_1,\epsilon_2}^{\alpha,\beta}} \left(\frac{\epsilon_1}{|\zeta^\alpha|^2} + \frac{\epsilon_2}{|\zeta^\beta|^2}\right) \frac{1}{|\zeta_1| \cdots |\zeta_n|} \lesssim \left|(\epsilon_1, \epsilon_2)\right|^\omega$$

and

$$\int\limits_{\Delta^{\alpha,\beta}_{\epsilon_1,\epsilon_2}} \frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2}\frac{1}{|\zeta_1|\cdots|\zeta_n|} \lesssim \left|(\epsilon_1,\epsilon_2)\right|^\omega,$$

where $2\omega < \min\{|\alpha|^{-1}, |\beta|^{-1}\}$.

Proof. The second and third inequalities follow from the first one since it implies that the integral in the second one is of the size

$$\epsilon_1^{\tau+1/(2|\alpha|)} + \epsilon_2^{\tau+1/(2|\beta|)} \lesssim \left|(\epsilon_1,\epsilon_2)\right|^{\tau+\omega} \quad \text{for any } \tau > 0$$

and that the integral in the third is of the size

$$\min\left\{\epsilon_1^{1/(2|\alpha|)}|\log\epsilon_1|^{n-1}, \epsilon_2^{1/(2|\beta|)}|\log\epsilon_2|^{n-1}\right\}.$$

To prove the first inequality we proceed as in the previous lemma and we see that the integral in question equals

$$(4\pi)^n\epsilon\int\limits_{Q_\epsilon}\frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}\,dx = (4\pi)^n\epsilon\int\limits_{Q_\epsilon\cap\{|x|\leqslant R\}}\frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}\,dx + (4\pi)^n\epsilon\int\limits_{Q_\epsilon\cap\{|x|\geqslant R\}}\frac{e^{\sum x_j}}{e^{2\sum\alpha_j x_j}}\,dx, \tag{16}$$

where $Q_\epsilon = \{x \in (-\infty,0]^n;\ 2\sum\alpha_j x_j \geqslant \log\epsilon\}$. We choose $2R = |\log\epsilon|/|\alpha|$, and then $Q_\epsilon \cap \{|x| \leqslant R\} = \{x \in (-\infty,0]^n;\ |x| \leqslant R\}$. If all $x_j \leqslant 0$ we have $\sum x_j \leqslant -|x|$ and by the Cauchy–Schwarz inequality we also have $-\sum\alpha_j x_j \leqslant |\alpha||x|$. Hence we may estimate the integrand in the second to last integral in (16) by $\exp((2|\alpha|-1)|x|)$. In the last integral we integrate where $\epsilon/\exp(2\sum\alpha_j x_j) \leqslant 1$ and so we see that the right-hand side of (16) is less then or equal to

$$(4\pi)^n\epsilon\int\limits_{\{|x|\leqslant R\}} e^{(2|\alpha|-1)|x|}\,dx + (4\pi)^n\int\limits_{\{|x|\geqslant R\}} e^{-|x|}\,dx.$$

By changing to polar coordinates this is seen to be of the size $\epsilon^{1/(2|\alpha|)}|\log\epsilon|^{n-1}$. □

The proof of the following proposition contains the technical core of this paper.

Proposition 11. *Assume that* $\chi_1, \chi_2 \in C^\infty([0,\infty])$ *vanish to orders* $k \geqslant 0$ *and* $\ell \geqslant 0$ *at* 0, *respectively, and that* $\chi_1(\infty) = 1$. *Then for any test form* $\varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ *we have*

$$\int\frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\left(\Phi\left|\zeta^\alpha\right|^2/\epsilon_1\right)\chi_2\left(\Psi\left|\zeta^\beta\right|^2/\epsilon_2\right)\varphi \to \begin{cases}[\frac{1}{\zeta^{k\alpha+\ell\beta}}].\varphi, & \chi_2(\infty)=1,\\ 0, & \chi_2(\infty)=0\end{cases}$$

as $\epsilon_1, \epsilon_2 \to 0^+$. *Moreover, as a function of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0, \infty)^2$, *the integral belongs to all* ω*-Hölder classes with* $2\omega < \min\{|\alpha|^{-1}, |\beta|^{-1}\}$.

Remark 12. The values of the integral at points $(\epsilon_1, 0)$ and $(0, \epsilon_2)$, $\epsilon_j \neq 0$, are

$$\chi_2(\infty)\frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}\left[\frac{1}{\zeta^{\ell\beta}}\right].\varphi \quad \text{and} \quad \frac{\chi_2(\Phi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}\left[\frac{1}{\zeta^{k\alpha}}\right].\varphi,$$

respectively.

Remark 13. The modulus of continuity can be improved by sharpening the estimates in the Lemmas 9 and 10 but we will not bother about this. This is because the multi-indices α and β will be implicitly given by Hironaka's theorem and so we can only be sure of the existence of some positive Hölder exponent when we prove our main theorems anyway.

Proof. We prove Hölder continuity for a path $(\epsilon_1, \epsilon_2) \to 0$, $\epsilon_j \neq 0$. For a general path (inside $[0, \infty)^2$) to an arbitrary point in $[0, \infty)^2$ one proceeds in a similar way. Let K be the set of indices j such that $k\alpha_j + \ell\beta_j \geqslant 2$ and let $M = M_K^r$ be the operator defined in Lemma 6 with $r_j = k\alpha_j + \ell\beta_j - 2$ for $j \in K$. Let also Δ be a polydisc containing the support of φ. In this proof we will identify φ with its coefficient function with respect to the volume form in $\mathbb{C}^n$. We make a preliminary decomposition

$$\int \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2\varphi = \int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2(\varphi - M\varphi) + \int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2 M\varphi. \tag{17}$$

Denote by Δ_ϵ the set $\{\zeta \in \Delta;\ |\zeta^\alpha|^2 \geqslant \epsilon_1, |\zeta^\beta|^2 \geqslant \epsilon_2\}$. Since $\varphi - M\varphi = \mathcal{O}(|\zeta^{r+1}|)$, according to Lemma 6, and $\chi_1(\infty) = 1$ we get

$$\begin{aligned}
&\left|\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2(\varphi - M\varphi) - \chi_2(\infty)\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}(\varphi - M\varphi)\right| \\
&\quad\lesssim \int_\Delta \frac{1}{|\zeta_1|\cdots|\zeta_n|}\big|\chi_1\chi_2 - \chi_2(\infty)\big| \\
&\quad\lesssim \int_{\Delta_\epsilon} \frac{1}{|\zeta_1|\cdots|\zeta_n|}\big|\chi_1\chi_2 - \chi_2(\infty)\big| + \int_{\Delta\setminus\Delta_\epsilon} \frac{1}{|\zeta_1|\cdots|\zeta_n|}.
\end{aligned} \tag{18}$$

It follows from Lemma 9 that the last integral is of order $|\epsilon|^\omega$ as $\epsilon_1, \epsilon_2 \to 0^+$. On the other hand, for $\zeta \in \Delta_\epsilon$ both $|\zeta^\alpha|^2/\epsilon_1 \geqslant 1$ and $|\zeta^\beta|^2/\epsilon_2 \geqslant 1$ and by Taylor expanding at infinity we see that

$$\chi_1\big(\Phi\big|\zeta^\alpha\big|^2/\epsilon_1\big) = \chi_1(\infty) + \frac{\epsilon_1}{|\zeta^\alpha|^2}B_1\big(\epsilon_1/\big|\zeta^\alpha\big|^2, \zeta\big),$$

$$\chi_2\big(\Psi\big|\zeta^\beta\big|^2/\epsilon_2\big) = \chi_2(\infty) + \frac{\epsilon_2}{|\zeta^\beta|^2}B_2\big(\epsilon_2/\big|\zeta^\beta\big|^2, \zeta\big),$$

where B_1 and B_2 are bounded. Using that $\chi_1(\infty)=1$ we thus get that $|\chi_1\chi_2-\chi_2(\infty)|$ is of the size $\epsilon_1/|\zeta^\alpha|^2+\epsilon_2/|\zeta^\beta|^2$. Hence, by Lemma 10, the second to last integral in (18) is also of order $|\epsilon|^\omega$ as $\epsilon_1,\epsilon_2\to 0^+$. In view of Lemma 7, we have thus showed that the first integral on the right-hand side of (17) tends to $[1/\zeta^{k\alpha+\ell\beta}].\varphi$ if $\chi_2(\infty)=1$ and to zero if $\chi_2(\infty)=0$ and, moreover, belongs to the stated Hölder classes. We will be done if we can show that the last integral in (17) is of order $|\epsilon|^\omega$. We know that $M\varphi=\sum_{IJ}\varphi_{IJ}\zeta^I\bar{\zeta}^J$ where each φ_{IJ} is independent of at least one variable and $I_j+J_j\leqslant k\alpha_j+\ell\beta_j-2$ for $j\in K$. Hence, if Φ and Ψ are constants (or only depend on the modulus of the ζ_j) then the last integral in (17) is zero for all $\epsilon_1,\epsilon_2>0$ by anti-symmetry. For the general case, consider one term,

$$\int_{\Delta}\frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\chi_2\varphi_{IJ}\zeta^I\bar{\zeta}^J, \tag{19}$$

and let L be the set of indices $j\in K$ such that $\zeta_j\mapsto\varphi_{IJ}(\zeta)$ is constant. Let also $\mathscr{M}=M_L^\rho$ be the operator defined in Lemma 6 with $\rho_j=k\alpha_j+\ell\beta_j-I_j-J_j-2$ for $j\in L$. We introduce the independent (real) variables, or "smoothing parameters," $t_1=|\zeta^\alpha|^2/\epsilon_1$ and $t_2=|\zeta^\beta|^2/\epsilon_2$. Below, $\mathscr{M}(\chi_1\chi_2)$ denotes the function we obtain by letting $\mathscr{M}$ operate on $\zeta\mapsto\chi_1(t_1\Phi(\zeta))\chi_2(t_2\Psi(\zeta))$ and then substituting $|\zeta^\alpha|^2/\epsilon_1$ and $|\zeta^\beta|^2/\epsilon_2$ for t_1 and t_2, respectively. We rewrite the integral (19) as

$$\int_{\Delta_\epsilon}\frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}\big(\chi_1\chi_2-\mathscr{M}(\chi_1\chi_2)\big)+\int_{\Delta\setminus\Delta_\epsilon}\frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}\big(\chi_1\chi_2-\mathscr{M}(\chi_1\chi_2)\big)$$
$$+\int_{\Delta}\frac{\varphi_{IJ}\zeta^I\bar{\zeta}^J}{\zeta^{k\alpha+\ell\beta}}\mathscr{M}(\chi_1\chi_2). \tag{20}$$

Now, $\mathscr{M}(\chi_1\chi_2)$ is a sum of terms which, at least for some $j\in L$, are monomials in ζ_j and $\bar{\zeta}_j$ times coefficient functions depending on $|\zeta_j|$ and the other variables. The degrees of these monomials do not exceed $\rho_j=k\alpha_j+\ell\beta_j-I_j-J_j-2$ and since $\zeta_j\mapsto\varphi_{IJ}(\zeta)$ is constant for $j\in L$ we see, by counting exponents, that the last integral in (20) vanishes by anti-symmetry for all $\epsilon_1,\epsilon_2>0$. By Lemma 8 we have

$$\chi_1(t_1\Phi)\chi_2(t_2\Psi)-\mathscr{M}\big(\chi_1(t_1\Phi)\chi_2(t_2\Psi)\big)=\big|\zeta^{\rho+1}\big|B(t_1,t_2,\zeta), \tag{21}$$

where B is bounded on $(0,\infty)^2\times\Delta$. We note also that by Lemma 6, $\varphi_{IJ}(\zeta)=\mathcal{O}(\prod_{j\in L\setminus K}|\zeta_j|^{r_j+1})$. From (21) we thus see that the modulus of the second integral in (20) can be estimated by

$$C\int_{\Delta\setminus\Delta_\epsilon}\frac{1}{|\zeta_1|\cdots|\zeta_n|},$$

which is of order $|\epsilon|^\omega$ by Lemma 9. It remains to consider the first integral in (20). On the set Δ_ϵ we have that $\Phi|\zeta^\alpha|^2/\epsilon_1$ and $\Psi|\zeta^\beta|^2/\epsilon_2$ are larger then some positive constant and so by multiplying the Taylor expansions of the functions $t_1\mapsto\chi_1(t_1\Phi)$ and $t_2\mapsto\chi_2(t_2\Psi)$ at infinity we get

$$\begin{aligned}\chi_1\bigl(\Phi\big|\zeta^\alpha\big|^2/\epsilon_1\bigr)\chi_2\bigl(\Psi\big|\zeta^\beta\big|^2/\epsilon_2\bigr) &= \chi_2(\infty)+\frac{\epsilon_2}{|\zeta^\beta|^2}\tilde\chi_2\bigl(\big|\zeta^\beta\big|^2/\epsilon_2,\zeta\bigr)\\ &\quad+\chi_2(\infty)\frac{\epsilon_1}{|\zeta^\alpha|^2}\tilde\chi_1\bigl(\big|\zeta^\alpha\big|^2/\epsilon_1,\zeta\bigr)\\ &\quad+\frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2}\tilde\chi_1\bigl(\big|\zeta^\alpha\big|^2/\epsilon_1,\zeta\bigr)\tilde\chi_2\bigl(\big|\zeta^\beta\big|^2/\epsilon_2,\zeta\bigr),\end{aligned}$$

where $\tilde\chi_j$ are smooth on $[1,\infty]\times\Delta$. Now since $|\zeta^\alpha|^2/\epsilon_1=t_1$ and $|\zeta^\beta|^2/\epsilon_2=t_2$ are independent variables we conclude that

$$\begin{aligned}\chi_1\chi_2-\mathscr{M}(\chi_1\chi_2) &= \frac{\epsilon_2}{|\zeta^\beta|^2}(\tilde\chi_2-\mathscr{M}\tilde\chi_2)+\frac{\epsilon_1}{|\zeta^\alpha|^2}\chi_2(\infty)(\tilde\chi_1-\mathscr{M}\tilde\chi_1)\\ &\quad+\frac{\epsilon_1\epsilon_2}{|\zeta^\alpha|^2|\zeta^\beta|^2}\bigl(\tilde\chi_1\tilde\chi_2-\mathscr{M}(\tilde\chi_1\tilde\chi_2)\bigr)\end{aligned}$$

for $\zeta\in\Delta_\epsilon$. By Lemmas 6 and 10 we see that the first integral in (20) also is of order $|\epsilon|^\omega$ as $\epsilon_1,\epsilon_2\to 0^+$ and the proof is complete. □

Remark 14. Let us assume that the function Φ is identically 1 in the previous proposition. Then, instead of adding and subtracting $\mathscr{M}(\chi_1\chi_2)$ in (20), it is enough to add and subtract $\chi_1\mathscr{M}(\chi_2)$. This suggests that one can relax the smoothness assumption on χ_1. It is actually possible to take χ_1 to be the characteristic function of $[1,\infty]$. If we define the value of the integral in Proposition 11 at a point $(\epsilon_1,0)$ to be

$$\int_\Delta \frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\bigl(\big|\zeta^\alpha\big|^2/\epsilon_1\bigr)(\varphi-M\varphi), \tag{22}$$

where Δ and M are as in the proof above, then the conclusions of Proposition 11 hold for this choice of χ_1. Only minor changes in the proof are needed to see this. One can also check that (22) is a way of computing

$$\chi_1\bigl(|\zeta^\alpha|^2/\epsilon_1\bigr)\left[\frac{1}{\zeta^{k\alpha+\ell\beta}}\right].\varphi.$$

The product $\chi_1(|\zeta^\alpha|^2/\epsilon_1)[1/\zeta^{k\alpha+\ell\beta}]$ is well defined because the wave front sets of the two currents behave in the right way, at least for almost all ϵ_1, see [6].

We make another useful observation. Since the function $\tilde\chi(s)=\chi(1/s)$ is smooth at zero and $\tilde\chi'(s):=-\frac{1}{s^2}\chi'(1/s)$, it follows that $s\mapsto\chi'(1/s)/s$ is smooth at zero and vanishes for $s=0$. Hence, $t\mapsto\chi'(t)t$ is smooth on $[0,\infty]$, vanishes to the same order at zero as χ, and maps ∞ to 0. From Proposition 11 we thus see that we have

Corollary 15. *Assume that $\chi_1,\chi_2\in C^\infty([0,\infty])$ vanish to orders k and ℓ at zero, respectively, and satisfy $\chi_j(\infty)=1$. For any smooth and strictly positive functions Φ and Ψ on $\mathbb{C}^n$ and any test form $\varphi\in\mathscr{D}_{n,n}(\mathbb{C}^n)$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0^+}\int\frac{1}{\zeta^{k\alpha+\ell\beta}}\chi_1\bigl(\Phi\big|\zeta^\alpha\big|^2/\epsilon_1\bigr)\chi_2'\bigl(\Psi\big|\zeta^\beta\big|^2/\epsilon_2\bigr)\frac{|\zeta^\beta|^2}{\epsilon_2}\varphi=0, \tag{23}$$

and, moreover, as a function of $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$, *the integral belongs to all* ω*-Hölder classes with* $2\omega < \min\{|\alpha|^{-1}, |\beta|^{-1}\}$.

6. Regularizations of products of Cauchy–Fantappiè–Leray type currents

We are now in a position to prove our main results. We start with a regularization of the product $U^f \wedge U^g$. Recall that if f is function then $U^f = [1/f]$ times some basis element.

Theorem 16. *Let f and g be holomorphic sections (locally non-trivial) of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Let $\chi_1, \chi_2 \in C^\infty([0,\infty])$ be any functions vanishing to orders m_1 and m_2 at zero, respectively, and satisfying $\chi_j(\infty) = 1$. Then, for any test form φ we have*

$$\int \chi_1\big(|f|^2/\epsilon_1\big)u^f \wedge \chi_2\big(|g|^2/\epsilon_2\big)u^g \wedge \varphi \to U^f \wedge U^g.\varphi,$$

as $\epsilon_1, \epsilon_2 \to 0^+$. Moreover, as a function of $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ the integral on the left-hand side belongs to some Hölder class independently of φ.

Proof. Recall that $U^f \wedge U^g.\varphi$ is defined as the value at zero of the meromorphic function

$$\lambda \mapsto \int |f|^{2\lambda} u^f \wedge |g|^{2\lambda} u^g \wedge \varphi.$$

Assuming only that χ_1 and χ_2 vanish to orders $k \leqslant m_1$ and $\ell \leqslant m_2$ at zero, respectively, we will show that

$$\int \chi_1 u^f_{k,k-1} \wedge \chi_2 u^g_{\ell,\ell-1} \wedge \varphi \to \int |f|^{2\lambda} u^f_{k,k-1} \wedge |g|^{2\lambda} u^g_{\ell,\ell-1} \wedge \varphi\Big|_{\lambda=0} \tag{24}$$

and that the left-hand side belongs to some Hölder class. This will clearly imply the theorem. We may assume that φ has arbitrarily small support after a partition of unity. If φ has support outside $f^{-1}(0) \cup g^{-1}(0)$ it is easy to check that (24) holds and hence we can restrict to the case that φ has support in a small neighborhood $\mathcal{U}$ of a point $p \in f^{-1}(0) \cup g^{-1}(0)$. We may also assume that $\mathcal{U}$ is contained in a coordinate neighborhood and that all bundles are trivial over $\mathcal{U}$. We let $(f_1, \ldots, f_{m_1})$ and $(g_1, \ldots, g_{m_2})$ denote the components of f and g, respectively, with respect to some holomorphic frames. It follows from Hironaka's theorem, possibly after another localization, that there is an n-dimensional complex manifold $\tilde{\mathcal{U}}$ and a proper holomorphic map $\Pi : \tilde{\mathcal{U}} \to \mathcal{U}$ such that Π is biholomorphic outside the nullset $\Pi^*\{f_1 \cdots f_{m_1} \cdot g_1 \cdots g_{m_2} = 0\}$ and that this hypersurface has normal crossings in $\tilde{\mathcal{U}}$. Hence we can cover $\tilde{\mathcal{U}}$ by local charts, each centered at the origin, such that $\Pi^* f_j$ and $\Pi^* g_j$ are monomials times non-vanishing functions. The support of $\Pi^*\varphi$ is compact because Π is proper and hence, we can cover the support of $\Pi^*\varphi$ by finitely many of these charts. We let ρ_k be a partition of unity on $\operatorname{supp}(\Pi^*\varphi)$ subordinate to this cover. Now, following [4,16], given monomials $\mu_1, \ldots, \mu_\nu$, one can construct an n-dimensional toric manifold $\mathcal{X}$ and a proper holomorphic map $\tilde{\Pi} : \mathcal{X} \to \mathbb{C}^n_t$ which is monoidal when expressed in local coordinates in each chart. Moreover, $\tilde{\Pi}$ is biholomorphic outside $\tilde{\Pi}^*\{t_1 \cdots t_n = 0\}$ and in each chart one of the monomials $\tilde{\Pi}^*\mu_1, \ldots, \tilde{\Pi}^*\mu_\nu$ divides all the others. By repeating this process, if necessary, and localizing with partitions of unity at each step, we may actually assume that $f_j = \mu_{f,j}\tilde{f}_j$ and $g_j = \mu_{g,j}\tilde{g}_j$ where $\tilde{f}_j$ and $\tilde{g}_j$ are non-vanishing and $\mu_{f,j}$ and $\mu_{g,j}$ are monomials with the property that μ_{f,ν_1} divides all $\mu_{f,j}$ and μ_{g,ν_2} divides all $\mu_{g,j}$ for some indices ν_1 and ν_2. Denote μ_{f,ν_1} by ζ^α and μ_{g,ν_2} by ζ^β. It follows that $|f|^2 = |\zeta^\alpha|^2\Phi$ and $|g|^2 = |\zeta^\beta|^2\Psi$ where Φ and Ψ are strictly positive functions. Moreover, $s_f = \bar{\zeta}^\alpha\tilde{s}_f$ and

$$u^f_{k,k-1} = \frac{s_f \wedge (\bar\partial s_f)^{k-1}}{|f|^{2k}} = \frac{1}{\zeta^{k\alpha}}\frac{\tilde{s}_f \wedge (\bar\partial\tilde{s}_f)^{k-1}}{\Phi^k} = \frac{1}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1},$$

where $\tilde{u}^f_{k,k-1}$ is a smooth form and similarly for $u^g_{\ell,\ell-1}$. In order to prove (24) it thus suffices to prove

$$\begin{aligned}\int \frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1} \wedge \frac{\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi \\ \to \int \frac{|\zeta^\alpha|^{2\lambda}}{\zeta^{k\alpha}}\Phi^\lambda\tilde{u}^f_{k,k-1} \wedge \frac{|\zeta^\beta|^{2\lambda}}{\zeta^{\ell\beta}}\Psi^\lambda\tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi\Big|_{\lambda=0},\end{aligned} \tag{25}$$

where $\tilde\varphi = \rho_{k_j}\Pi^*_j \cdots \rho_{k_1}\Pi^*_1\varphi$ and that the integral on the left-hand side belongs to some Hölder class. But by Proposition 11 it does belong to some Hölder class and tends to $[1/\zeta^{k\alpha+\ell\beta}].\tilde{u}^f_{k,k-1} \wedge \tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi$. One can verify that this indeed is equal to the right-hand side of (25) by integrations by parts as in e.g. [1]. □

Remark 17. This theorem can actually be generalized to any number of factors U^f. One first checks that the analogue of Proposition 11 holds for any number of functions χ_j and then reduces to this case just as in the proof above. In particular, if f_j, $j = 1, \ldots, p$, are holomorphic functions and χ_j vanish at 0, we have

$$\int \frac{\chi_1(|f_1|^2/\epsilon_1)}{f_1}\cdots\frac{\chi_p(|f_p|^2/\epsilon_p)}{f_p}\varphi \to \left[\frac{1}{f_1}\cdots\frac{1}{f_p}\right].\varphi$$

unrestrictedly as all $\epsilon_j \to 0^+$. However, we focus on the two factor case since we do not know how to handle more than two residue factors.

To prove our regularization results for the currents $U^f \wedge R^g$ and $R^f \wedge R^g$ we have to structure the information obtained from an application of Hironaka's theorem more carefully and then use Proposition 11 and Corollary 15 in the right way. The technical part of this is contained in the following proposition.

Proposition 18. *Assume that $\chi_1, \chi_2 \in C^\infty([0,\infty])$ vanish to orders k and ℓ at zero, respectively, and satisfy $\chi_j(\infty) = 1$. Let α', α'', β' and β'' be multi-indices such that α', α'' and β' have pairwise disjoint supports, and $\alpha''_j = 0$ if and only if $\beta''_j = 0$. Assume also that $\varphi \in \mathscr{D}_{n,n-1}(\mathbb{C}^n)$ has the property that $d\bar\zeta_j/\bar\zeta_j \wedge \varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ for all j such that $\alpha''_j \neq 0$. Then for any smooth and strictly positive functions Φ and Ψ on $\mathbb{C}^n$ we have*

$$\lim_{\epsilon_1,\epsilon_2\to 0^+}\int \frac{1}{\mu_1^k\mu_2^\ell}\chi_1(\Phi|\mu_1|^2/\epsilon_1)\bar\partial\chi_2(\Psi|\mu_2|^2/\epsilon_2) \wedge \varphi = \left[\frac{1}{\mu_1^k\zeta^{\ell\beta''}}\right] \otimes \bar\partial\left[\frac{1}{\zeta^{\ell\beta'}}\right].\varphi,$$

where $\mu_1 = \zeta^{\alpha'+\alpha''}$ *and* $\mu_2 = \zeta^{\beta'+\beta''}$. *Moreover, as a function of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$, *the integral belongs to all* ω*-Hölder classes with* $2\omega < \min\{|\alpha' + \alpha''|^{-1}, |\beta' + \beta''|^{-1}\}$.

Remark 19. Note that the hypotheses on the multi-indices imply that a factor ζ_j divides both the monomials μ_1 and μ_2 if and only if $\alpha''_j \neq 0$ (or equivalently $\beta''_j \neq 0$). In particular, the tensor product of the currents is well defined.

Remark 20. We may let k or ℓ or both of them be equal to zero and the conclusions of the proposition still hold. In case $\ell = 0$ one should interpret $\bar\partial[1/\zeta^{\ell\beta'}]$ as zero.

Proof. Let K, L and K^c be the set of indices j such that $\beta'_j \neq 0$, $\beta''_j \neq 0$ and $\beta'_j = 0$, respectively. Clearly $L \subseteq K^c$. We write $\bar\partial = \bar\partial_K + \bar\partial_{K^c}$ and integrate by parts with respect to $\bar\partial_K$ to see that

$$\begin{aligned}
&\int \frac{1}{\mu_1^k \mu_2^\ell} \chi_1 (\bar\partial_K + \bar\partial_{K^c}) \chi_2 \wedge \varphi \\
&\quad = -\int \frac{1}{\mu_1^k \mu_2^\ell} \chi_1' \frac{|\mu_1|^2}{\epsilon_1} \chi_2 \bar\partial_K \Phi \wedge \varphi - \int \frac{1}{\mu_1^k \mu_2^\ell} \chi_1 \chi_2 \bar\partial_K \varphi \\
&\qquad + \int \frac{1}{\mu_1^k \mu_2^\ell} \chi_1 \chi_2' \frac{|\mu_2|^2}{\epsilon_2} \left(\Psi \sum_{j \in L} \beta''_j \frac{d\bar\zeta_j}{\bar\zeta_j} + \bar\partial_{K^c} \Psi \right) \wedge \varphi. \qquad (26)
\end{aligned}$$

Note that $\bar\partial_K$ does not fall on $|\mu_1|^2$ because of the hypotheses on the multi-indices. By assumption, $d\bar\zeta_j/\bar\zeta_j \wedge \varphi \in \mathscr{D}_{n,n}(\mathbb{C}^n)$ for $j \in L$ and so the first and the last integral on the right-hand side of (26) tend to zero and has the right modulus of continuity by Corollary 15. The second to last integral in (26) tends to $-[1/(\mu_1^k \mu_2^\ell)].\bar\partial_K \varphi = [1/(\mu_1^k \zeta^{\ell\beta''})] \otimes \bar\partial[1/\zeta^{\ell\beta'}].\varphi$ and has the right modulus of continuity by Proposition 11. □

Theorem 21. *Let* f *and* g *be holomorphic sections* (*locally non-trivial*) *of the holomorphic* m_j*-bundles* $E_j^* \to X$, $j = 1, 2$, *respectively. Assume that the section* $f \oplus g$ *of* $E_1^* \oplus E_2^* \to X$ *defines a complete intersection. Let* $\chi_1, \chi_2 \in C^\infty([0,\infty])$ *be any functions vanishing to orders* m_1 *and* m_2 *at zero, respectively, and satisfying* $\chi_j(\infty) = 1$. *Then, for any test form* φ *we have*

$$\int \chi_1\big(|f|^2/\epsilon_1\big) u^f \wedge \bar\partial \chi_2\big(|g|^2/\epsilon_2\big) \wedge u^g \wedge \varphi \to U^f \wedge R^g.\varphi \qquad (27)$$

as $\epsilon_1, \epsilon_2 \to 0^+$. *Moreover, as a function of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0,\infty)^2$ *the integral on the left-hand side belongs to some Hölder class independently of* φ.

Proof. We will assume that χ_1 and χ_2 only vanish to orders $k \leqslant m_1$ and $\ell \leqslant m_2$, respectively, and show that

$$\int \chi_1 u^f_{k,k-1} \wedge \bar\partial \chi_2 \wedge u^g_{\ell,\ell-1} \wedge \varphi \to \int |f|^{2\lambda} u^f_{k,k-1} \wedge \bar\partial |g|^{2\lambda} \wedge u^g_{\ell,\ell-1} \wedge \varphi \Big|_{\lambda=0}. \qquad (28)$$

By arguing as in the proof of Theorem 16 we may assume that $|f|^2 = |\zeta^\alpha|^2 \Phi$ and $|g|^2 = |\zeta^\beta|^2 \Psi$, where Φ and Ψ are strictly positive functions and, moreover, that $u^f_{k,k-1} = \tilde u^f_{k,k-1}/\zeta^{k\alpha}$ for a smooth form $\tilde u^f_{k,k-1}$ and similarly for $u^g_{\ell,\ell-1}$. What we have to prove is thus

$$\int \frac{\chi_1(\Phi|\zeta^\alpha|^2/\epsilon_1)}{\zeta^{k\alpha}}\tilde{u}^f_{k,k-1} \wedge \frac{\bar\partial\chi_2(\Psi|\zeta^\beta|^2/\epsilon_2)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi$$
$$\to \int \frac{|\zeta^\alpha|^{2\lambda}}{\zeta^{k\alpha}}\Phi^\lambda\tilde{u}^f_{k,k-1} \wedge \frac{\bar\partial(|\zeta^\beta|^{2\lambda}\Psi^\lambda)}{\zeta^{\ell\beta}}\tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi\Big|_{\lambda=0}, \tag{29}$$

where $\tilde\varphi = \rho_{k_j}\Pi_j^* \cdots \rho_{k_1}\Pi_1^*\varphi$. After the resolutions of singularities we can in general no longer say that the pull-back of $f \oplus g$ defines a complete intersection. On the other hand, we claim that if ζ_j divides both ζ^α and ζ^β then $d\bar\zeta_j/\bar\zeta_j \wedge \tilde\varphi$ is smooth. In fact, let z be local coordinates on our original manifold. In order that the integrals in (28) should be non-zero, φ has to have degree $n-k-\ell+1$ in $d\bar z$ and so we can assume that

$$\varphi = \sum_{\#J=n-k-\ell+1} \varphi_J \wedge d\bar z_J.$$

Since the variety $V_{f\oplus g} = f^{-1}(0) \cap g^{-1}(0)$ has dimension $n-m_1-m_2 < n-k-\ell+1$ we see that $d\bar z_J$ vanishes on $V_{f\oplus g}$. The pull-back of $d\bar z_J$ through all the resolutions Π_j can be written $\sum_I C_I(\zeta)\,d\bar\zeta_I$ and it must vanish on the pull-back of $V_{f\oplus g}$. In particular it has to vanish on $\{\zeta_j = 0\}$ if ζ_j divides both ζ^α and ζ^β. If $d\bar\zeta_j$ does not occur in $d\bar\zeta_I$ it must be that the coefficient function $C_I(\zeta)$ vanishes on $\{\zeta_j = 0\}$. But these functions are anti-holomorphic and so $\bar\zeta_j$ must divide $C_I(\zeta)$. The claim is established. We now write $\zeta^\alpha = \zeta^{\alpha'+\alpha''}$ and $\zeta^\beta = \zeta^{\beta'+\beta''}$ where α', α'' and β' have pairwise disjoint supports and $\alpha'' = 0$ if and only if $\beta'' = 0$. Thus, ζ_j divides both ζ^α and ζ^β if and only if $\alpha''_j \neq 0$, or equivalently, $\beta''_j \neq 0$. According to Proposition 18 the left-hand side of (29) belongs to some Hölder class and tends to

$$-\left[\frac{1}{\zeta^{k\alpha+\ell\beta''}}\right] \otimes \bar\partial\left[\frac{1}{\zeta^{\ell\beta'}}\right].\tilde{u}^f_{k,k-1} \wedge \tilde{u}^g_{\ell,\ell-1} \wedge \tilde\varphi.$$

One can compute the right-hand side of (29) by integrations by parts as in e.g. [1] to see that it equals the same thing. □

Remark 22. The form $\bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g$ is actually smooth even if χ_2 only vanishes to order m_2 at 0. The only possible problem is with the top degree term $\bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g_{m_2,m_2-1}$. But we have

$$C^\infty(X) \ni \bar\partial\big(\chi_2\big(|g|^2/\epsilon_2\big)u^g_{m_2,m_2-1}\big) = \bar\partial\chi_2\big(|g|^2/\epsilon_2\big) \wedge u^g_{m_2,m_2-1} + \chi_2\big(|g|^2/\epsilon_2\big)\bar\partial u^g_{m_2,m_2-1},$$

and since $u^g_{m_2,m_2-1}$ is $\bar\partial$-closed (outside V_g) it follows that $\bar\partial\chi_2(|g|^2/\epsilon_2) \wedge u^g_{m_2,m_2-1}$ is smooth as well.

Corollary 23. *With the same hypotheses as in Theorem* 21 *we have*

$$\int \bar\partial\chi_1\big(|f|^2/\epsilon_1\big) \wedge u^f \wedge \bar\partial\chi_2\big(|g|^2/\epsilon_2\big) \wedge u^g \wedge \varphi \to R^f \wedge R^g.\varphi,$$
$$\int \bar\partial\chi_1\big(|f|^2/\epsilon_1\big) \wedge u^f\,\chi_2\big(|g|^2/\epsilon_2\big) \wedge \varphi \to R^f.\varphi, \tag{30}$$

and

$$\int \chi_1\big(|f|^2/\epsilon_1\big) \wedge u^f \wedge \bar\partial\chi_2\big(|g|^2/\epsilon_2\big) \wedge \varphi \to 0 \tag{31}$$

as $\epsilon_1, \epsilon_2 \to 0^+$*, and as functions of* $\epsilon = (\epsilon_1, \epsilon_2) \in [0, \infty)^2$ *the integrals on the left-hand sides belong to some Hölder classes independently of* φ.

Proof. We have the following equality of smooth forms:

$$\nabla\big(\bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g\big) = -\bar\partial\chi_1 \wedge \chi_2 u^g - \bar\partial\chi_1 \wedge u^f \wedge \bar\partial\chi_2 \wedge u^g + \bar\partial\chi_1 \wedge u^f \chi_2. \tag{32}$$

The computation rules established in [22], and Theorem 21 now imply that, for any test form φ (of complementary total degree), we have

$$\begin{aligned} R^f.\varphi - R^f \wedge R^g.\varphi = \nabla\big(R^f \wedge U^g\big).\varphi &= -R^f \wedge U^g.\nabla\varphi \\ &= \lim -\int \bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g \wedge \nabla\varphi \\ &= \lim \int \nabla\big(\bar\partial\chi_1 \wedge u^f \wedge \chi_2 u^g\big) \wedge \varphi. \end{aligned}$$

The integral on the second row is Hölder continuous by Theorem 21 and so, also the integral on the third row is. By choosing φ of appropriate bidegrees the corollary now follows from (32). □

The statements (30) and (31) actually hold with no assumptions on the behavior of χ_2 at zero. This can be seen by using that we know this when $\chi_2 \equiv 1$ by Corollary 5, and when χ_2 vanishes to high enough order by the previous corollary.

Assume that f defines a complete intersection and pick a holomorphic function g such that $f \oplus g$ also defines a complete intersection and such that g is zero on the singular part of V_f. After resolving singularities in the proof of Theorem 21 we can find coordinates such that g is a monomial times a non-vanishing holomorphic function $\tilde g$. But $\tilde g$ can be incorporated in some coordinate and we can therefore assume that $\tilde g \equiv 1$. Repeating the proof of Theorem 21 and using Remark 14 one shows that (30) holds for χ_2 equal to the characteristic function of $[1, \infty]$. Then, if we first let ϵ_1 tend to zero, keeping ϵ_2 fixed, and after that let ϵ_2 tend to zero we get that

$$\lim_{\epsilon_2 \to 0^+} \chi_2\big(|g|^2/\epsilon_2\big) R^f = R^f.$$

We remark that the product $\chi_2(|g|^2/\epsilon_2)R^f$ is well defined since the wave front sets of $\chi_2(|g|^2/\epsilon_2)$ and R^f behave properly, see e.g. [6]. Since $\chi_2(|g|^2/\epsilon_2)$ equals the characteristic function of $\{|g|^2 > \epsilon_2\}$ we have

Corollary 24. *If* f *defines a complete intersection then the Cauchy–Fantappiè–Leray current* R^f *has the standard extension property.*

This is a well-known result and follows from the fact that R^f equals the Coleff–Herrera current in the sense of (7). It is even true that $\chi_{\rho g}(\epsilon)R^f \to R^f$, $\epsilon \to 0^+$, where ρ is a positive smooth function and $\chi_{\rho g}(\epsilon)$ is the characteristic function of $\{|\rho g| > \epsilon\}$. In fact, via Hironaka and toric resolutions one reduces to the case of one function and then one can proceed as in [6].

We know from [22] that if $f \oplus g$ defines a complete intersection then $R^f \wedge R^g$ consists of one term of top degree. Hence, it is only the top degree term of $\bar\partial\chi_1 \wedge u^f \wedge \bar\partial\chi_2 \wedge u^g$ which gives a contribution in the limit. With the natural choices $\chi_1(t) = t^{m_1}/(t+1)^{m_1}$ and $\chi_2(t) = t^{m_2}/(t+1)^{m_2}$, Corollary 23 and Remark 22 thus give

Corollary 25. *Let f and g be holomorphic sections* (*locally non-trivial*) *of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Assume that the section $f \oplus g$ of $E_1^* \oplus E_2^* \to X$ defines a complete intersection. Then, for any test form φ we have*

$$\int \bar\partial \frac{s_f \wedge (\bar\partial s_f)^{m_1-1}}{(|f|^2+\epsilon_1)^{m_1}} \wedge \bar\partial \frac{s_g \wedge (\bar\partial s_g)^{m_2-1}}{(|g|^2+\epsilon_2)^{m_2}} \wedge \varphi \to R^f \wedge R^g.\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

For sections f and g of the trivial line bundle we get the result announced in [18].

Corollary 26. *Let f and g be holomorphic functions defining a complete intersection. Then for any test form φ we have*

$$\int \bar\partial \frac{\bar f}{|f|^2+\epsilon_1} \wedge \bar\partial \frac{\bar g}{|g|^2+\epsilon_2} \wedge \varphi \to \left[\bar\partial \frac{1}{f} \wedge \bar\partial \frac{1}{g}\right].\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

Proof. We consider f and g as sections of (different copies of) the trivial line bundle $X \times \mathbb{C} \to X$ with the standard metric. Then, suppressing the natural global frame elements, we have $s_f = \bar f$ and $s_g = \bar g$. By Corollary 25 we are done since $R^f \wedge R^g$ is the Coleff–Herrera current. $\square$

So far, in this section, we have used one function χ to regularize all terms of u^f. One could try to take different χ's for different terms. We recall the natural choices $t^k/(t+1)^k$ from Corollary 4 and we let $u_\epsilon^f = s_f/(\nabla s_f + \epsilon) = \sum s_f \wedge (\bar\partial s_f)^{k-1}/(|f|^2+\epsilon)^k$. The next theorem says that, in the complete intersection case, the product of two such regularized currents goes unrestrictedly to the product, in the sense of [22], of the currents.

Theorem 27. *Let f and g be holomorphic sections* (*locally non-trivial*) *of the holomorphic m_j-bundles $E_j^* \to X$, $j = 1, 2$, respectively. Assume that the section $f \oplus g$ of $E_1^* \oplus E_2^* \to X$ defines a complete intersection. Then, for any test form φ we have*

$$\int u_{\epsilon_1}^f \wedge \nabla u_{\epsilon_2}^g \wedge \varphi \to \big(U^f - U^f \wedge R^g\big).\varphi$$

as $\epsilon_1, \epsilon_2 \to 0^+$, and the integral to the left belongs to some Hölder class independently of φ.

Proof. We first note that

$$\nabla u^g_{\epsilon_2} = 1 - \epsilon_2 \sum_{\ell \geqslant 1} \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell},$$

see the proof of Corollary 4. As $U^f \wedge R^f$ is defined as the value at zero of the analytic continuation (in the sense of currents) of $|f|^{2\lambda} u^f \wedge \bar\partial |g|^{2\lambda} \wedge u^g$, what we have to prove is that

$$\int \frac{s_f \wedge (\bar\partial s_f)^{k-1}}{(|f|^2+\epsilon_1)^k} \wedge \epsilon_2 \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell} \wedge \varphi \\ \to \int |f|^{2\lambda} u^f_{k,k-1} \wedge \bar\partial |g|^{2\lambda} \wedge u^g_{\ell-1,\ell-2} \wedge \varphi \Big|_{\lambda=0} \tag{33}$$

and that the integral on the left belongs to some Hölder class. We first consider the case $\ell = 1$. The right-hand side of (33) should then be interpreted as zero. We write the integrand on the left-hand side of (33) as $\chi_1(|f|^2/\epsilon_1)\chi_2(|g|^2/\epsilon_2) u^f_{k,k-1} \wedge \varphi$ where $\chi_1(t) = t^k/(t+1)^k$ and $\chi_2(t) = 1/(t+1)$. As in the proof of Theorem 16 we may assume that $u^f_{k,k-1} = \tilde u^f_{k,k-1}/\zeta^{k\alpha}$, where $\tilde u^f_{k,k-1}$ is a smooth form, that $|f|^2 = |\zeta^\alpha|\Phi$ and that $|g|^2 = |\zeta^\beta|^2\Psi$, where Φ and Ψ are strictly positive smooth functions. Since $\chi_2(\infty) = 0$ the left-hand side of (33) tends to zero and belongs to some Hölder class by Proposition 11. For $\ell \geqslant 2$ we proceed as in the proof of Theorem 21 and we see that we may assume that $f = (f_1, \dots, f_m)$ and $g = (g_1, \dots, g_{m_2})$ with $f_j = \zeta^{\alpha^j} f'_j$ and $g_j = \zeta^{\beta^j} g'_j$ where all f'_j and g'_j are non-vanishing and, moreover, that for some indices ν_1 and ν_2 it holds that $\zeta^\alpha := \zeta^{\alpha^{\nu_1}}$ divides all ζ^{α^j} and $\zeta^\beta := \zeta^{\beta^{\nu_2}}$ divides all ζ^{β^j}. From the same proof we also see that we may assume that $d\bar\zeta_j/\bar\zeta_j \wedge \varphi$ is smooth (and compactly supported) for all ζ_j which divide both ζ^α and ζ^β, since $f \oplus g$ defines a complete intersection. We use the notation from the proof of Theorem 21, e.g. $|f|^2 = |\zeta^\alpha|^2\Phi = |\zeta^{\alpha'+\alpha''}|^2\Phi$, $u^f_{k,k-1} = \tilde u^f_{k,k-1}/\zeta^{k(\alpha'+\alpha'')}$ and $|g|^2 = |\zeta^\beta|^2\Psi = |\zeta^{\beta'+\beta''}|^2\Psi$, etc. We also introduce the notation $\chi_j(t)$ for the function $t^j/(t+1)^j$, and so, in particular, we can write $1/(t+\epsilon)^j = \chi_j(t/\epsilon)/t^j$. For $\ell \geqslant 2$, one can verify that

$$\epsilon_2 \frac{(\bar\partial s_g)^{\ell-1}}{(|g|^2+\epsilon_2)^\ell} = \frac{1}{\zeta^{(\ell-1)\beta}} \bar\partial \chi_{\ell-1}\big(|\zeta^\beta|^2\Psi/\epsilon_2\big) \wedge \tilde u^g_{\ell-1,\ell-2} \\ + \frac{1}{\zeta^{(\ell-1)\beta}} \chi'_{\ell-1}\big(|\zeta^\beta|^2\Psi/\epsilon_2\big) \frac{|\zeta^\beta|^2}{\epsilon_2} \frac{\Psi}{\ell-1} \bar\partial \tilde u^g_{\ell-1,\ell-2}. \tag{34}$$

Using this identity we see that the integral on the left-hand side of (33) splits into two integrals. The integral corresponding to the last term in (34) tends to zero as $\epsilon_1, \epsilon_2 \to 0$ and belongs to some Hölder class according to Corollary 15. By Proposition 18, the integral corresponding to the first term on the right-hand side of (34) also belongs to some Hölder class and tends to

$$-\left[\frac{1}{\zeta^{k\alpha+(\ell-1)\beta''}}\right] \otimes \bar\partial \left[\frac{1}{\zeta^{(\ell-1)\beta'}}\right] . \tilde u^f_{k,k-1} \wedge \tilde u^g_{\ell-1,\ell-2} \wedge \varphi \tag{35}$$

as $\epsilon_1, \epsilon_2 \to 0$. This is seen to be equal to the right-hand side of (33) by using the methods in [22]. $\square$

7. The Passare–Tsikh example

Let $f = z_1^4$, $g = z_1^2 + z_2^2 + z_1^3$ and $\varphi = \rho \bar{z}_2 g \, dz_1 \wedge dz_2$ where ρ has compact support and is identically 1 in a neighborhood of the origin. Since the common zero set of f and g is just the origin, f and g define a complete intersection. In [15] Passare and Tsikh show that the residue integral

$$(\epsilon_1, \epsilon_2) \mapsto I_{f,g}^{\varphi}(\epsilon_1, \epsilon_2) = \int\limits_{\substack{|f|^2=\epsilon_1 \\ |g|^2=\epsilon_2}} \frac{\varphi}{fg}$$

is discontinuous at the origin. More precisely, they show that for any fixed positive number $c \neq 1$ one has $\lim_{\epsilon \to 0} I_{f,g}^{\varphi}(\epsilon^4, c\epsilon^2) = 0$ but $\lim_{\epsilon \to 0} I_{f,g}^{\varphi}(\epsilon^4, \epsilon^2) \neq 0$. On the other hand, by Fubini's theorem we have

$$\int\limits_{[0,\infty)^2} \frac{\epsilon_2 \epsilon_2 I_{f,g}^{\varphi}(t_1, t_2) \, dt_1 \, dt_2}{(t_1 + \epsilon_1)^2 (t_2 + \epsilon_2)^2} = \int \frac{\epsilon_1 \, d|f|^2}{(|f|^2 + \epsilon_1)^2} \wedge \frac{\epsilon_2 \, d|g|^2}{(|g|^2 + \epsilon_2)^2} \wedge \frac{\varphi}{fg}$$

$$= \int \bar{\partial} \frac{\bar{f}}{|f|^2 + \epsilon_1} \wedge \bar{\partial} \frac{\bar{g}}{|g|^2 + \epsilon_2} \wedge \varphi. \tag{36}$$

Hence, this average of the residue integral is continuous at the origin by Corollary 26. In this section we will examine the last integral in (36) as $\epsilon_1, \epsilon_2 \to 0$ explicitly. We will see that it is continuous at the origin with Hölder exponent at least $1/8$ and that it tends to zero. Morally, the value of $I_{f,g}^{\varphi}(\epsilon_1, \epsilon_2)$ at 0 should be the Coleff–Herrera current associated to f and g multiplied by $\bar{z}_2 g$ acting on $\rho \, dz_1 \wedge dz_2$. But both g and $\bar{z}_2$ annihilate the Coleff–Herrera current since g belongs to the ideal generated by f and g, and z_2 belongs to the radical of this ideal. We will thus verify Corollary 26 explicitly in this special case.

Our first objective is to resolve singularities to obtain normal crossings. This is accomplished by a blow-up of the origin. The map $\pi : \mathcal{B}_0\mathbb{C}^2 \to \mathbb{C}^2$ looks like $\pi(u, v) = (u, uv)$ and $\pi(u', v') = (u'v', u')$ in the two standard coordinate systems on $\mathcal{B}_0\mathbb{C}^2$. The exceptional divisor, E, corresponds to the sets $\{u = 0\}$ and $\{u' = 0\}$ and π is a biholomorphism $\mathcal{B}_0\mathbb{C}^2 \setminus E \to \mathbb{C}^2 \setminus \{0\}$. In the (u, v)-coordinates we have $\pi^* f = u^4$ and $\pi^* g = u^2(1 + v^2 + u)$. The function $1 + v^2 + u$ has non-zero differential and its zero locus intersects E normally in the two points $v = i$ and $v = -i$. Moreover, in the (u', v')-coordinates we have $\pi^* f = u'^4 v'^4$ and $\pi^* g = u'^2(v'^2 + 1 + u'v'^3)$. The zero locus of $v'^2 + 1 + u'v'^3$ intersects E normally in the points $v' = -i$ and $v' = i$, which we already knew, and it does not intersect $v' = 0$. Also, the differential of $v'^2 + 1 + u'v'^3$ is non-zero on the zero locus of $v'^2 + 1 + u'v'^3$. Hence, $\{\pi^* f \cdot \pi^* g = 0\}$ has normal crossings. We assume that φ has support so close to the origin that $\operatorname{supp}(\pi^*\varphi) \cap \{1 + v^2 + u = 0\}$ has two (compact) components, K_1 and K_2, and that these components together with the compacts $K_3 = \operatorname{supp}(\pi^*\varphi) \cap \{v = 0\}$ and $K_4 = \operatorname{supp}(\pi^*\varphi) \cap \{v =' 0\}$ are pairwise disjoint. We can then choose a partition of unity $\{\rho_j\}_1^4$ such that $\sum \rho_j \equiv 1$ on the support of $\pi^*\varphi$ and for each $j = 1, 2, 3, 4$, the support of ρ_j intersects only one of the compacts K_1, K_2, K_3 and K_4. We choose the numbering such that the support of ρ_j intersects K_j. The last integral in (36) now equals

$$\sum_1^4 \int \bar\partial \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \wedge \bar\partial \frac{\pi^*\bar g}{|\pi^* g|^2+\epsilon_2} \wedge \rho_j \pi^*\varphi := I_1 + I_2 + I_3 + I_4. \tag{37}$$

In fact, it is only in I_3 we have resonance and we start by considering the easier integrals I_1, I_2 and I_4. The integrals I_1 and I_2 are similar and we only consider I_1. The support of ρ_1 is contained in a neighborhood of $p_1 = (0, i)$ in the (u, v)-coordinates and $\rho_1\pi^*\varphi = \rho_1\pi^*\rho\bar u\bar v\pi^* g u\, du \wedge dv$. Integrating by parts we thus see that

$$I_1 = -\int \bar\partial \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \frac{|\pi^* g|^2}{|\pi^* g|^2+\epsilon_2} \wedge u\bar\partial(\bar u\bar v\rho_1\pi^*\rho\, du\wedge dv).$$

Since $\pi^* f = u^4$ depends on u only, the term of $\bar\partial(\bar u\bar v\rho_1\pi^*\rho)$ involving $d\bar u$ does not give any contribution to I_1. Hence we can replace $\bar\partial(\bar u\bar v\rho_1\pi^*\rho)$ by $\bar u\varphi_1$ where φ_1 is smooth and supported where ρ_1 is. We put $\zeta_1 = u$ and $\zeta_2 = 1 + v^2 + u$, which defines a change of variables on the support of ρ_1. In these coordinates $\pi^* f = \zeta_1^4$ and $\pi^* g = \zeta_1^2\zeta_2$ and so we get

$$I_1 = -\int \frac{1}{\zeta_1^3} \bar\partial\chi\left(\left|\zeta_1^4\right|^2/\epsilon_1\right)\chi\left(\left|\zeta_1^2\zeta_2\right|^2/\epsilon_2\right) \wedge \bar\zeta_1\varphi_1,$$

where $\chi(t) = t/(t+1)$. We also write $\bar\partial\chi(|\zeta_1^4|^2/\epsilon_1) = 4\tilde\chi(|\zeta_1^4|^2/\epsilon_1)\, d\bar\zeta_1/\bar\zeta_1$, where $\tilde\chi(t) = t/(t+1)^2$. To proceed we replace (the coefficient function of) $d\bar\zeta_1/\bar\zeta_1 \wedge \bar\zeta_1\varphi_1$ by its Taylor expansion of order one, considered as a function of ζ_1 only, plus a remainder term $|\zeta_1|^2 B(\zeta)$, with B bounded. The terms corresponding to the Taylor expansion do not give any contribution to I_1 since we have anti-symmetry with respect to ζ_1 for these terms. Hence, we obtain

$$|I_1| \lesssim \int_\Delta \left| \frac{|\zeta_1|^2 B(\zeta)}{\zeta_1^3} \tilde\chi\left(\left|\zeta_1^4\right|^2/\epsilon_1\right)\chi\left(\left|\zeta_1^2\zeta_2\right|^2/\epsilon_2\right)\right|, \tag{38}$$

where Δ is a polydisc containing the support of φ_1. We estimate $|B(\zeta)|$ and $\chi(|\zeta_1^2\zeta_2|^2/\epsilon_2)$ by constants, and on the sets $\Delta_\epsilon = \{\zeta \in \Delta;\ |\zeta_1^4|^2 \geqslant \epsilon_1\}$ and $\Delta \setminus \Delta_\epsilon$ we use that $\tilde\chi(|\zeta_1^4|^2/\epsilon_1) \lesssim \epsilon_1/|\zeta_1^4|^2$ and $\tilde\chi(|\zeta_1^4|^2/\epsilon_1) \lesssim |\zeta_1^4|^2/\epsilon_1$, respectively, to see that the right-hand side of (38) is of the size $|\epsilon|^{1/8}$.

To deal with I_4 we proceed as follows. The support of ρ_4 is contained in a neighborhood of $p_4 = (0, 0)$ in the (u', v')-coordinates and $\pi^* f = u'^4 v'^4$ and $\pi^* g = u'^2(1 + v'^2 + u'v'^3) := u'^2\tilde g$. On the support of ρ_4 we have $\tilde g \neq 0$. The multi-indices $(4, 4)$ and $(2, 0)$ are linearly independent and so we can make the factor $\tilde g$ disappear. Explicitly, choose a square root $\tilde g^{1/2}$ of $\tilde g$ and put $\zeta_1 = u'\tilde g^{1/2}$ and $\zeta_2 = v'\tilde g^{-1/2}$. In these coordinates $\pi^* f = \zeta_1^4\zeta_2^4$ and $\pi^* g = \zeta_1^2$. One also checks that $\rho_4\pi^*\varphi = |\zeta_1|^2\pi^* g\varphi_4$ where φ_4 is a test form of bidegree $(2, 0)$. After an integration by parts we see that

$$I_4 = \int \frac{\pi^*\bar f}{|\pi^* f|^2+\epsilon_1} \bar\partial \frac{|\pi^* g|^2}{|\pi^* g|^2+\epsilon_2} \wedge \bar\partial\left(|\zeta_1|^2\varphi_4\right). \tag{39}$$

Since $\pi^* g = \zeta_1^2$ only depends on ζ_1 we may replace $\bar\partial(|\zeta_1|^2\varphi_4)$ by $|\zeta_1|^2\bar\partial\varphi_4$ in (39). Computing $\bar\partial(|\pi^*g|^2/(|\pi^*g|^2+\epsilon_2))$ we find that

$$I_4 = 2\int \frac{1}{\zeta_1^3\zeta_2^4}\chi\left(\left|\zeta_1^4\zeta_2^4\right|^2/\epsilon_1\right)\tilde\chi\left(\left|\zeta_1^2\right|^2/\epsilon_2\right)d\bar\zeta_1\wedge\bar\partial\varphi_4.$$

With abuse of notation we write the test form $d\bar\zeta_1 \wedge \bar\partial\varphi_4$ as $\varphi_4\, d\zeta \wedge d\bar\zeta$. Let $M = M^{1,2}_{1,2}$ be the operator defined in Lemma 6. Explicitly, we have

$$\begin{aligned} M\varphi_4 &= M_1^1\varphi_4 + M_2^2\varphi_4 - M_1^1M_2^2\varphi_4 \\ &= M_1^1\left(\varphi_4 - M_2^2\varphi_4\right) + M_2^2\left(\varphi_4 - M_1^1\varphi_4\right) + M_1^1M_2^2\varphi_4. \end{aligned}$$

All of the following properties will not be important for this computation but to illustrate Lemma 6 we note that the second expression of $M\varphi$ reveals that $M\varphi_4$ can be written as a sum of terms $\phi_{IJ}(\zeta)\zeta^I\bar\zeta^J$ with $I_1 + J_1 \leqslant 1$ and $I_2 + J_2 \leqslant 2$ and, moreover, that ϕ_{IJ} is independent of at least one variable and is of the size $\mathcal{O}(|\zeta_1|^2)$ if it depends on ζ_1 and of the size $\mathcal{O}(|\zeta_2|^3)$ if it depends on ζ_2. By Lemma 6 we also have $\varphi_4 = M\varphi_4 + |\zeta_1|^2|\zeta_2|^3B(\zeta)$ for some bounded function B and so

$$I_4 = \int_\Delta \frac{1}{\zeta_1^3\zeta_2^4}\chi\,\tilde\chi\, M\varphi_4 + \int_\Delta \frac{1}{\zeta_1^3\zeta_2^4}\chi\,\tilde\chi\,|\zeta_1|^2|\zeta_2|^3B(\zeta) =: I_{4.1} + I_{4.2},$$

where Δ is a polydisc containing the support of φ_4. By anti-symmetry $I_{4.1} = 0$. To estimate $I_{4.2}$ we use that $|\chi B|$ is bounded by a constant and that $\tilde\chi(\Psi|\zeta_1^2|^2/\epsilon_2) \lesssim \epsilon_2/|\zeta_1^2|^2$ and $\tilde\chi(\Psi|\zeta_1^2|^2/\epsilon_2) \lesssim |\zeta_1^2|^2/\epsilon_2$ on the sets $\Delta_\epsilon = \{\zeta \in \Delta;\ |\zeta_1^2|^2 \geqslant \epsilon_2\}$ and $\Delta \setminus \Delta_\epsilon$, respectively. Hence,

$$|I_{4.2}| \lesssim \int_{\Delta_\epsilon} \frac{\epsilon_2}{|\zeta_1^2|^2|\zeta_1||\zeta_2|} + \int_{\Delta\setminus\Delta_\epsilon} \frac{|\zeta_1^2|^2}{\epsilon_2|\zeta_1||\zeta_2|}, \tag{40}$$

which is seen to be of the size $|\epsilon|^{1/4}$.

It remains to take care of I_3. We are now working close to $u = v = 0$ and $\pi^* f = u^4$ and $g = u^2(1 + v^2 + u) := u^2\tilde g$. The multi-indices are linearly dependent and we cannot dispose of the non-zero factor $\tilde g$. We rename our variables $(u, v) = (\zeta_1, \zeta_2)$ and proceed in precisely the same way as we did when we were considering I_1. We get

$$I_3 = -4\int \frac{1}{\zeta_1^3}\tilde\chi\left(\left|\zeta_1^4\right|^2/\epsilon_1\right)\chi\left(\Phi\left|\zeta_1^2\right|^2/\epsilon_2\right)\varphi_3\, d\zeta \wedge d\bar\zeta,$$

where $\Phi = |\tilde g|^2$ is a strictly positive smooth function and φ_3 is smooth with compact support. As before, we replace φ_3 by $M^1_{\zeta_1}\varphi_3 + |\zeta_1|^2B(\zeta)$. The integral corresponding to $|\zeta_1|^2B(\zeta)$ satisfies the same estimate as the one in (38) and hence is of the size $|\epsilon_1|^{1/8}$. We cannot use anti-symmetry directly to conclude that the integrals corresponding to the other terms in the Taylor expansion tend to zero since the factor $\tilde g$ is present. We illustrate why this is true anyway by considering the integral corresponding to the term $\varphi_3(0,\zeta_2)$. Let Δ be a polydisc containing the support of φ_3 and consider

$$\int_{\Delta} \frac{1}{\zeta_1^3}\tilde{\chi}\left(\left|\zeta_1^4\right|^2/\epsilon_1\right)\chi\left(\Phi\left|\zeta_1^2\right|^2/\epsilon_2\right)\varphi_3(0,\zeta_2). \tag{41}$$

We introduce the smoothing parameter $t=|\zeta_1^2|^2/\epsilon_2$ as an independent variable and write

$$\chi(\Phi t)=\chi(\Phi t)-M^1_{\zeta_1}\chi(\Phi t)+M^1_{\zeta_1}\chi(\Phi t):=|\zeta_1|^2B(t,\zeta)+M^1_{\zeta_1}\chi(\Phi t).$$

Here B is bounded on $[0,\infty]\times\Delta$. Substituting into (41) we obtain one integral corresponding to $|\zeta_1|^2B(|\zeta_1^2|^2/\epsilon_2,\zeta)$, which satisfies an estimate like (38), while the integral corresponding to $M^1_{\zeta_1}\chi(\Phi|\zeta_1^2|^2/\epsilon_2)$ is zero since we have anti-symmetry with respect to ζ_1. Hence $|I_3|\lesssim|\epsilon|^{1/8}$.

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