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On Hochschild Cohomology of Preprojective Algebras, II

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We study the Hochschild cohomology of a finite-dimensional preprojective algebra; this is periodic by a result of A. Schofield. We determine the ring structure of the Hochschild cohomology ring given by the Yoneda product. As a result we obtain an explicit presentation by generators and relations.

Let Λ be a preprojective algebra of type A_n over a field K. In [ES] bases and dimensions of the Hochschild cohomology groups $HH^i(\Lambda)$ were obtained. In this second paper we determine the ring structure of HH^* given by the Yoneda product for type A_n. By [GS, Sect. 13], this ring structure is the same as that given by the cup product.

We use the notation and the results of the first part; a summary of the main facts needed here is given in Subsection 1.1 below. As far as the results of this paper are concerned, Table I at the end describes the multiplication of homogeneous elements. Moreover, we obtain a presentation of $HH^{ev}(\Lambda)$ (the part of even degree) and of $HH^*(\Lambda)$ by generators and relations (Section 5). In particular, the ring is commutative. We note that the space $HH^1(\Lambda) \times HH^2(\Lambda)$ depends on the characteristic of the field. Namely, those fields whose characteristic divides $n + 1$ are different.

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Using the isomorphism $HH^i(\Lambda) \cong \underline{\mathrm{Hom}}(\Omega^i\Lambda, \Lambda)$, we may consider an element of $HH^i(\Lambda)$ as the class of a map $\Omega^i(\Lambda) \to \Lambda$. For $[f] \in HH^i(\Lambda)$ and $[g] \in HH^j(\Lambda)$ the product $[f][g] := [f \circ \Omega^i g]$ in $HH^{i+j}(\Lambda)$. We usually omit $[-]$; this should not cause confusion. For $1 \le i \le j \le 5$, we compute the products of $HH^i(\Lambda)$ with $HH^j(\Lambda)$. The remaining products then follow since the projective resolution is periodic, and since the multiplication is graded-commutative.

1. SOME PRODUCTS $HH^i \times HH^j$ WITH i OR j ODD

1.1. Let Λ be a preprojective algebra of type A_n over a field K. Then Λ is given by quiver and relations in Sections I.1.1 and I.3.3. (We reference [ES] by I throughout.) First we will summarize results needed. The $\Lambda - \Lambda$ bimodule Λ is periodic of period 6 (or 2 if $n = 2$); in [S] a minimal projective resolution of Λ was determined which we will now describe.

Let $\tilde{\nu}$ be the automorphism of Λ of order two as defined in Section I.3.3. Define projective $\Lambda - \Lambda$ bimodules $P_0 = P_2 = \bigoplus_i \Lambda(e_i \otimes e_i)\Lambda$ and $P_1 = \bigoplus_\alpha \Lambda(e_{i\alpha} \otimes e_{t\alpha})\Lambda$, and define homomorphisms $\delta : P_1 \to P_0$ and $R : P_2 \to P_1$ by

$$\delta(e_{i\alpha} \otimes e_{t\alpha}) = \alpha \otimes e_{t\alpha} - e_{i\alpha} \otimes \alpha =: x_\alpha$$
$$R(e_i \otimes e_i) = \sum_{i\alpha = i} e_{i\alpha} \otimes \overline{\alpha} + \alpha \otimes e_{t\overline{\alpha}} =: \sigma_i.$$

Moreover, let $u : P_0 \to \Lambda$ be the multiplication map.

THEOREM [S]. *We have an exact sequence of $\Lambda - \Lambda$ bimodules*

$$0 \to {}_1\Lambda_{\tilde{\nu}} \to P_2 \xrightarrow{R} P_1 \xrightarrow{\delta} P_0 \xrightarrow{u} \Lambda \to 0,$$

where ${}_1\Lambda_{\tilde{\nu}}$ is the bimodule structure on Λ where the action on the right is twisted by the automorphism $\tilde{\nu}$ of Λ, of order 2.

Now one observes that ${}_1\Lambda_{\tilde{\nu}} \otimes_\Lambda ({}_1\Lambda_{\tilde{\nu}}) \cong \Lambda$ and hence if one tensors the above exact sequence with ${}_1\Lambda_{\tilde{\nu}}$ one obtains the other half of a projective resolution of Λ and it has length 6.

As in Section I.2.2, we define elements $\zeta_i \in \oplus \Lambda(e_i \otimes e_i)\Lambda$ by the formula

$$\zeta_i := \sum_{x \in e_i B} (-1)^{\deg(x)} x \otimes x^*,$$

where B is a basis of Λ as described in Section I.4. We have fixed generators ω_i of the socle of $e_i\Lambda$ and then $*$ satisfies $vv^* = \omega_i$. Moreover, $N := {}_1\Lambda_{\tilde{\nu}}$ is generated by the ζ_i as a bimodule.

We shall use further notation. Define

$$w = \sum_i e_i \otimes e_{i+1} \in P_1, \qquad \overline{\omega} = \sum_i e_{e+1} \otimes e_i \in P_1,$$

$$\eta = \sum_i a_i \in \Lambda, \qquad \overline{\eta} = \sum_i \bar{a}_i \in \Lambda.$$

Let also $m = (n-1)/2$ if n is odd and $m = (n-2)/2$ otherwise.

1.1.1. In [ES] the following bases of the spaces $HH^k(\Lambda)$ were obtained.

(0) (I.5.2) $HH^0(\Lambda)$ has basis $\{z_i : 0 \le i \le m\}$ with $z_0 = 1$ and z_j of degree $2j$. Moreover, $HH^6(\Lambda) \cong HH^0(\Lambda)$ if n is even. For n odd we have $HH^6(\Lambda) \cong HH^0(\Lambda)/\langle z_m \rangle$.

(1) (I.6.3) $HH^1(\Lambda)$ has basis $\{g_i : 0 \le i \le n-m-2\}$ where g_i is identified with the map $g_i(w) = 0$, $g_i(\overline{w}) = z_i \overline{\eta}$.

(2) (I.7.2) $HH^2(\Lambda)$ has basis $\{f_i : 0 \le i \le n-m-2\}$ where

$$f_i(\sigma_j) = \begin{cases} e_i, & j = i \\ (-1)^n e_{\nu(i)}, & j = \nu(i) \\ 0, & \text{otherwise.} \end{cases}$$

(3) (I.7.5) $HH^3(\Lambda)$ is a quotient of (N, Λ). A basis of (N, Λ) is given by elements h_i, $0 \le i \le n-1$, where $h_i(\zeta_j) = \delta_{ij}\omega_j$. Then a basis of $HH^3(\Lambda)$ is given by the classes of $h_0, \dots, h_{n-m-2}$. Moreover, we have $[h_i] = (-1)^n[h_{\nu(i)}]$ in $HH^3(\Lambda)$ and if n is odd then $[h_m] = 0$.

(4) (I.9.4) $HH^4(\Lambda)$ has basis $\{\psi_0, \dots, \psi_{n-m-2}\}$ where for n odd

$$\psi_i(x_a \otimes \zeta_{ta}) = \begin{cases} z_i \bar{a}_{m-1}, & a = a_m \\ -z_i \bar{a}_m, & a = \bar{a}_m \\ 0, & \text{otherwise} \end{cases}$$

and for n even

$$\psi_i(x_a \otimes \zeta_{ta}) = \begin{cases} z_i e_m, & a = a_m \\ z_i e_{m+1}, & a = \bar{a}_m \\ 0, & \text{otherwise.} \end{cases}$$

(5) (I.8.5) and (I.1.2) $HH^5(\Lambda)$, as a quotient of $(\operatorname{Im} R \otimes N, \Lambda)$, has basis $\{\theta_i : 0 \le i \le n-m-2\}$ where $\theta_i(\sigma_j \otimes \zeta_j) = \delta_{jm} a_m z_i$ if n is even, and for n odd $\theta_i(\sigma_j \otimes \zeta_j) = \delta_{jm} a_m \bar{a}_m z_i$.

1.1.2. The ring $HH^0(\Lambda)$ is local, with radical generated by z_1. We have $z_j z_k = (-1)^{jk} z_{j+k}$ if $j + k \leq m$ and 0 otherwise. The $Z(\Lambda)$-modules $HH^i(\Lambda)$ are cyclic for $i = 1, 4, 5$ generated by g_0, ψ_0, θ_0, respectively. The $Z(\Lambda)$-modules $HH^i(\Lambda)$ for $i = 2, 3$ are annihilated by the radical of $Z(\Lambda)$. We shall use these facts tacitly for simplifying calculations.

Now we shall determine products of homogeneous elements where one factor has odd degree; these are very often zero and they are relatively easy to find.

1.2. $HH^1(\Lambda) \times HH^1(\Lambda) = 0$.

Let $f, g \in HH^1(\Lambda)$. Then there are elements $z_f, z_g \in Z(\Lambda)$ such that $f(w) = 0 = g(w)$ and $f(\overline{w}) = z_f \overline{\eta}$, $g(\overline{w}) = z_g \overline{\eta}$. Then f may be lifted to the map $\hat{f}: P_1 \to P_0$ where $w \mapsto 0, \overline{w} \mapsto z_f(\sum_{i=0}^{n-1} e_i \otimes e_i)\overline{\eta}$. Now $\hat{f}(\sigma_i) = a_i z_f(e_{i+1} \otimes e_{i+1})\overline{a}_i + z_f(e_i \otimes e_i)\overline{a}_{i-1} a_{i-1} = z_f(a_i \otimes e_{i+1} - e_i \otimes a_i)\overline{a}_i = z_f x_{a_i} \overline{a}_i$. Thus $\Omega f : \operatorname{Ker} \delta \to \operatorname{Ker} u$ is given by $\sigma_i \mapsto \hat{f}(\sigma_i) = z_f x_{a_i} \overline{a}_i$ for $i = 0, \ldots, n-2$. So $g \cdot f = [g \circ \Omega f] = 0$.

1.3. $HH^1(\Lambda) \times HH^3(\Lambda) = 0$.

Let $h_i \in HH^3(\Lambda)$ be one of the basis elements as given in Subsection 1.1. The map $\hat{h}_i : P_0 \otimes N \to P_0$ given by

$$\hat{h}_i : (e_j \otimes e_j) \otimes \zeta_j \mapsto \begin{cases} e_i \otimes \omega_i & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

is a lifting of h_i. Then Ωh_i is the restriction of $\hat{h}_i$ to $\operatorname{Ker} u \otimes N$. One finds that

$$\Omega h_i(x_a \otimes \zeta_{ta}) = \begin{cases} x_a \omega_i & \text{if } ta = i \\ 0 & \text{otherwise.} \end{cases}$$

To compute $HH^1(\Lambda) \times HH^3(\Lambda)$, let $g \in HH^1(\Lambda)$; then $(g \circ \Omega h_i)(x_{\overline{a}_i} \otimes \zeta_i) \in g(x_{\overline{a}_i})\operatorname{soc} \Lambda$ and this lies in $Z(\Lambda)J \operatorname{soc} \Lambda$ which is zero.

1.4. $HH^2(\Lambda) \times HH^3(\Lambda)$ *is* 1-*dimensional. We have* $f_i \cdot h_j = 0$ *for* $i \neq j$ *and, for* $0 \leq i \leq n - m - 2$, $f_i \cdot h_i = \theta_{n-m-2}$.

Proof. Suppose $h_i \in HH^3(\Lambda)$ is as in Subsection 1.1; we take Ωh_i as in Subsection 1.3. The first step is to find $\Omega^2 h_i$. The map $\tilde{h}_i : P_1 \otimes N \to P_1$ where

$$\tilde{h}_i : (e_{ia} \otimes e_{ta}) \otimes \zeta_{ta} \mapsto \begin{cases} e_{ia} \otimes \omega_i & \text{if } ta = i \\ 0 & \text{otherwise} \end{cases}$$

provides a lifting of Ωh_i. It is then easy to verify that the restriction of $\tilde{h}_i$ to Ker $\delta \otimes N$ is given by

$$\Omega^2 h_i : \sigma_j \otimes \zeta_j \mapsto \begin{cases} a_i \otimes \omega_i + \bar{a}_{i-1} \otimes \omega_i = \sigma_i \omega_i & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

Now we may compute $HH^2(\Lambda) \times HH^3(\Lambda)$. Let $f_l \in HH^2(\Lambda)$ be a basis element as given in Subsection 1.1. Then $f_l \circ \Omega^2 h_i = 0$ if $l \neq i$ and, if $l = i$, then

$$f_i \circ \Omega^2 h_i : \sigma_j \otimes \zeta_j \mapsto \begin{cases} \omega_i & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$

For $0 \leq k \leq m$, define $\beta_k \in (P_0, N)$ by $\beta_k(e_k \otimes e_k) = \omega_k \zeta_{\nu(k)}$ and $\beta_k(e_j \otimes e_j) = 0$ for $j \neq k$. Then for $i \leq n - m - 2$ the map $\beta_i \circ R$ is identified with $f_i \circ \Omega^2 h_i$, and moreover $\beta_m = \theta_{n-m-2}$, a basis element in $HH^5(\Lambda)$. So we are done if we show that the class of $\beta_i - \beta_{i+1}$ in $HH^5(\Lambda)$ is zero.

For $i < m$, let $\mu_i = a_i^* \zeta_{\nu(i)} \in e_{i+1} N e_i$; then $\omega_i \zeta_{\nu(i)} = a_i \mu_i$. With the notation of Section I.8.3, the Λ^3-homomorphism with

$$\phi_{\bar{a}_i} \circ R : e_k \otimes e_k \mapsto \begin{cases} a_i \mu_i & \text{if } k = i \\ \mu_i a_i & \text{if } k = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

is in the image of the map $\iota^* : (P_1, N) \to (\text{Im } R, N)$. We have, using Section I.3.3, that

$$\mu_i a_i = a_i^* \zeta_{\nu(i)} a_i = a_i^* \left(-\bar{a}_{\nu(i)-1} \zeta_{\nu(i)-1} \right) = -\omega_{i+1} \zeta_{\nu(i)-1} = -a_{i+1} \mu_{i+1}.$$

Hence $\beta_i - \beta_{i+1} = \phi_{\bar{a}_i} \circ R$.

1.5. $HH^3(\Lambda) \times HH^3(\Lambda) = 0$.

Let $h, h' \in HH^3(\Lambda)$, then $h, h' : N \to \Lambda$. The image is contained in soc Λ (I.7.3). We have that $\Omega^3 h' : N \otimes N \to N$ can be identified with a map $h'' \otimes 1$ for some $h'' \in (N, \Lambda)$. Then the image of h'' is contained in the socle of Λ, hence in J, and we deduce that the image of $\Omega^3 h'$ is contained in JN and then $h \circ \Omega^3 h' = 0$.

1.6. $HH^3(\Lambda) \times HH^4(\Lambda) = 0$ *for n odd. If n is even then*

$$h_i \cdot \psi_0 = \begin{cases} g_m \cdot 1 & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\psi_0 \in HH^*(\Lambda)$ as defined in Subsection 1.1; recall that it is a map from Ker $u \otimes N$ to Λ. We consider first $\Omega^3 h_i \circ \psi_0$. If n is odd then this is zero. Namely, the image of ψ_0 is contained in J. By the argument of Subsection 1.5, $h \circ \psi_0 = 0$ for any homomorphism $h : \Lambda \to N$.

Assume now that n is even. In Subsection 1.6.1 below we give a formula for a lifting $\Omega^3 h_i$. We see for $i < m$, that $\Omega^3 h_i \circ \psi_0 = 0$, since the image of ψ_0 is generated by elements in $(e_m + e_{m+1})\Lambda(e_m + e_{m+1}) \subseteq \mathrm{Ker}(\Omega^3 h_i)$.

Suppose now that $i = m$. Then $\Omega^3 h_m$ takes $\zeta_{m+1} \otimes \zeta_m$ (i.e., e_{m+1}) to $\zeta_{m+1}\omega_m$ and all other e_i are mapped to zero. So the composition with ψ_0 takes $x_{\bar{a}_m} \otimes \zeta_m$ to $\zeta_{m+1}\omega_m$ and all other generators to zero. This is now a map from $\mathrm{Ker}\, u \otimes N$ to N. Let $\beta = \Omega^3 h_m \circ \psi_0$; we need $\Omega^{-3}\beta$.

Define $\beta_1 : P_0 \otimes N \to P_0$ by $\beta_i(e_i \otimes e_i \otimes \zeta_i) = \delta_{im}(-(\bar{a}_m)^*(e_m \otimes e_m)\omega_m)$. Then β_1 is a lifting of $\Omega^{-1}\beta$ and hence $\Omega^{-1}\beta(\zeta_i) = \delta_{im}(-(\bar{a}_m)^* a_m \otimes \omega_m)$.

Define $\beta_2 : P_0 \to P_1$ by $\beta_2(e_i \otimes e_i) = \delta_{im}(-(\bar{a}_m)^* a_m \otimes e_m)$. Then β_2 is a lifting of $\Omega^{-2}\beta$ and so we have $\Omega^{-2}\beta(\sigma_i) = \delta_{im}((\bar{a}_m)^* a_m \otimes \bar{a}_m)$.

Define $\beta_3 : P_1 \to P_0$ by

$$e_{ia} \otimes e_{ta} \mapsto \begin{cases} (\bar{a}_m)^* a_m(e_{m+1} \otimes e_{m+1}) & \text{if } a = a_m \\ 0 & \text{otherwise.} \end{cases}$$

Then β_3 is a lifting of $\Omega^{-3}\beta$ and hence $\Omega^{-3}\beta$ takes x_a to $(\bar{a}_m)^* a_m$ if $a = a_m$ and to zero otherwise. One finds that $(\bar{a}_m)^* a_m = \omega_m$. So this is the map which takes $w \mapsto \omega_m$ and $\bar{w} \mapsto 0$. Now consider the basis element $g_m \in HH^1(\Lambda)$. It takes $w \mapsto 0$ and $\bar{w} \mapsto z_m \bar{\eta} = \omega_{m+1}$. Let $\gamma := \Omega^{-3}\beta - g_m$. By Subsection I.6.1 we see that the class of γ is zero in $HH^1(\Lambda)$. This completes the proof.

1.6.1. *Let $h_i : N \to \Lambda$. Then $\Omega^3 h_i$ can be taken as the map $N \otimes N \to N$ which takes $\zeta_j \otimes \zeta_{\nu(j)}$ to $\zeta_{\nu(i)}\omega_i$ if $j = \nu(i)$ and to zero otherwise.*

Proof. In Subsection 1.4 we found $\Omega^2 h_i$. Define $\gamma : P_0 \otimes N \to P_0$ by $\gamma(e_j \otimes e_j \otimes \zeta_j) = \delta_{ij}(e_i \otimes \omega_i)$. This is a lifting of $\Omega^2 h_i$. The statement follows, using $\omega_{\nu(i)} \otimes \omega_i = (-1)^{n-1}\zeta_{\nu(i)}\omega_i$ which is easy to check (with Section I.2.2).

1.7. $HH^1(\Lambda) \times HH^4(\Lambda) = HH^5(\Lambda)$. *If $g_0 \in HH^1(\Lambda)$ and $\psi_0 \in HH^4(\Lambda)$ then*

$$g_0 \cdot \psi_0 = (-1)^n \theta_0.$$

Proof. It suffices to prove the formula for the product since the elements generate the corresponding Hochschild cohomology groups. Let $\psi = \psi_0$; then the homomorphism $\hat{\psi} : P_1 \otimes N \to P_0$ given by

$$\hat{\psi} : (e_{ia} \otimes e_{ta}) \otimes \zeta_{ta} \mapsto \begin{cases} e_{ia} \otimes e_{ia}\psi(x_a \otimes \zeta_{ta}) & \text{if } a = a_m \text{ or } \bar{a}_m \\ 0 & \text{otherwise} \end{cases}$$

is a lifting for ψ. Thus $\Omega\psi : \operatorname{Ker}\delta \otimes N \to \operatorname{Ker} u$ can be taken as

$$\Omega\psi : \sigma_i \otimes \zeta_i \mapsto \begin{cases} x_{a_m}\psi\left(x_{\bar{a}_m} \otimes \zeta_m\right) & \text{if } i = m \\ x_{\bar{a}_m}\psi\left(x_{a_m} \otimes \zeta_{m+1}\right) & \text{if } i = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now let $g_0 \in HH^1(\Lambda)$. Then the composition $g_0 \circ \Omega\psi$ is the map

$$\sigma_i \otimes \zeta_i \mapsto \begin{cases} \bar{a}_m F & \text{if } i = m+1 \\ 0 & \text{otherwise;} \end{cases}$$

here we write $F = \psi(x_{a_m} \otimes \zeta_{m+1})$. In order to identify the class of this map as an element of $HH^5(\Lambda)$, we consider the Λ^e-homomorphism $\beta : \operatorname{Im} R \to N$ such that $\beta(\sigma_{m+1}) = \bar{a}_m F\zeta_{\nu(m+1)}$ and $\beta(\sigma_j) = 0$ otherwise.

$HH^5(\Lambda)$ is a quotient of $(\operatorname{Im} R, N)$. Noting that $F\zeta_{\nu(m+1)} \in e_m N e_{m+1}$, we get from Section I.8.3 that there is a map $\phi \in (\operatorname{Im} R, N)$ whose class in $HH^5(\Lambda)$ is zero which is given by

$$\phi(\sigma_i) = \begin{cases} \left(F\zeta_{\nu(m+1)}\right)\bar{a}_m & \text{if } i = m \\ \bar{a}_m\left(F\zeta_{\nu(m+1)}\right) & \text{if } i = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\beta - \phi$ is zero except at σ_m, and it takes σ_m to $-F\zeta_{\nu(m+1)}\bar{a}_m$. If n is odd then $F = \bar{a}_{m-1}$ and one finds that this value is equal to $-a_m\bar{a}_m\zeta_m$. If n is even then $F = e_m$ and the value is $a_m\zeta_{m+1}$. This gives the statement.

1.8. *If i is odd then $HH^i(\Lambda) \times HH^5(\Lambda) = 0$.*

From Subsection 1.7 and using associativity, $HH^1(\Lambda) \times HH^5(\Lambda) = (HH^1(\Lambda) \times HH^1(\Lambda)) \times HH^4(\Lambda)$. Thus, from Subsection 1.2, $HH^1(\Lambda) \times HH^5(\Lambda) = 0$. Similarly one proves the other parts.

2. ON $HH^1 \times HH^2$

2.1. We start with some technical preparation; this will be used in the following two sections. The first part is a tool to calculate images under maps $\operatorname{Ker} u \to \Lambda$. Such maps are defined on elements of the form $x_b = b \otimes e_{tb} - e_{ib} \otimes b$ for b an arrow. Suppose $p \in K\mathcal{Q}$ is a monomial of degree k, say $p = b_1 b_2 \cdots b_k$; then we set

$$x_p = x_{b_1} b_2 \cdots b_k + b_1 x_{b_2} b_3 \cdots b_k + \cdots + b_1 b_2 \cdots b_{k-1} x_{b_k}.$$

Then $x_p \in \operatorname{Ker} u$. Note that this depends on the monomial.

COROLLARY. *Suppose $g \in HH^1(\Lambda)$, where $g(w) = 0$ and $g(\overline{w}) = z_g \overline{\eta}$ with $z_g \in Z(\Lambda)$. Then $g(x_p) = tz_g p$ where t is the number of bar letters of the monomial p.*

2.2. We will choose pre-images under δ and also identify elements of Ker δ in terms of the generators of Im R. Given x_p as in Subsection 2.1, we fix a canonical pre-image under δ as

$$\delta^{-1}(x_p) = \sum_{i=1}^{k} b_1 \cdots b_{i-1} \tilde{w} b_{i+1} \cdots b_k,$$

where $\tilde{w} = w$ if b_i is a non-bar letter and $= \overline{w}$ otherwise. Then for β, γ monomials such that $\beta\gamma$ is defined we have $\delta^{-1}(x_{\beta\gamma}) = \beta\delta^{-1}(x_\gamma) + \delta^{-1}(x_\beta)\gamma$. Moreover, if $\sigma_i = R(e_i \otimes e_i)$ then $\sigma_i = \delta^{-1}(x_{\overline{a}_{i-1}a_{i-1}}) + \delta^{-1}(x_{a_i\overline{a}_i})$. More generally:

LEMMA. *Suppose ρ_1, ρ_2 are monomials such that $\rho_1 \overline{a}_{i-1} a_{i-1} \rho_2$ is defined. Then*

$$\delta^{-1}(x_{\rho_1\overline{a}_{i-1}a_{i-1}\rho_2}) + \delta^{-1}(x_{\rho_1 a_i \overline{a}_i \rho_2}) = \rho_1 \sigma_i \rho_2.$$

2.2.1. Suppose $p = b_1 \cdots b_u$ is a monomial. Define

$$y(p) := \sum_{k=0}^{u} (-1)^k (b_1 \cdots b_k) \sigma_{tb_k} (b_{k+1} \cdots b_u)$$

(where $tb_0 := ib_1$). By applying the above Lemma repeatedly one obtains

LEMMA. *Let $\rho = (a_i a_{i+1} \cdots a_{i+u-1})$ of degree u. Then we have*

$$\delta^{-1}(x_{\overline{a}_{i-1}a_{i-1}\rho}) + (-1)^u \delta^{-1}(x_{\rho a_{i+u}\overline{a}_{i+u}}) = y(\rho).$$

Similarly, if $\overline{\rho} = \overline{a}_{i+u-1} \cdots \overline{a}_i$ has degree u with only bar letters then

$$\delta^{-1}(x_{a_{i+u}\overline{a}_{i+u}\overline{\rho}}) + (-1)^u \delta^{-1}(x_{\overline{\rho}\overline{a}_{i-1}a_{i-1}}) = y(\overline{\rho}).$$

2.3. We shall now determine Ωf for $f \in HH^2(\Lambda)$.

(1) We work with the basis for $HH^2(\Lambda)$ as given in Subsection 1.1. Fix i with $0 \leq i \leq n - m - 2$. The homomorphism $\hat{f}_i : P_0 \to P_0$ defined by

$$\hat{f}_i(e_k \otimes e_k) = \begin{cases} e_k \otimes e_k, & k = i \\ (-1)^n (e_k \otimes e_k), & k = \nu(i) \\ 0, & \text{otherwise} \end{cases}$$

is a lifting of f_i, and we take for Ωf_i the restriction of $\hat{f}_i$ to N.

(2) Fix j with $0 \le j \le n-1$; then one finds

$$(\Omega f_i)(\zeta_j) = \hat{f}_i(\zeta_j) = (-1)^{i-j}\left(\sum_{v \in e_j B e_i} v \otimes v^* - \sum_{v \in e_j B e_{\nu(i)}} v \otimes v^* \right).$$

(3) Suppose M is a $\Lambda - \Lambda$ bimodule and $x \in e_j M e_{\nu(j)}$ where $j \le \nu(j)$, and let $0 \le s \le j$. Define a "trace" by the following formula.

$$\mathrm{Tr}^s(x) := \sum_{u=0}^{s} (-1)^{u\nu(s)} L_u x R_{s-u},$$

where L_u is the left-normalized monomial of degree $2u$ in $e_j \Lambda e_j$, and R_{s-u} is the right-normalized monomial of degree $2(s-u)$ in $e_{\nu(j)} \Lambda e_{\nu(j)}$. The starting vertex and ending vertex of L_u, R_{i-u} will always be clear from the context.

2.4. We fix now the following notation for monomials $(0 \le i \le n - m - 2)$.

(a) Suppose $0 \le j < i$; then $\nu(i) < \nu(j)$. Let $p \in e_j \Lambda e_i$, $b \in e_i \Lambda e_{\nu(i)}$, and $q \in e_{\nu(i)} \Lambda e_{\nu(j)}$ be the monomials of smallest degree. Then p, b, q have no bar letters.

(b) Suppose now $i \le j \le \nu(i)$; then $\nu(j) \le \nu(i)$ and $i \le \nu(j)$. Let $q \in e_j \Lambda e_{\nu(i)}$, $p \in e_i \Lambda e_{\nu(j)}$ and $\bar{p}_1 \in e_j \Lambda e_i$, $\bar{q}_1 \in e_{\nu(i)} \Lambda e_{\nu(j)}$ be the monomials of smallest degree. Then p, q have no bar letters, and $\bar{p}_1, \bar{q}_1$ have only bar letters.

(c) The remaining case occurs when $\nu(i) < j$. Then let $\bar{p} \in e_j \Lambda e_{\nu(i)}$, $\bar{b} \in e_{\nu(i)} \Lambda e_i$, and $\bar{q} \in e_i \Lambda e_{\nu(j)}$ be the monomials of smallest degree. Then $\bar{p}, \bar{b}, \bar{q}$ have only bar letters.

LEMMA. *With the notation as above, we have*

$$(\Omega f_i)(\zeta_j) = \begin{cases} (-1)^{i-j} \mathrm{Tr}^j(p \otimes bq - pb \otimes q), \\ \qquad 0 \le j < i \\ \epsilon_{j,n} (-1)^{\nu(i)(i-j)} \mathrm{Tr}^i\left(\bar{p}_1 \otimes p - (-1)^{n(j-i)} q \otimes \bar{q}_1\right), \\ \qquad i \le j \le \nu(i) \\ \epsilon_{j,n} (-1)^{\nu(j)(n-1)+i-j} \mathrm{Tr}^{\nu(j)}\left(\bar{p}\bar{b} \otimes \bar{q} - \bar{p} \otimes \bar{b}\bar{q}\right), \\ \qquad \nu(i) < j \le n-1, \end{cases}$$

where $\epsilon_{j,n} = (-1)^{j-m}$ *if* n *is odd and* $j \ge m$; *otherwise* $\epsilon_{j,n} = 1$.

Proof. Let $\tilde{\omega}_j$ be the right-normalized monomial in $\operatorname{soc}(e_j\Lambda)$; then we have for v in the basis that

$$\omega_j = vv^* = \begin{cases} (-1)^{j-m}\tilde{\omega}_j, & n \text{ odd}, j \geq m \\ \tilde{\omega}_j, & \text{otherwise}. \end{cases}$$

This accounts for the factor $\epsilon_{j,n}$. For the case $0 \leq j < i$, it suffices to show that $(-1)^{u\nu(j)} L_u pbqR_{j-u} = \tilde{\omega}_j$ and this is straightforward.

Now consider the case when $i \leq j \leq \nu(i)$. First, $\bar{p}_1 p = (-1)^{n(j-i)} q\bar{q}_1$. So we need

$$(-1)^{\nu(i)(i-j)+u\nu(i)} L_u \bar{p}_1 pR_{i-u} = (-1)^{i-j}\tilde{\omega}_j.$$

One finds that $L_u \bar{p}_1 pR_{i-u} = (-1)^{\alpha}\tilde{\omega}_j$ where

$$a \equiv (j-i)(\nu(j) - u) + u\nu(j) \equiv u\nu(i) + \nu(j)(j-i) \pmod 2$$

and we are done since $(\nu(i) + \nu(j))(i-j) \equiv i - j \pmod 2$. Similarly one deals with the last case.

2.5. PROPOSITION. *Let $g \in HH^1(\Lambda)$ with $g(w) = 0$ and $g(\bar{w}) = z_g\bar{\eta}$ for $z_g \in Z(\Lambda)$. Then*

$$(g \circ \Omega f_i)(\zeta_i) = \begin{cases} 0, & 0 \leq j < i \\ (-1)^{j-i}(j-i)(i+1)z_g\omega_j, & i \leq j \leq \nu(i) \\ (-1)^{j-i}(\nu(i) - i)(\nu(j) + 1)z_g\omega_j, & \nu(i) < j. \end{cases}$$

In particular if $z_g \in J$ then $g \circ \Omega f_i = 0$.

2.5.1. Hence in $HH^3(\Lambda)$ we have

$$g_0 \cdot f_i = (-1)^{i+1}\Bigg[(n-2i-1)\Bigg(\sum_{j=0}^{i-1}(-1)^j(j+1)[h_j]\Bigg) + (i+1)\Bigg(\sum_{j=i}^{n-m-2}(-1)^j(n-2j-1)[h_j]\Bigg)\Bigg].$$

Proof of 2.5. Since q is a homomorphism, it commutes with the trace.

(1) Assume $0 \leq j < i$; then we get from Subsections 2.4 and 2.1 that $g \circ \Omega f_i(\zeta_j) = 0$ since $p \otimes bq - pb \otimes q = -px_b q$.

(2) Now let $i \leq j < \nu(i)$; then $\bar{p}_1 \otimes p - (-1)^{n(j-i)}(q \otimes \bar{q}_1) = -\bar{p}_1 x_p + (-1)^{n(j-i)} qx_{\bar{q}_1}$. It follows that

$$(g \circ \Omega f_i)(\zeta_j) = \epsilon_{n,j}(-1)^{\nu(i)(j-i)}(-1)^{n(j-i)}\operatorname{Tr}^i\big(g(qx_{\bar{q}_1})\big).$$

From Subsection 2.1 we know that $g(qx_{\bar{q}_1}) = (j-i)q\bar{q}_1 z_g$ where $j-i$ is the length of $\bar{q}_1$; and one finds $\mathrm{Tr}^i(q\bar{q}_1) = (-1)^{i(j-i)}(i+1)\tilde{\omega}_j$. This gives the stated formula.

(3) Let $\nu(i) < j \le n-1$; then $g(\bar{p}x_{\bar{b}}\bar{q}) = (\nu(i)-i)\bar{p}\bar{b}\bar{q}$ and the claim follows from

$$\mathrm{Tr}^{\nu(j)}\big(\bar{p}\bar{b}\bar{q}\big) = (-1)^{\nu(j)(n-1)}(\nu(j)+1)\,\tilde{\omega}_j.$$

2.6. Lemma. *For $0 \le i < n-m-2$, define*

$$\tilde{f}_i := \begin{cases} f_i + (-1)^{m-i+1}(\nu(i)-i)f_m, & n \text{ even} \\ f_i + (-1)^{m-i}(m-i)f_{m-1}, & n \text{ odd}. \end{cases}$$

Then in $HH^3(\Lambda)$ we have $g_0 \circ \tilde{f}_i = \sum_{j=i}^{n-m-2}(-1)^{j-i}(j-i)(n+1)[h_j]$. Moreover we have

$$g_0 \cdot f_{n-m-2} = \begin{cases} (-1)^{m+1}\sum_{j=0}^{m}(-1)^j(j+1)\big[h_j\big], & n \text{ even} \\ (-1)^m 2\sum_{j=0}^{m-1}(-1)^j(j+1)\big[h_j\big], & n \text{ odd}. \end{cases}$$

This follows from Subsection 2.5.1 by a straightforward calculation.

2.6.1. Proposition. (a) *Assume that* $\mathrm{char}(K)$ *does not divide* $n+1$. *Then* $HH^1 \times HH^2 = HH^3$.

(b) *Assume that* $\mathrm{char}(K)$ *divides* $n+1$ *and that* $\mathrm{char}(K) \ne 2$ *in case n is odd. Then* $HH^1 \times HH^2$ *is* 1*-dimensional, spanned by* $g_0 \cdot f_{n-m-2}$.

(c) *If* $\mathrm{char}(K) = 2$ *and n is odd then* $HH^1 \times HH^2 = 0$.

Proof. The elements $\tilde{f}_i$, $0 \le i < n-m-2$, and f_{n-m-2} form a basis for $HH^2(\Lambda)$. Suppose first that $n+1$ is non-zero in K. Let n be even. Then $g_0 \cdot \tilde{f}_{m-1} = -(n+1)h_m$ in $HH^3(\Lambda)$, so h_m belongs to the span of the products. For $1 \le k$ we have

$$g_0 \cdot \tilde{f}_{m-k} \in -(n+1)h_{m-k+1} + \mathrm{sp}\{h_m, \dots, h_{m-k+2}\}$$

and by induction we get that the space spanned by the products contains h_i for $1 \le i \le m$. By considering $g_0 \cdot f_m$ it follows that also h_0 belongs to the space. Similarly the claim follows if n is odd.

Now assume that $n+1$ is zero in K. Then in $HH^3(\Lambda)$, we have $g_0 \cdot \tilde{f}_i = 0$ for $0 \le i < n-m-2$, and $HH^1 \times HH^2$ is spanned by $g_0 \cdot f_{n-m-2}$. This is non-zero except with n is odd and the characteristic of the field is 2.

3. ON $HH^2 \times HH^2$

3.1. The following may be deduced directly from Section I.9.4.1.

LEMMA. *Suppose* $f, f_i \in \mathrm{Hom}(\Omega^2\Lambda, \Lambda)$. *Then the class* $[f][f_i]$ *in* $HH^4(\Lambda)$ *is completely determined by the values* $f \circ F_i(x_{a_m} \otimes \zeta_{m+1})$ *and* $f \circ F'(x_{\bar{a}_m} \otimes \zeta_m)$ *where* $F_i, F_i' : P_0 \otimes N \to P_1$ *satisfy* $\delta \circ F_1 = \delta \circ F_i' = \Omega f_i \circ (u \otimes 1)$.

3.2. We shall now define the homomorphisms $F_i, F_i' : P_0 \otimes N \to P_1$ as in Subsection 3.1 which lift the maps Ωf_i as determined in Subsection 2.4. With the notation of Subsection 2.4, we have that in case $i \le j \le \nu(i)$

$$\bar{p}_1 \otimes p - (-1)^{n(j-i)} q \otimes \bar{q}_1 = -\bar{p}_1 x_p + (-1)^{n(j-i)} q x_{\bar{q}_1}$$
$$= x_{\bar{p}_1} p - (-1)^{n(j-i)} x_q \bar{q}_1.$$

In the first case, $p \otimes bq - pb \otimes q = -p x_b q$, and a similar formula holds in the last case, Now we define F_i by

$$F_i(e_j \otimes e_j \otimes \zeta_i) = \begin{cases} (-1)^{i-j}\mathrm{Tr}^j\big(-p\delta^{-1}(x_b)q\big), \\ \qquad 0 \le j < i \\ \epsilon_{j,n}(-1)^{\nu(i)(i-j)}\mathrm{Tr}^i\Big[-\bar{p}_1\delta^{-1}(x_p) + (-1)^{n(j-i)} q\delta^{-1}(x_{\bar{q}_1})\Big], \\ \qquad i \le j \le \nu(i) \\ \epsilon_{j,n}(-1)^{\nu(j)(n-1)+i-j}\mathrm{Tr}^{\nu(j)}\big(\bar{p}\delta^{-1}(x_b)\bar{q}\big), \\ \qquad \text{otherwise,} \end{cases}$$

where pre-images under δ are as in Subsection 2.2. Moreover, we define F_i' similarly; namely

$$F_i'(e_j \otimes e_j \otimes \zeta_j) = \begin{cases} e_{j,n}(-1)^{\nu(i)(i-j)}\mathrm{Tr}^i\Big[\delta^{-1}(x_{\bar{p}_1})p - (-1)^{n(j-i)}\delta^{-1}(x_q)\bar{q}_1\Big], \\ \qquad i \le j \le \nu(i) \\ F_i(e_j \otimes e_j \otimes \zeta_j), \\ \qquad \text{otherwise.} \end{cases}$$

3.3. Assume first that n is even, then $a_m \zeta_{m+1} = -\zeta_m \bar{a}_m$ and $\bar{a}_m \zeta_m = -\zeta_{m+1} a_m$. We fix i with $0 \le i \le m$. Here $\epsilon_{j,n} = 1$, so we ignore it. Then

$$F_i(x_{a_m} \otimes \zeta_{m+1}) = a_m F_i(e_{m+1} \otimes e_{m+1} \otimes \zeta_{m+1}) + F_i(e_m \otimes e_m \otimes \zeta_m)\bar{a}_m$$

$$F_i'(x_{\bar{a}_m} \otimes \zeta_m) = \bar{a}_m F_i'(e_m \otimes e_m \otimes \zeta_m) + F_i'(e_{m+1} \otimes e_{m+1} \otimes \zeta_{m+1}) a_m.$$

We need the monomials as in Subsection 2.4 for $j = m, m+1$ explicitly. First, we always have $i \le j \le \nu(i)$. Let $p \in e_i \Lambda e_{m+1}, q \in e_m \Lambda e_{\nu(i)}$ be the monomials of smallest degree. Then $\bar{p} \in e_{m+1}\Lambda e_i, \bar{q} \in e_{\nu(i)}\Lambda e_m$ are the monomials of smallest degree. Moreover, let $q_1 \in e_{m+1}\Lambda e_{\nu(i)}, p_1 \in e_i \Lambda e_m$ be the monomials of smallest degree; then $\bar{q}_1 \in e_{\nu(i)}\Lambda e_{m+1}, \bar{p}_1 \in e_m \Lambda e_i$ are the monomials of smallest degree. Then we get from Subsection 3.2

$$F_i\big(x_{a_m} \otimes \zeta_{m+1}\big) = (-1)^{\nu(i)(i-m-1)} a_m \mathrm{Tr}^i\big[-\bar{p}\delta^{-1}(x_{p_1}) + q_1\delta^{-1}(x_{\bar{q}})\big]$$
$$+(-1)^{\nu(i)(i-m)}\mathrm{Tr}^i\big[-\bar{p}_1\delta^{-1}(x_p) + q\delta^{-1}(x_{\bar{q}_1})\big]\bar{a}_m \quad (1)$$

$$F_i'\big(x_{\bar{a}_m} \otimes \zeta_m\big) = (-1)^{\nu(i)(i-m)}\Big[\bar{a}_m \mathrm{Tr}^i\big[\delta^{-1}(x_{\bar{p}_1})p - \delta^{-1}(x_q)\bar{q}_1\big]$$
$$+(-1)^{\nu(i)}\mathrm{Tr}^i\big[\delta^{-1}(x_{\bar{p}})p_1 - \delta^{-1}(x_{q_1})\bar{q}\big]a_m\Big]. \quad (2)$$

3.4. Proposition. *Assume n is even, and fix $0 \le i \le m$.*

(a) *For $0 \le k \le i$ define $\rho_k = a_k \cdots a_{m-1}$, of length $m-k$. Then*

$$F_i\big(x_{a_m} \otimes \zeta_{m+1}\big) = (-1)^{i+1}\sum_{k=0}^{i}(-1)^{k(m+1)}\bar{\rho}_k y(\rho_k)R_k.$$

(b) *For $0 \le k \le i$ define $\rho_k' = a_{m+1} \cdots a_{m+(m-k)}$, of length $m-k$. Then*

$$F_i'\big(x_{\bar{a}_m} \otimes \zeta_m\big) = (-1)^{m+\nu(i)}\sum_{k=0}^{i}(-1)^{km}L_k y(\rho_k')\bar{\rho}_k'.$$

Proof. (a) This is a lengthy calculation whose main difficulty consists of keeping track of the signs. As the first step, we simplify the expressions from Subsection 3.3(1), (2).

(i) Let $\beta = \bar{p}_1\delta^{-1}(x_{p_1})$. Then $F_i(x_{a_m} \otimes \zeta_{m+1})$ *is equal to*

$$(-1)^{\nu(i)(i-m)}\big[-\beta a_m R_i \bar{a}_m - \mathrm{Tr}^i(\bar{p}_1 p_1 w)\bar{a}_m + (-1)^{\nu(i)}a_m \mathrm{Tr}^i(q_1\bar{q}_1\bar{w})\big].$$

(ii) Let $\beta' = \delta^{-1}(x_{q_1})\bar{q}_1$. Then $F_i'(x_{\bar{a}_m} \otimes \zeta_m)$ is equal to

$$(-1)^{\nu(i)(i-m)}\big[-\bar{a}_m L_i a_m \beta' - \bar{a}_m \mathrm{Tr}^i(wq_1\bar{q}_1) + (-1)^{\nu(i)}\mathrm{Tr}^i(\overline{w}\bar{p}_1 p_1)a_m\big].$$

To prove (i), take out the factor of $(-1)^{\nu(i)(i-m)}$.

(I) We have $p = p_1 a_m$, consequently $\delta^{-1}(x_p) = \delta^{-1}(x_{p_1})a_m + p_1 w$ and $\bar{p} = \bar{a}_m \bar{p}_1$. We combine the p-terms in Subsection 3.3(1) and get

$$(-1)^{\nu(i)+1}a_m \mathrm{Tr}^i_{m+1}(\bar{a}_m\beta) - \mathrm{Tr}^i_m(\beta a_m)\bar{a}_m - \mathrm{Tr}^i(\bar{p}_1 p_1 w)\bar{a}_m. \quad (*)$$

Now, $a_m L_u(\bar{a}_m\beta)R_{i-u} = -L_{u+1}\beta a_m R_{i-u-1}\bar{a}_m$. Moreover if $u = i$ then $L_{i+1}\bar{p}_1 = 0$, by the relations, and we deduce that $a_m \mathrm{Tr}^i(\bar{a}_m\beta)$ is equal to

$$-\left(\sum_{u=0}^{i-1}(-1)^{u\nu(i)}L_{u+1}(\beta a_m)R_{i-(u+1)}\right)\bar{a}_m$$
$$= (-1)^{\nu(i)+1}\left(\sum_{k=1}^{i}(-1)^{k\nu(i)}L_k(\beta a_m)R_{i-k}\right)\bar{a}_m.$$

Hence this cancels against most of $\mathrm{Tr}^i(\beta a_m)\bar{a}_m$, leaving only the term with $k = 0$. Therefore $(*)$ is equal to $-\beta a_m R_i\bar{a}_m - \mathrm{Tr}^i(\bar{p}_1 p_1 w)\bar{a}_m$.

(II) Now we combine the remaining terms of the two traces in Subsection 3.3(1). We have $q = a_m q_1$, so $\bar{q} = \bar{q}_1\bar{a}_m$ and $\delta^{-1}(x_{\bar{q}}) = \delta^{-1}(x_{\bar{q}_1})\bar{a}_m + \bar{q}_1\bar{w}$. Set $\alpha = q_1\delta^{-1}(x_{\bar{q}_1})$, then we get

$$(-1)^{\nu(i)}a_m\mathrm{Tr}^i(\alpha\bar{a}_m) + (-1)^{\nu(i)}a_m\mathrm{Tr}^i(q_1\bar{q}_1\bar{w}) + \mathrm{Tr}^i(a_m\alpha)\bar{a}_m. \quad (**)$$

We claim that the first and the last trace cancel out. We rewrite the first trace. Since $a_m L_u = (-1)^u L_u a_m$, $\bar{a}_m R_{i-u} = (-1)^{i-u}R_{i-u}\bar{a}_m$ we have $a_m\mathrm{Tr}^i(\alpha\bar{a}_m) = (-1)^i\mathrm{Tr}^i(a_m\alpha)\bar{a}_m$ and also $i \equiv \nu(i) \pmod 2$ since n is even, and the claim follows. So $(**)$ is equal to $(-1)^{\nu(i)}a_m\mathrm{Tr}^i(q_1\bar{q}_1\bar{w})$ and we are done. The proof of part (ii) is similar; we omit details.

The second step is to show that the stated formula is the same as (i), (ii), respectively. Take out the factor $(-1)^{i+1}$. Using Subsection 2.2.1 one sees that the stated formula can be written as

$$\sum_{k=0}^{i}(-1)^{k(m+1)}\Big[\bar{\rho}_{k-1}\delta^{-1}(x_{\rho_{k-1}}) + \bar{\rho}_k\bar{w}\rho_{k-1}$$
$$+(-1)^{m-k}\big[\bar{\rho}_k\delta^{-1}(x_{\rho_k})a_m\bar{a}_m + \bar{\rho}_k\rho_k w\bar{a}_m + \bar{\rho}_k\rho_k a_m\bar{w}\big]\Big]R_k.$$

We split the sum into three parts. Define Σ_1 to be the sum of all terms where δ^{-1} occurs. Moreover, let Σ_2 be the sum of all terms with w as a factor, and let Σ_3 be the sum of all terms with $\bar{w}$ as a factor.

(1) *We claim that* $\Sigma_1 = (-1)^{m\nu(i)+i}\beta a_m R_i\bar{a}_m$ *where* $\beta = p_1\delta^{-1}(x_{p_1})$. Since $a_m\bar{a}_m R_k = (-1)^k R_{k+1}$, one gets

$$\Sigma_1 = \sum_{k=0}^{i}(-1)^{k(m+1)}\big[\bar{\rho}_{k-1}\delta^{-1}(x_{\rho_{k-1}})R_k + (-1)^m\bar{\rho}_k\delta^{-1}(x_{\rho_k})R_{k+1}\big].$$

Hence the second summand of the kth term cancels against the first summand of the $(k+1)$st term. The first term for $k = 0$ is zero, and this leaves us with the last term for $k = i$. Now note that $\rho_i = p_1$, and the statement follows.

(2) *Next we claim that* $\Sigma_2 = (-1)^{m\nu(i)+i}\mathrm{Tr}^i(\bar{p}_1 p_1 w)\bar{a}_m$. By definition,

$$\Sigma_2 = \sum_{k=0}^{i} (-1)^{k(m+1)}(-1)^{m-k}\bar{\rho}_k \rho_k w \bar{a}_m R_k.$$

We have that $\bar{\rho}_k \rho_k = (-1)^{(i-k)(m-i)} L_{i-k}\bar{p}_1 p_1$. Moreover $\bar{a}_m R_k = (-1)^k R_k \bar{a}_m$; substituting everything gives

$$\Sigma_2 = \sum_{k=0}^{i} (-1)^{m(k+1)+(i-k)(m-i)+k} L_{i-k}(\bar{p}_1 p_1 w) R_k \bar{a}_m.$$

Now set $u = i - k$; the statement follows.

(3) *We claim that* $\Sigma_3 = (-1)^{m\nu(i)} a_m \mathrm{Tr}^i(q_1 \bar{q}_1 \bar{w})$. First, observe that $\rho_{k-1} R_k = 0$ since ρ_{k-1} starts at vertex $k-1$ and R_k has k bar letters. Therefore

$$\Sigma_3 = \sum_{k=0}^{i} (-1)^{m(k+1)}\bar{\rho}_k \rho_k a_m \bar{w} R_k.$$

Next we claim that $\bar{\rho}_k \rho_k a_m = (-1)^{(m-k)(m-i+1)} a_m L_{i-k} q_1 \bar{q}_1$. Both monomials start at m and end at $m+1$, so they differ only by a sign, and to get this make $a_m L_{i-k} q_1 \bar{q}_1$ left-normalized. Substitute and change variables; the statement follows. Combining (1) to (3) gives the result (a).

Part (b) is similar; we omit details.

3.5. Now assume that n is odd; then $\nu(m) = m$ and moreover $a_m \zeta_{m+1} = \zeta_m \bar{a}_{m-1}$ and $\bar{a}_m \zeta_m = -\zeta_{m+1} a_{m-1}$. Fix i with $0 \le i \le m-1$; then

$$F_i(x_{a_m} \otimes \zeta_{m+1}) = a_m F_i(e_{m+1} \otimes e_{m+1} \otimes \zeta_{m+1}) - F_i(e_m \otimes e_m \otimes \zeta_m)\bar{a}_{m-1}$$

$$F_i'(x_{\bar{a}_m} \otimes \zeta_m) = \bar{a}_m F_i'(e_m \otimes e_m \otimes \zeta_m) + F_i'(e_{m+1} \otimes e_{m+1} \otimes \zeta_{m+1}) a_{m-1}.$$

We need the monomials as in Subsection 2.4 for $j = m, m+1$. First, we always have $i \le j \le \nu(i)$. Let $\rho \in e_i \Lambda e_m, q \in e_m \Lambda e_{\nu(i)}$ be the monomials of smallest degree; then $\bar{p} \in e_m \Lambda e_i$ and $\bar{q} \in e_{\nu(i)} \Lambda e_m$ are the monomials of smallest degree. Moreover, let $p_1 \in e_i \Lambda e_{m-1}, q_1 \in e_{m+1}\Lambda e_{\nu(i)}$ and $\bar{p}_2 \in e_{m+1}\Lambda e_i, \bar{q}_2 \in e_{\nu(i)}\Lambda e_{m-1}$ be the monomials of smallest degree. Then

$$\begin{aligned} F_i(x_{a_m} \otimes \zeta_{m+1}) = (-1)^{\nu(i)(i-m-1)+1} a_m & \\ \times \mathrm{Tr}^i\Big[\Big(-\bar{p}_2 \delta^{-1}(x_{p_1}) + (-1)^{i-m-1} q_1 \delta^{-1}(x_{\bar{q}_2})\Big)\Big] & \\ -(-1)^{\nu(i)(i-m)} & \\ \times \mathrm{Tr}^i\Big[\Big(-\bar{p}\delta^{-1}(x_p) + (-1)^{i-m} q \delta^{-1}(x_{\bar{q}})\Big)\Big]\bar{a}_{m-1}. & \end{aligned}$$

Similarly one gets the corresponding images under F_i', and

$$F_i'\left(x_{\bar{a}_m} \otimes \zeta_m\right) = (-1)^{\nu(i)(i-m)} \bar{a}_m \mathrm{Tr}^i\left[\left(\delta^{-1}(x_{\bar{p}})p - (-1)^{i-m}\delta^{-1}(x_q)\bar{q}\right)\right] - (-1)^{\nu(i)(i-m-1)} \times \mathrm{Tr}^i\left[\left(\delta^{-1}(x_{\bar{p}_2})p_1 - (-1)^{i-m-1}\delta^{-1}\bar{q}_2\right)\right] a_{m-1}.$$

3.6. PROPOSITION. *Assume n is odd, and fix* $0 \le i \le m-1$.

(a) *For* $0 \le k \le i$ *define* $\rho_k = a_k a_{k+1} \cdots a_{m-2}$, *of length* $m-k-1$. *Then*

$$F_i\left(x_{a_m} \otimes \zeta_{m+1}\right) = (-1)^{i+m+1} \sum_{k=0}^{i} (-1)^{mk} \bar{a}_{m-1} \bar{\rho}_k y(\rho_k) R_k.$$

(b) *For* $0 \le k \le i$ *define* $\rho_k' = a_{m+1} \cdots a_{m+(m-k-1)}$, *of length* $m - k - 1$. *Then*

$$F_i'\left(x_{\bar{a}_m} \otimes \zeta_m\right) = (-1)^{i+m+1} \sum_{k=0}^{i} (-1)^{k(m+1)} L_k y(\rho_k') \bar{\rho}_k' \bar{a}_m.$$

The proof is similar to that of Subsection 3.4. First, to simplify the expression in Subsection 3.5 one shows

(i) Let $\beta = \bar{p}\delta^{-1}(x_{p_1})$; then $F_i(x_{a_m} \otimes \zeta_{m+1})$ is equal to

$$(-1)^{\nu(i)(i-m)}\left[\beta a_{m-1} R_i \bar{a}_{m-1} + \mathrm{Tr}^i(\bar{p}p_1 w)\bar{a}_{m-1} + (-1)^m a_m \mathrm{Tr}^i\left(q_1 \overline{qw}\right)\right].$$

(ii) Let $\beta' = \delta^{-1}(x_{q_1})\bar{q}_1$; then $F_i'(x_{\bar{a}_m} \otimes \zeta_m)$ is equal to

$$(-1)^{\nu(i)(i-m)}\left[(-1)^{m-1}\bar{a}_m L_i a_m(\beta'\bar{a}_m) + (-1)^{\nu(i)+1}\mathrm{Tr}^i\left(\overline{wp}p_1\right)a_{m-1} + (-1)^{i-m+1}\bar{a}_m \mathrm{Tr}^i\left(wq_1\bar{q}_1\bar{a}_m\right)\right].$$

Then one shows that the stated formula is the same. We omit the details.

3.7. PROPOSITION. *Let* $f_i, f_t \in HH^2(\Lambda)$.

(a) *If n is even, let* $\gamma \in e_m \Lambda e_m$ *and* $\gamma' \in e_{m+1}\Lambda e_{m+1}$ *be the right-normalized monomials of length* $2m$, *then*

$$f_t \circ F_i\left(x_{a_m} \otimes \zeta_{m+1}\right) = (-1)^{i+t+1+m}[\min(i,t)+1]\gamma,$$

$$f_t \circ F_i'\left(x_{\bar{a}_m} \otimes \zeta_m\right) = (-1)^{i+t+1}[\min(i,t)+1]\gamma'.$$

(b) *If n is odd, let $\gamma \in e_m \Lambda e_{m-1}$ and $\gamma' \in e_{m+1} \Lambda e_m$ be right-normalized monomials of length $2m-1$, then*

$$f_t \circ F_i(x_{a_m} \otimes \zeta_{m+1}) = (-1)^{m+i+t+1}[\min(i,t)+1]\gamma,$$

$$f_t \circ F_i'(x_{\bar{a}_m} \otimes \zeta_m) = (-1)^{i+t+1}[\min(i,t)+1]\gamma'.$$

Proof. Assume first that n is even and fix $0 \le i \le m$. Since f_t is a homomorphism it suffices to find $f_t(y(\rho_k))$ and $f_t(y(\rho_k'))$. We get from Subsection 2.2.1 that

$$f_t(y(\rho_k)) = \begin{cases} (-1)^{t-k}\rho_k, & k \le t \le m \\ 0, & \text{otherwise}. \end{cases}$$

Similarly, using the fac that ρ_k' is a path from $m+1$ to $\nu(k) = n-k-1$ and recalling that $f_t(\sigma_{\nu(t)}) = e_{\nu(t)}$ we have

$$f_t(y(\rho_k')) = \begin{cases} (-1)^{\nu(t)-m-1}\rho_k', & m+1 \le \nu(t) \le \nu(k) \\ 0, & \text{otherwise}. \end{cases}$$

It follows that $f_t \circ F_i$ takes $x_{a_m} \otimes \zeta_{m+1}$ to

$$(-1)^{i+1}\sum_{k=0}^{\min(i,t)}(-1)^{k(m+1)}\bar{\rho}_k\,\rho_k R_k(-1)^{t-k}.$$

Moreover, we have $\bar{\rho}_k\,\rho_k R_k = (-1)^{m(m-k)}\gamma$. Substituting this gives the first part of (a), and the second part is similar.

Part (b) is similar (but note that here $f_t(\sigma_{\nu(t)}) = -1$).

3.8. COROLLARY. *For $0 \le i, t \le n-m-2$ we have*

$$f_t \cdot f_t = (-1)^{i+t+1}(-1)^{nm}(\min(i,t)+1)\psi_{n-m-2}.$$

This is a direct translation of Subsection 3.7 into the terminology of Section I.9.

4. PRODUCTS INVOLVING HH^4

In this section we find $HH^2(\Lambda) \times HH^4(\Lambda)$ and $HH^4(\Lambda) \times HH^4(\Lambda)$. The remaining products in $HH^*(\Lambda)$, viz. $HH^2(\Lambda) \times HH^5(\Lambda)$ and $HH^4(\Lambda) \times HH^4(\Lambda)$, are then determined using Subsection 1.7. We start with $HH^2(\Lambda) \times HH^4(\Lambda)$.

4.1. Let $\psi = \psi_0 \in HH^4(\Lambda)$; then $\Omega\psi$ is given in Subsection 1.7. Define the map $\tilde{\psi} : P_0 \otimes N \to P_1$ by

$$\tilde{\psi} : (e_i \otimes e_i) \otimes \zeta_i \mapsto \begin{cases} e_m \otimes e_{m+1}\psi(x_{\bar{a}_m} \otimes \zeta_m) & \text{if } i = m \\ e_{m+1} \otimes e_m \psi(x_{a_m} \otimes \zeta_{m+1}) & \text{if } i = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{\psi}$ is a lifting for $\Omega\psi$. Thus

$$\begin{aligned} \Omega^2\psi(\zeta_j \otimes \zeta_{\nu(j)}) &= \sum_{v \in e_j B e_m} (-1)^{\deg v} v(e_m \otimes e_{m+1})\psi(x_{\bar{a}_m} \otimes \zeta_m)\tilde{\nu}(v^*) \\ &\quad + \sum_{v \in e_j B e_{m+1}} (-1)^{\deg v} v(e_{m+1} \otimes e_m)\psi(x_{a_m} \otimes \zeta_{m+1})\tilde{\nu}(v^*). \end{aligned}$$

For $i = 0, \dots, n-m-2$, let f_i be the basis element of $HH^2(\Lambda)$ as in Subsection 1.1.

LEMMA. $f_i \circ \Omega^2\psi(\zeta_j \otimes \zeta_{\nu(j)}) = 0$ *whenever* $j < m$ *or* $j > \nu(m)$.

Proof. The image of $\Omega^2\psi$ is in Ker δ and so $\Omega^2\psi(\zeta_j \otimes \zeta_{\nu(j)})$ is a sum of elements of the form $\lambda_k \sigma_k \lambda'_k$ with $\deg(\lambda_k \lambda'_k) = 2m$. Thus $f_i \circ \Omega^2\psi(\zeta_j \otimes \zeta_{\nu(j)})$ is an element λ in $e_j \Lambda e_j$ with λ of degree $2m$. This element λ is a linear combination (over K) of monomials which start and end at j, and so each monomial has precisely m bar letters and m non-bar letters. If $j < m$ then each monomial starts at j with strictly more than j bar letters and so is zero; if $j > \nu(m)$ then each monomial starts at j with strictly more than $\nu(j)$ non-bar letters and so is also zero (I.3.3). Thus $f_i \circ \Omega^2\psi(\zeta_j \otimes \zeta_{\nu(j)}) = 0$ whenever $j < m$ or $j > \nu(m)$.

4.2. PROPOSITION. $HH^2(\Lambda) \times HH^4(\Lambda) = 0$ *if n is odd. If n is even then it is* 1-*dimensional, and we have, for* $0 \le i \le m$,

$$f_i \cdot \psi_0 = (-1)^{m+i+1}(i+1)z_m \cdot 1.$$

Proof. We fix some $i \le n-m-2$ and let $\beta = f_i \circ \Omega^2\psi_0$. The argument in Subsection 4.1 shows that elements in the image of $\Omega^2\psi_0$ are spanned by elements $\lambda_k \sigma_k \lambda'_k$ where $\deg(\lambda_k \lambda'_k) = 2m$.

Assume first that n is odd; then the only elements in $Z(\Lambda)$ of degree $2m$ are scalar multiples of z_m which are zero in $HH^6(\Lambda)$. So the claim follows in this case.

Assume now that n is even. By Subsection 4.1 we only need the value on $\zeta_m \otimes \zeta_{m+1}$ and $\zeta_{m+1} \otimes \zeta_m$. Moreover, the degree argument gives that there is some $c \in K$ such that

$$f_i \cdot \psi_0 = cz_m \cdot 1 \in HH^6(\Lambda), \qquad (\cong HH^0(\Lambda)).$$

We determine now the scalar factor c by using associativity. We multiply by g_0, and get

$$g_0 \cdot (f_i \cdot \psi_0) = g_0(cz_m \cdot 1) = cg_m \cdot 1 \qquad \text{in degree 7.}$$

This is also equal to $(g_0 \cdot f_i) \cdot \psi_0$. Now, $g_0 \cdot f_i$ is given in Subsection 2.5.1 and $h_i \cdot \psi_0$ is given in Subsection 1.6, and we get

$$(g_0 \cdot f_i) \cdot \psi_0 = (-1)^{i+1}(i+1)(-1)^m g_m \cdot 1$$

and $c = (-1)^{i+m+1}(i+1)$.

4.3. $HH^2(\Lambda) \times HH^5(\Lambda) = 0$ *if n is odd. If n is even then it is* 1-*dimensional and, for* $0 \le i \le m$,

$$f_i \cdot \theta_0 = (-1)^{m+i+1}(i+1)g_m \cdot 1.$$

Proof. Using associativity, graded-commutativity, and Subsection 1.7, $HH^2(\Lambda) \times HH^5(\Lambda) = (HH^2(\Lambda) \times HH^4(\Lambda)) \times HH^1(\Lambda)$. If n is odd then from Subsection 4.2, $HH^2(\Lambda) \times HH^5(\Lambda) = 0$.

Suppose n is even; then since $HH^5(\Lambda)$ is cyclic as a module over $Z(\Lambda)$ and $HH^2(\Lambda)$ is annihilated by the radical of the center we see that $HH^2(\Lambda) \times HH^5(\Lambda)$ is spanned by the products $f_i \cdot \theta_0$, and $f_i \cdot \theta_j = 0$ for $j \neq 0$. Moreover, Subsections 1.7 and 4.2 give the formula for $f_i \cdot \theta_0$.

4.4. PROPOSITION. *If n is even then* $HH^4(\Lambda) \times HH^4(\Lambda) = \mathrm{sp}\{f_m \cdot 1\}$ *and* $\psi_0 \cdot \psi_0 = f_m \cdot 1$. *If n is odd then* $HH^4(\Lambda) \times HH^4(\Lambda) = 0$.

Proof. It suffices to determine $\psi_0 \cdot \psi_0$.

Let $\beta = \psi_0 \circ \Omega^4 \psi_0$. Then the image of β is contained in the image of ψ_0, so it follows that for $i < m$ or $i > m+1$ one has $\beta(\sigma_i) = e_i\, \beta(\sigma_i) = 0$. This leaves us to study the values on σ_m, σ_{m+1}.

Assume first that n is odd. Then similarly $\beta(\sigma_{m+1}) = \beta(\sigma_{m+1})e_{m+1} = 0$. So we are left with σ_m. But for n odd, each element of $HH^2(\Lambda)$ maps σ_m to zero, so we are done in this case.

Now assume that n is even. By the above argument we know that $\psi_0 \cdot \psi_0 = cf_m \cdot 1$ for some $c \in K$. We determine c by using associativity and commutativity. From Subsectin 3.8,

$$f_0 \cdot (\psi_0 \cdot \psi_0) = cf_0 \cdot f_m \cdot 1 = c(-1)^{m+1}\psi_m \cdot 1.$$

On the other hand, from Subsection 4.2 we have $(f_0 \cdot \psi_0) \cdot \psi_0 = (-1)^{m+1} z_m \psi_0 \cdot 1 = (-1)^{m+1} \psi_m \cdot 1$ and $c = 1$.

4.5. *$HH^4(\Lambda) \times HH^5(\Lambda) = 0$ if n is odd. If n is even then*

$$\psi_0 \cdot \theta_0 = (-1)^{m+1} \left(\sum_{j=0}^{m} (-1)^j (j+1) h_j \right) \cdot 1.$$

In particular the space is 1-dimensional.

Proof. Using associativity, graded-commutativity, and Subsection 1.7, $HH^4(\Lambda) \times HH^5(\Lambda) = HH^1(\Lambda) \times (HH^4(\Lambda) \times HH^4(\Lambda))$. If n is odd then, from Subsection 4.4, $HH^4(\Lambda) \times HH^5(\Lambda) = 0$.

So suppose n is even. Then the space of products is spanned by $\psi_0 \cdot \theta_0$ and from Subsection 2.5.1 we get the formula since

$$\psi_0 \cdot \theta_0 = \psi_0 \cdot (g_0 \cdot \psi_0) = g_0 \cdot f_m \cdot 1.$$

5. SUMMARY OF THE RING STRUCTURE OF $HH^*(\Lambda)$ FOR TYPE A_n

The cohomology is periodic with periodicity 6, so we may identify $HH^i(\Lambda) \times HH^6(\Lambda)$ and $HH^i(\Lambda)$ as K-spaces for $i \geq 1$. Note also that $HH^i(\Lambda) \times HH^0(\Lambda) \cong HH^i(\Lambda)$ although $HH^0(\Lambda)$ and $HH^6(\Lambda)$ are not necessarily isomorphic spaces (see Sections I.5.3 and I.5.5). From [GS, G], $HH^*(\Lambda)$ is graded-commutative. In fact the ring $HH^*(\Lambda)$ is commutative, since we have shown in Section 1 that, for i, j odd, $HH^i(\Lambda) \times HH^j(\Lambda) = 0$.

The information given in this paper about the composition of elements in $HH^*(\Lambda)$ is summarized in Table I. From the previous remarks, only the composition of elements $HH^i(\Lambda)$ with $HH^j(\Lambda)$ for $1 \leq i \leq j \leq 5$ need be given. Moreover, we give a presentation of $HH^{ev}(\Lambda)$ by generators and relations, and then we also give a presentation of $HH^*(\Lambda)$ as an algebra over $HH^{ev}(\Lambda)$. These presentations may be deduced directly from Table I. The notation is given in Subsection 1.1.

THEOREM. *The even Hochschild cohomology ring $HH^{ev}(\Lambda)$ has a presentation given as follows. It is commutative, generated by elements*

$$1, z, f_0, \ldots, f_{n-m-2}, \psi_0, X,$$

where 1, z are in degree zero, the f_i are in degree 2, ψ_0 is in degree 4, and X is in degree 6. The subring generated by X is the polynomial ring in X. The other generators $\neq 1$ are nilpotent. The relations are as follows.

TABLE I
Composition of Elements in $HH^*(\Lambda)$

	$HH^1(\Lambda)$	$HH^2(\Lambda)$	$HH^3(\Lambda)$	$HH^4(\Lambda)$	$HH^5(\Lambda)$
$HH^1(\Lambda)$	0	$\begin{cases} HH^3(\Lambda) & \text{if char } K \nmid n+1 \\ g_0 \cdot f_{n-m-2} & \text{if char } K \mid n+1 \text{ and char } K \neq 2 \text{ in case } n \text{ odd} \\ 0 & \text{if char } K = 2 \text{ and } n \text{ odd} \end{cases}$	0	$HH^5(\Lambda)$	0
$HH^2(\Lambda)$		ψ_{n-m-2}	θ_{n-m-2}	$\begin{cases} 0, & n \text{ odd} \\ z_m \cdot 1, & n \text{ even} \end{cases}$	$\begin{cases} 0, & n \text{ odd} \\ g_m \cdot 1, & n \text{ even} \end{cases}$
$HH^3(\Lambda)$			0	$\begin{cases} 0, & n \text{ odd} \\ g_m \cdot 1, & n \text{ even} \end{cases}$	0
$HH^4(\Lambda)$				$\begin{cases} 0, & n \text{ odd} \\ f_m, & n \text{ even} \end{cases}$	$\begin{cases} 0, & n \text{ odd} \\ \Sigma_{j=0}^{m}(-1)^j(j+1)h_j, & n \text{ even} \end{cases}$
$HH^5(\Lambda)$					

(a) $z^{m+1} = 0 = zf_i$ *for* $0 \le i \le n - m - 2$. *If* n *is odd then* $z^m \psi_0 = 0$ *and* $z^m X = 0$.

(b) *Let* $\epsilon_k = -1$ *if* $k \equiv 2, 3 \pmod 4$ *and* $\epsilon_k = 1$ *otherwise; moreover let* $c(t, i) = (-1)^{i+t+l+nm} \epsilon_{n-m-2}(\min(i, t) + 1)$. *Then*

$$\psi_0 \cdot \psi_0 = \begin{cases} f_m X, & n \text{ even} \\ 0, & n \text{ odd} \end{cases}$$

$$f_t \cdot f_i = c(t, i) z^{n-m-2} \psi_0$$

$$f_i \cdot \psi_0 = \begin{cases} (-1)^{m+i+1} \epsilon_m (i + 1) z^m X, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

THEOREM. *The Hochschild cohomology ring* $HH^*(\Lambda)$ *has a presentation over* $HH^{ev}(\Lambda)$ *as follows. It is commutative, generated by elements*

$$g_0, h_0, h_1, \dots, h_{n-m-2}$$

where g_0 *is in degree* 1 *and the* h_i *are in degree* 3 *for* $0 \le i \le n - m - 2$. *The relations are as follows.*

(a) *For* $0 \le i, j \le n - m - 2$ *we have* $zh_i = 0$, $g_0 \cdot g_0 = 0$, $h_i \cdot h_j = 0$, *and* $g_0 \cdot h_i = 0$. *If* n *is odd then* $g_0 z^m = 0$.

(b) *We have*

$$g_0 \cdot f_1 = (-1)^{i+1} \left[(n - 2i - 1) \left(\sum_{j=0}^{i-1} (-1)^j (j + 1) h_j \right) + (i + 1) \left(\sum_{j=i}^{n-m-2} (-1)^j (n - 2j - 1) h_j \right) \right]$$

$$h_i \cdot f_j = \left(\delta_{ij} z^{n-m-2} \epsilon_{n-m-2} \right) (-1)^n g_0 \cdot \psi_0, \qquad 0 \le i, j \le n - m - 2$$

$$h_i \cdot \psi_0 = \begin{cases} \delta_{im} \epsilon_m z^m g_0 X, & n \text{ even} \\ 0, & n \text{ odd}. \end{cases}$$

REFERENCES

[ES] K. Erdmann and N. Snashall, *On Hochschild cohomology of preprojective algebras*, I, *J. Algebra*.

[G] M. Gerstenhaber, The cohomology structure of an associative ring, *Ann. of Math.* **78** (1963), 267–288.

[GS] M. Gerstenhaber and S. D. Schack, Algebraic cohomology and deformation theory, *in* “Deformation Theory of Algebras and Applications” (M. Hazewinkel and M. Gerstenhaber, Eds.), Kluwer ASI Series Vol. 247, pp. 11–264, Kluwer Academic, Dordrecht, 1988.

[S] A. Schofield, Wild algebras with periodic Auslander–Reiten translate, manuscript.