Multiple solutions for semilinear totally characteristic elliptic equations with subcritical or critical cone Sobolev exponents

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A B S T R A C T

In this paper, we study the Dirichlet problem for a class of semilinear totally characteristic elliptic equations with subcritical or critical cone Sobolev exponents and get the existence of infinitely many solutions in both case.

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1. Introduction

In this paper, we consider the following Dirichlet problem

\begin{equation}
\begin{cases}
-\Delta_\mathbb{B} u = g(x, u), & x \in \text{int} \mathbb{B}, \\
u = 0 & \text{on } \partial \mathbb{B}
\end{cases}
\end{equation}

where \( g(x, u) \) satisfies some subcritical or critical hypothesis. For the critical case we mainly discuss \( g(x, u) = \lambda u + |u|^{2^* - 2}u \) for \( 2^* = \frac{2n}{n-2} \). Here \( 2^* \) is the critical cone Sobolev exponents, and the domain \( \mathbb{B} \)
is \([0, 1) \times X\) for \(X \subset \mathbb{R}^{n-1}\) compact, which is regarded as the local model near the conical points on stretched manifolds with conical singularities, \(\text{int} \mathbb{B}\) is interior of \(\mathbb{B}\) and \(\partial \mathbb{B} = \{0\} \times X\). Moreover, the operator \(\Delta_{\mathbb{B}}\) in (1.1) is defined by \((x_1 \partial_{x_1})^2 + \partial_{n_1}^2 + \cdots + \partial_{n_k}^2\), which is an elliptic operator with totally characteristic degeneracy on the boundary \(x_1 = 0\) (we also call it Fuchsian type Laplace operator), and the corresponding gradient operator is denoted by \(\nabla_{\mathbb{B}} := (x_1 \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_k})\). We will often use coordinates \((x_1, x) = (x_1, x_2, \ldots, x_n)\) for \(0 \leq x_1 < 1, x \in X \) near \(\partial \mathbb{B}\). Our main result is the existence of infinitely many solutions for Eq. (1.1) in the cone Sobolev space \(\mathcal{H}^{\frac{1}{2}, 0}_{1,0}(\mathbb{B})\). The definition of such distribution spaces will be given in the next section.

The analysis on manifolds with conical singularities and the properties of elliptic, parabolic and hyperbolic equations in this setting are intensively studied in the last decades. More specifically, in aspects of partial differential equations and pseudo-differential theory of configurations with piecewise smooth geometry, the work of Kondrat’ev (see [14]) has to be mentioned here as the starting point of the analysis of operators on manifolds with conical singularities. The foundations of our paper have been developed through the fundamental works by B.-W. Schulze, and subsequently further expended by him and his collaborators, such as J.B. Gil, J. Seiler, T. Krainer and so on. The main subject of their work is the calculus of pseudo-differential operators on manifolds with singularities (see [23] and the references therein). On the other hand, R. Melrose and his collaborators gave various methods and ideas in the pseudo-differential calculus on manifolds with singularities, cf. Melrose and Mendoza [16], Melrose and Piazza [17], Melrose and Nistor [18] and Mazzeo [15]. All these mathematicians investigated deeply the underlying pseudo-differential calculi and the connected functional spaces. While these theories are nowadays well-established, many aspects are still to be interested, for instance, the existence theorem for the corresponding nonlinear elliptic equations on manifolds with singularities. In particular this is the main aim of our present paper.

Our work is in fact motivated by the work of Schrohe and Seiler in [22], where they introduced the so-called \(L_p\)-theory for the cone Sobolev spaces. Moreover, in [7–9], Coriasco, Schrohe and Seiler discussed the applications of those theory for linear and nonlinear parabolic equations on manifolds with conical singularities (with or without boundary). In [11], Dreher and Witt introduced the edge Sobolev spaces and dealt with the hyperbolic operators of degenerate type. They proved the well-posedness of the associated linear and semilinear Cauchy problems and considered the propagation of singularities for solutions to semilinear problems. All the mathematicians mentioned above provided important progress, by different approaches, on the general theory of totally characteristic operators, equations on singular manifolds, wave-front sets and propagation of singularities, etc.

Recently, the authors established the so-called cone Sobolev inequality (see Proposition 2.3) and Poincaré inequality (see Proposition 2.4) for the weighted Sobolev spaces (2.1) (see [5] for details). Such kind of inequalities are fundamental to prove the existence of the solutions for such nonlinear problems with totally characteristic degeneracy, and they are expected to be very useful in solving some geometry problem, e.g. Yamabe problem on manifolds with conical singularities. In [5], by using these inequalities and the variational method we already got the existence theorem for a class of semilinear degenerate equations on manifolds with conical singularities, that is, for the following Dirichlet problem

\[
\begin{align*}
-\Delta_{\mathbb{B}} u &= |u|^{p-2} u, \quad x \in \text{int} \mathbb{B}, \\
u &= 0, \quad \text{on } \partial \mathbb{B},
\end{align*}
\]

there exists a non-trivial solution \(u\) in \(\mathcal{H}^{\frac{1}{2}, 0}_{1,0}(\mathbb{B})\) with \(2 < p < 2^* = \frac{2n}{n-2}\). In this paper, we consider that the nonlinearity \(g(x, u)\) satisfies the subcritical condition and we get the multiplicity result as follows.

**Theorem 1.1.** Let \(g : \mathbb{B} \times \mathbb{R} \to \mathbb{R}\) be a Carathéodory function with primitive \(G(\cdot, u) = \int_0^u g(\cdot, v) dv\). Suppose that

1. \(g\) is odd: \(g(x, -u) = -g(x, u)\).

2. there exist \(p < 2^*\) and \(C > 0\) such that \(|g(x, u)| \leq C(1 + |u|^{p-1})\) almost everywhere,
(3) there exist \( q > 2 \) and a constant \( R_0 > 0 \) such that \( 0 < qG(x, u) \leq g(x, u)u \) holds for almost every \( x \) and \( |u| \geq R_0 \).

Then problem (1.1) admits infinitely many solutions in \( \mathcal{H}^{1, \frac{n}{2}}_{2,0}(B) \).

In their famous paper [4], Brezis and Nirenberg studied the following equation

\[
\begin{aligned}
-\Delta u &= \lambda u + |u|^{2^*_n - 2}u, & x \in \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

and get a positive solution of (1.3) in \( H^1_0(\Omega) \) for \( \lambda \in (0, \lambda_1) \) when \( n \geq 3 \), where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) with Dirichlet boundary condition. Later on Devillanova and Solimini [10] proved that (1.3) has infinitely many solutions for \( \lambda > 0 \) when \( n \geq 7 \) and Schechter and Zou [21] got infinitely many sign-changing solutions for \( \lambda > 0 \) when \( n \geq 7 \). Based on these achievements, we focus on the following Dirichlet problem

\[
\begin{aligned}
-\Delta_B u &= \lambda u + |u|^{2^* - 2}u, & x \in \text{int } B, \\
u &= 0 & \text{on } \partial B,
\end{aligned}
\]

where \( \lambda > 0 \), and \( 2^* = \frac{2n}{n-2} \). In [6] we have already proved that there exists a positive solution of (1.4) in \( \mathcal{H}^{1, \frac{n}{2}}_{2,0}(B) \) for \( \lambda \in (0, \lambda_1) \) when \( n \geq 4 \), where \( \lambda_1 \) is the first eigenvalue of \( -\Delta_B \) with Dirichlet boundary condition. However, the multiplicity result is also interesting and we obtain the following result.

**Theorem 1.2.** If \( n \geq 7 \), then problem (1.4) admits infinitely many solutions.

The proof of Theorem 1.2 is based on a uniform bound theorem of solutions to

\[
\begin{aligned}
-\Delta_B u &= \lambda u + |u|^{p-2}u, & x \in \text{int } B, \\
u &= 0 & \text{on } \partial B,
\end{aligned}
\]

where \( p \) varies in \([2, 2^*]\), that is,

**Theorem 1.3.** Let \( n \geq 7 \) and \( U \) be a bounded set in \( \mathcal{H}^{1, \frac{n}{2}}_{2,0}(B) \) whose elements are solutions, for a fixed \( \lambda > 0 \), to problems (1.4), for \( p \) varying in \([2, 2^*]\). Then \( U \) is uniformly bounded, i.e., there exists a constant \( C > 0 \) such that

\[
\sup_{u \in U} \sup_{x \in B} |u(x)| \leq C.
\]

In Section 2 we give some preliminaries, such as the definition of weighted Sobolev spaces and some lemmas which will be used in the later sections.

In Section 3 we discuss the existence of multiple solutions to the degenerate elliptic equations with subcritical exponent and give the proof of Theorem 1.1.

In Section 4 we will deal with the existence of infinitely many solutions to the degenerate elliptic equations with critical exponent. Since the Sobolev embedding is noncompact in this case, the idea to get this result is to extract a convergent subsequence from a noncompact (PS) sequence, which depends mainly on a uniformly bound estimate of solutions to the problem with subcritical exponent and Morse index theory and min–max method. Section 4.1 is dealt with some integral estimates for controlled concentrating sequences. In Section 4.2 we mainly discuss the local uniform bounds on controlled concentrating sequences. In Section 4.3 we give the proof of Theorem 1.3 of solutions to the degenerate elliptic equations with subcritical case. Section 4.4 concerns the proof of Theorem 1.2.
2. Preliminaries

2.1. Cone Sobolev spaces and inequalities

Here we introduce the manifolds with conical singularities and the corresponding cone Sobolev spaces.

Let $X$ be a closed, compact $C^\infty$ manifold of dimension $n-1$, and set $X^\Delta = (\mathbb{R}_+ \times X)/\{(0) \times X\}$ which is the local model interpreted as a cone with the base $X$. Since the analysis is formulated off the singularity it makes sense to pass to $X^\Delta = \mathbb{R}_+ \times X$ the open stretched cone with the base $X$.

A finite dimensional manifold $B$ with conical singularities is a topological space with a finite subset $B_0 = \{b_1, \ldots, b_M\} \subset B$ of conical singularities, which has the following two properties:

1. $B \setminus B_0$ is a $C^\infty$ manifold.
2. Every $b \in B_0$ has an open neighborhood $U$ in $B$, such that there is a homeomorphism $\varphi : U \rightarrow X^\Delta$ for some closed compact $C^\infty$ manifold $X = X(b)$, and $\varphi$ restricts a diffeomorphism $\varphi' : U \setminus \{b\} \rightarrow X^\Delta$.

From now on, we assume that the manifold $B$ is paracompact and of dimension $n$. By this assumption we can define the stretched manifold $B$ associated with $B$. Let $\mathbb{B}$ be a $C^\infty$ manifold with compact $C^\infty$ boundary $\partial \mathbb{B} = \bigcup_{b \in B_0} X(b)$, such that there is a diffeomorphism $B \setminus B_0 \cong \mathbb{B} \setminus \partial \mathbb{B} := \text{int } \mathbb{B}$, the restriction of which to $U_1 \setminus B_0 \cong V_1 \setminus \partial \mathbb{B}$ for an open neighborhood $U_1 \subset B$ near the points of $B_0$ and a collar neighborhood $V_1 \subset \mathbb{B}$ with $V_1 \cong \bigcup_{b \in B_0} ((0,1) \times X_b)$. In this paper, we shall consider $\mathbb{B} = [0,1] \times X$, and use the coordinates $(x_1, x') \in \mathbb{B}$.

**Definition 2.1.** For $(x_1, x') \in \mathbb{R}_+ \times \mathbb{R}^{n-1}$, we say that $u(x_1, x') \in L_p(\mathbb{R}_+, \frac{dx_1}{x_1}, dx')$ if

$$
\|u\|_{L_p} = \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} |u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < +\infty.
$$

Moreover, the weighted $L_p$-spaces with weight data $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}_+, \frac{dx_1}{x_1}, dx')$, namely, if $u(x_1, x') \in L_p^\gamma(\mathbb{R}_+, \frac{dx_1}{x_1}, dx')$, then $x_1^{-\gamma} u(x_1, x') \in L_p(\mathbb{R}_+, \frac{dx_1}{x_1}, dx')$, and

$$
\|u\|_{L_p^\gamma} = \left( \int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} |x_1^{-\gamma} u(x_1, x')|^p \frac{dx_1}{x_1} dx' \right)^{\frac{1}{p}} < +\infty.
$$

Now we can define the weighted Sobolev space for $1 \leq p < +\infty$.

**Definition 2.2.** For $m \in \mathbb{N}$, and $\gamma \in \mathbb{R}$, the spaces

$$
\mathcal{H}^m_{L_p} \left( \mathbb{R}_+ \right) := \left\{ u \in D' \left( \mathbb{R}_+ \right) : x_1^{-\gamma} (x_1 \partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p \left( \mathbb{R}_+, \frac{dx_1}{x_1}, dx' \right) \right\},
$$

for arbitrary $\alpha \in \mathbb{N}$, $\beta \in \mathbb{N}^{n-1}$, and $|\alpha| + |\beta| \leq m$. In other words, if $u(x_1, x') \in \mathcal{H}^m_{L_p} \left( \mathbb{R}_+ \right)$, then $(x_1 \partial_{x_1})^\alpha \partial_{x'}^\beta u \in L_p^\gamma(\mathbb{R}_+, \frac{dx_1}{x_1}, dx')$. 


It is easy to see that \( \mathcal{H}_p^{m,Y}(\mathbb{R}^n_+) \) is a Banach space with norm
\[
\|u\|_{\mathcal{H}_p^{m,Y}(\mathbb{R}^n_+)} = \sum_{|\alpha|+|\beta|\leq m} \left( \int_{\mathbb{R}^n_+} x_1^{n-1} \left( \frac{\partial^\alpha \partial^\beta u(x_1,x')}{x_1} \right)^p \right)^{1/p}.
\]

In this paper by a cut-off function we understand any real-valued \( \omega(x_1) \in C_0^\infty(\mathbb{B}) \) which equals 1 near \( \partial \mathbb{B} \). Now we discuss the weighted Sobolev spaces \( \mathcal{H}_p^{m,Y}(X^\gamma) \) with \( 1 \leq p < \infty \) on manifolds with conical singularities. We have the following definition (cf. [5]):

**Definition 2.3.**

(i) Let \( X \) be closed compact \( C^\infty \) manifold, and \( \mathcal{U} = \{U_1, \ldots, U_N\} \) an open covering of \( X \) by coordinate neighborhoods. If we fix a subordinate partition of unity \( \{\psi_1, \ldots, \psi_N\} \) and charts \( \chi_j : U_j \to \mathbb{R}^n, \quad j = 1, \ldots, N \), then \( \mathcal{H}_p^{m,Y}(X^\gamma) \) denotes the closure of \( C_0^\infty(X^\gamma) \) with respect to the norm
\[
\|u\|_{\mathcal{H}_p^{m,Y}(X^\gamma)} = \left\{ \sum_{j=1}^N \left\| (1 \times \chi_j)^{-1} \psi_j u \right\|^p_{\mathcal{H}_p^{m,Y}(\mathbb{R}^n_+)} \right\}^{1/p}.
\]

Here \( 1 \times \chi_j \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_0^\infty(\mathbb{R}_+ \times U_j) \) is the pull-back function with respect to \( 1 \times \chi_j : \mathbb{R}_+ \times U_j \to \mathbb{R}_+ \times \mathbb{R}^n \). Moreover, \( \mathcal{H}_p^{m,Y}(X^\gamma) \) denotes the closure of \( C_0^\infty(X^\gamma) \) in the space \( \mathcal{H}_p^{m,Y}(X^\gamma) \).

(ii) Let \( B \) be the stretched manifold to a manifold \( B \) with conical singularities. Then \( \mathcal{H}_p^{m,Y}(B) \) for \( m \in \mathbb{N}, \gamma \in \mathbb{R} \) denotes the subspace of all \( u \in W^{m,p}_{\text{loc}}(\text{int } B) \), such that
\[
\mathcal{H}_p^{m,Y}(B) = \{ u \in W^{m,p}_{\text{loc}}(\text{int } B) \mid \omega u \in \mathcal{H}_p^{m,Y}(X^\gamma) \}
\]

for any cut-off function \( \omega \), supported by a collar neighborhood of \( (0,1) \times \partial \mathbb{B} \). Moreover, the subspace \( \mathcal{H}_p^{m,Y}(B) \) of \( \mathcal{H}_p^{m,Y}(B) \) is defined as follows:
\[
\mathcal{H}_p^{m,Y}(B) := [\omega] \mathcal{H}_p^{m,Y}(X^\gamma) + [1 - \omega] W^{m,p}_{0}(\text{int } B),
\]

where \( W^{m,p}_{0}(\text{int } B) \) denotes the closure of \( C_0^\infty(\text{int } B) \) in Sobolev spaces \( W^{m,p}(\tilde{X}) \) when \( \tilde{X} \) is a closed compact \( C^\infty \) manifold of dimension \( n \) containing \( B \) as a submanifold with boundary.

The following two propositions tell us some properties of Sobolev spaces \( \mathcal{H}_p^{m,Y}(X^\gamma) \) and \( \mathcal{H}_p^{m,Y}(B) \). The similar results can be found in [22,24].

**Proposition 2.1.** (See [22]) We have \( \mathcal{H}_p^{m,Y}(X^\gamma) \subset H^{m}_{p,\text{loc}}(X^\gamma) \) for all \( m \in \mathbb{N}, \gamma \in \mathbb{R} \), where \( H^{m}_{p,\text{loc}}(X^\gamma) \) denotes the subspace of all \( u \in \mathcal{D}'(X^\gamma) \) such that \( \varphi u \in H^{m}_{p}(X^\gamma) \) for every \( \varphi \in C_0^\infty(X^\gamma) \).

**Proposition 2.2.** (See [22,24]) We have the following properties.

1. \( \mathcal{H}_p^{m,Y}(B) \) is Banach space for \( 1 \leq p < \infty \), and is Hilbert space for \( p = 2 \).
2. \( L^p(B) := \mathcal{H}^{0,Y}_{p}(B) \).
3. \( L^p(B) := \mathcal{H}^{0,Y}_{p}(B) \).
4. \( t \cap \mathcal{H}^{m,Y}_{p}(B) = \mathcal{H}^{m,Y}_{p}(B) \).
5. The embedding \( \mathcal{H}_p^{m,Y}(B) \hookrightarrow \mathcal{H}_p^{m',Y}(B) \) is continuous if \( m \geq m', \gamma \geq \gamma' \); and is compact embedding if \( m > m', \gamma > \gamma' \).
We first recall the cone Sobolev inequality and Poincaré inequality. For details we refer to [5,6] and the references [12,16,19].

**Proposition 2.3** (Cone Sobolev Inequality). Assume that $1 \leq p < n$, $\frac{1}{p} = \frac{1}{p} - \frac{1}{n}$, and $\gamma \in \mathbb{R}$. Let $\mathbb{R}^n_+ := \mathbb{R}_+ \times \mathbb{R}^{n-1}$, $x_1 \in \mathbb{R}_+$, and $x' = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$. The following estimate

$$
\| u \|_{L^{\gamma_p^*}(\mathbb{R}^n_+)} \leq c_1 \left\| (x_1 \partial_{x_1}) u \right\|_{L^{\gamma_p}(\mathbb{R}^n_+)} + (c_1 + \alpha c_2) \sum_{i=2}^{n} \| \partial_{x_i} u \|_{L^{\gamma_p}(\mathbb{R}^n_+)} + c_2 \| u \|_{L^{\gamma_p}(\mathbb{R}^n_+)}
$$

holds for all $u \in C^\infty_0(\mathbb{R}^n_+)$, where $\gamma^* = \gamma - 1$, $c_1 = \frac{n-1}{n(n-p)}$, $\alpha = \frac{(n-1)p}{n-p}$ and $c_2 = \frac{|n-\gamma^*(n-1)|}{n}$. Moreover, if $u \in H^{1,\gamma_p}(\mathbb{R}^n_+)$, we have

$$
\| u \|_{L^{\gamma_p^*}(\mathbb{R}^n_+)} \leq c \| u \|_{H^{1,\gamma_p}(\mathbb{R}^n_+)}
$$

where the constant $c = c_1 + \alpha c_2$.

**Proof.** See [5, Theorem 2.1]. □

**Proposition 2.4** (Poincaré Inequality). Let $\mathbb{B} = (0, 1) \times X$ be a bounded subspace in $\mathbb{R}^n_+$, and $1 < p < +\infty$, $\gamma \in \mathbb{R}$. If $u(x_1, x') \in H^{1,\gamma_p}(\mathbb{B})$, then

$$
\left\| u(x_1, x') \right\|_{L^p(\mathbb{B})} \leq c \left\| \nabla u(x_1, x') \right\|_{L^p(\mathbb{B})},
$$

where $\nabla = (x_1 \partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$, and the constant $c$ depends only on $\mathbb{B}$ and $p$.

**Proof.** See [5, Theorem 2.5]. □

**Proposition 2.5.** For $2 < p < 2^*$, the embedding $H^{1,\frac{2}{p}}(\mathbb{B}) \hookrightarrow H^{0,\frac{2}{p}}(\mathbb{B})$ is compact.

**Proof.** See [5, Theorem 3.1]. □

**Proposition 2.6.** There exist $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \rightarrow +\infty$, such that for each $k \geq 1$, the following Dirichlet problem

$$
\begin{cases}
-\Delta_B \phi_k = \lambda_k \phi_k, & \text{in } \text{int}(\mathbb{B}), \\
\phi_k = 0 & \text{on } \partial \mathbb{B}
\end{cases}
$$

admits a non-trivial solution in $H^{1,\frac{2}{p}}(\mathbb{B})$. Moreover, $\{\phi_k\}_{k \geq 1}$ constitute an orthonormal basis of the Hilbert space $H^{1,\frac{2}{p}}(\mathbb{B})$.

**Proof.** See [6, Proposition 3.4]. □
2.2. The global compactness

We first introduce a scale operation on our Sobolev spaces.

**Proposition 2.7.** Given \( \sigma > 0 \) and \( \bar{x} = (\bar{x}_1, \bar{x}_1') \in \mathbb{R}^n_+ \), let us consider the following scaled function

\[
\rho(u) = u_\sigma : (x_1, x'_1) \mapsto \sigma^{n^2} u\left( \frac{x_1}{\bar{x}_1}, \bar{x}_1' + \sigma (x'_1 - \bar{x}_1') \right). \tag{2.5}
\]

This scaling operation \( \rho \) keeps constant the norms \( \| \nabla_B u_\sigma \|_{L^2_n} \) and \( \| u_\sigma \|_{L^2_n} \) and is determined by the “center” or “concentration” point \( \bar{x} \) and the “modulus” \( \sigma \).

**Proof.** We need to prove that \( \| \nabla_B u_\sigma \|_{L^2_n} \) and \( \| u_\sigma \|_{L^2_n} \) as required. In fact, let \( y_1 = (\frac{x_1}{\bar{x}_1})^\sigma \) and \( y' = \bar{x}_1' + \sigma (x'_1 - \bar{x}_1') \), then we have \( \frac{dx_1}{y_1} = \frac{1}{\sigma} \frac{dy_1}{y_1} \) and \( dx' = \frac{1}{\sigma^{n-1}} dy' \) and \( x_1 \partial x_1 = \sigma y_1 \partial y_1 \). Then

\[
\| \nabla_B u_\sigma \|_{L^2_n} \leq \int_B |\nabla_B u_\sigma|^2 \frac{dx_1}{y_1} \frac{dx'}{x_1'} = \sigma^{n^2} \int_B |(x_1 \partial x_1, \partial x_2, \ldots, \partial x_n u_\sigma)|^2 \frac{dx_1}{y_1} \frac{dx'}{x_1'}
\]

In an analogous manner, we can get \( \| u_\sigma \|_{L^2_n} \).

**Definition 2.4.** Let \( \{u_m\}_{m \in \mathbb{N}} \) be a given sequence, we shall say that \( \{u_m\}_{m \in \mathbb{N}} \) is

- a controlled sequence if each \( u_m \) is a solution to

\[
- \Delta_B u \leq b |u|^{2^*-1} + A, \tag{2.6}
\]

where \( b > 1 \) and \( A = - \inf(b s^{2^*-1} - s^{p-1} - \lambda s) \) (taken for \( 1 \leq p \leq 2^*, \ s > 0 \) is a constant which does not depend on \( u \),

- a balanced sequence if each \( u_n \) solves (1.5) for some \( p \in [2, 2^*) \).

The corresponding functional to (1.4) is

\[
I_\lambda(u) = \frac{1}{2} \| u \|_{\mathcal{H}^{1, \frac{n}{2}}_0(B)}^2 - \frac{\lambda}{2} \int_B |u|^2 \frac{dx_1}{x_1} \frac{dx'}{x_1'} - \lambda \int_B |u|^{2^*} \frac{dx_1}{x_1} \frac{dx'}{x_1'}, \tag{2.7}
\]

and for \( \lambda = 0 \) we denote

\[
I_0(u) = \frac{1}{2} \| u \|_{\mathcal{H}^{1, \frac{n}{2}}_0(B)}^2 - \frac{1}{2^*} \int_B |u|^{2^*} \frac{dx_1}{x_1} \frac{dx'}{x_1'}, \tag{2.8}
\]

for \( u \in \mathcal{H}^{1, \frac{n}{2}}_0(B) \).

We recall the definition of (PS) sequence.
**Definition 2.5.** Let $V$ be a Banach space and a functional $E \in C^1(V, \mathbb{R})$. We say that the functional $E$ satisfies the (PS)$_c$ condition, if for any sequence $\{u_m\}_{m \in \mathbb{N}} \subset V$ with the properties:

$$E(u_m) \to c \quad \text{and} \quad \|E'(u_m)\|_{V^*} \to 0,$$

there exists a subsequence which is convergent, where $E' (\cdot)$ is the Fréchet differentiation of $E$ and $V^*$ is the dual space of $V$. If (PS)$_c$ condition holds for every $c \in \mathbb{R}$, we say that $E$ satisfies the (PS) condition and $\{u_m\}_{m \in \mathbb{N}}$ is called the (PS) sequence.

**Proposition 2.8.** Let $\{u_m\}_{m \in \mathbb{N}}$ be a noncompact (PS) sequence for $I_\lambda$. Then there exist a finite number $k \in \mathbb{N}$, a sequence $\{\sigma_m\}_{m \in \mathbb{N}}$ of moduli $\sigma_m \to \infty$ as $m \to \infty$, and a sequence $\{x_m\}_{m \in \mathbb{N}}$ of concentration points $x_m \in \mathbb{R}$, $1 \leq i \leq k$, a solution $u_\infty \in H_{2,0}^1(\mathbb{B})$ to (1.4) and non-trivial solutions $\varphi_i$, $1 \leq i \leq k$, to the limiting problem

$$\begin{cases}
-\Delta \varphi = |\varphi|^{2^n-2}\varphi, & \varphi \in \text{int} \mathbb{B}, \\
\varphi = 0 & \text{on } \partial \mathbb{B},
\end{cases}$$

such that up to a subsequence, $\{u_m\}_{m \in \mathbb{N}}$ satisfies

$$u_m - \sum_{i=1}^k \rho_m^i(\varphi_i) \to u_\infty \quad \text{in } H_{2,0}^1(\mathbb{B}),$$

where $\rho_m^i(\varphi_i)$ denotes the rescaled function

$$\rho_m^i(\varphi_i) = \left(\sigma_m^i\right)^{\frac{n}{2n-2}} \varphi_i \left(\frac{x}{\sigma_m^i}, \frac{x}{\sigma_m^i} + \sigma_m^i (\bar{x} - \bar{x})\right),$$

as in (2.5).

We call **concentrating sequence** any bounded sequence which satisfies a weaker case of the property in Proposition 2.8. More precisely, we say that the sequence $\{u_m\}_{m \in \mathbb{N}}$ is a concentrating sequence if the limit (2.10) holds in the $L^2_{\mathbb{B}}$ strong topology with $1 \leq k \leq MS^{-\frac{1}{2}}$, $u_\infty$ solution to (1.4) and $\varphi_i$ multiple of a global solution by a constant $\alpha_i$.

Since the proof of Proposition 2.8 is similar to the proof of Theorem 3.1 in Chapter III of Struwe [26], so we omit it here. The similar proof can also be found in [25] and references therein. However, by Proposition 2.8, we know that from any noncompact (PS) sequence we can extract a concentrating sequence. In what follows we call $\rho_{m, \mathbb{N}}$ one of the basic scaling sequences $\rho_m$, which corresponds to a function $\varphi_i$ which concentrates in $x_m = x_m$ in the slowest way. So $\sigma_m = \sigma_m$ and $x_m$ will also be considered to be given when we have fixed any concentrating sequence. Now we can define the so-called “safe region” for (PS) sequences and they are the sets on which the local uniform bounds will be established. For $1 \leq C \leq 7k + 1 \leq 7MS^{-\frac{1}{2}} + 1$, we define

$$A_m^1 = \Omega_{(C+5)\sigma_m^{-1/2}}(x_m) \setminus \Omega_{C\sigma_m^{-1/2}}(x_m),$$

$$A_m^2 = \Omega_{(C+4)\sigma_m^{-1/2}}(x_m) \setminus \Omega_{(C+1)\sigma_m^{-1/2}}(x_m),$$

$$A_m^3 = \Omega_{(C+3)\sigma_m^{-1/2}}(x_m) \setminus \Omega_{(C+2)\sigma_m^{-1/2}}(x_m).$$
Let us denote the “ball” in $\mathbb{R}^n_+$ in the sense of measure $\frac{dx_1}{x_1} dx'$ with the center $y = (y_1, y')$ and radius $r$ as follows:

$$\Omega_r(y) = \left\{ (x_1, x') \in \mathbb{R}^n_+ : \ln \left( \frac{x_1}{y_1} \right)^2 + |x' - y'|^2 \leq r^2 \right\}. \quad (2.11)$$

Then it is easy to see that the “ball” in the “origin” with radius $r$ is indeed $\Omega_r(x_0)$ for $x_0 = (1, 0)$ and sometimes we denote $\Omega_r(x_0)$ or $\Omega_r(y)$ by $\Omega_r$. Fig. 1 is a graph of (2.11) in 3-dimension case.

3. Proof of Theorem 1.1

In this section we will give the proof of Theorem 1.1. The following propositions will be used in the proof. For details one can see Struwe [26] or Ambrosetti and Rabinowitz [1,20].

**Proposition 3.1.** (See [26].) Suppose that the functional $E$ has the following properties.

1. Any $(PS)$ sequence for $E$ is bounded in some Banach space $V$ with dual $V^*$.
2. For any $u \in V$ we can decompose

$$DE(u) = L + K(u),$$

where $L : V \to V^*$ is a fixed bounded invertible linear map and the operator $K$ maps bounded sets in $V$ to relatively compact sets in $V^*$.

Then $E$ satisfies $(PS)$ condition.

**Proposition 3.2.** (See [26].) Suppose $V$ is an infinite dimensional Banach space and suppose the functional $E \in C^1(V)$ satisfies $(PS)$ condition, $E(u) = E(-u)$ for all $u$, and $E(0) = 0$. Suppose $V = V^+ \oplus V^-$, where $V^-$ is finite dimensional, and assume the following conditions:

1. $\exists \alpha > 0, \rho > 0, \forall u \in V^+: \|u\| = \rho \Rightarrow E(u) \geq \alpha$.
2. For any finite dimensional subspace $W \subset V$ there is $R = R(W)$ such that $E(u) \leq 0$ for $u \in W, \|u\| \geq R$.

Then $E$ possesses an unbounded sequence of critical values.
Proof of Theorem 1.1. We define the functional

$$E(u) = \frac{1}{2} \int_{\mathbb{B}} |\nabla_B u|^2 \frac{dx_1}{x_1} \, dx' - \int_{\mathbb{B}} G(x, u) \frac{dx_1}{x_1} \, dx'.$$

Hypothesis (2) implies that $E$ is Fréchet differentiable on $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})$. The assertion of this theorem is equivalent to the assertion that $E$ admits an unbounded sequence of critical points.

Let $J(u) := \int_{\mathbb{B}} G(x, u) \frac{dx_1}{x_1} \, dx'$. In order to show that $E$ satisfies (PS) condition, we first note that the map $u \rightarrow g(\cdot, u)$ takes bounded sets in $L^p_2(\mathbb{B})$ into bounded sets in $L^2_0(\mathbb{B})$ by the hypothesis (2), which implies that $J$ is weakly continuous. If $\{u_m\}_{m \in \mathbb{N}}$ is bounded in $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})$, then along a subsequence, $u_m$ converges weakly to some $u \in \mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})$ and $u_m \rightarrow u$ in $L^p_2(\mathbb{B})$, hence, $u_m$ converges to $u$ almost everywhere on $\mathbb{B}$. Since the embedding $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B}) \hookrightarrow \mathcal{H}^{0, \frac{n}{2}}_2(\mathbb{B})$ is compact for $p < \frac{2n}{n-2}$, we have

$$\| J'(u_m) - J'(u) \| = \sup_{\| \varphi \|_{\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})} \leq 1} \left| \int_{\mathbb{B}} (g(x, u_m) - g(x, u)) \varphi(x) \frac{dx_1}{x_1} \, dx' \right| \leq C \left\| g(\cdot, u_m) - g(\cdot, u) \right\|_{L^p_2(\mathbb{B})} \rightarrow 0 \quad \text{as} \quad m \rightarrow \infty$$

which implies $J'$ is continuous. Since $J$ is weakly continuous and $J'$ is uniformly differentiable on bounded subsets of $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})$, an abstract theorem [13] implies that $J' : \mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B}) \rightarrow \mathcal{H}^{-1, -\frac{n}{2}}_2(\mathbb{B})$ is compact. Now denote $DE(u) = -\Delta_B u - J'(u)$, and hence by Proposition 3.1 it suffices to show that any (PS) sequence $\{u_m\}$ for $E$ is bounded in $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})$.

Let $\{u_m\}$ be a (PS) sequence. Then we obtain

$$C + o(1) \| u_m \|_{\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})}^2 \geq q E(u_m) - \langle u_m, DE(u_m) \rangle \geq \frac{q - 2}{2} \int_{\mathbb{B}} |\nabla_B u_m|^2 \frac{dx_1}{x_1} \, dx' + \int_{\mathbb{B}} (g(x, u_m) u_m - qG(x, u_m)) \frac{dx_1}{x_1} \, dx' \geq \frac{q - 2}{2} \| u_m \|_{\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})}^2 + |\mathbb{B}| \cdot \text{esssup}_{x \in \mathbb{B}, v \in \mathbb{R}} (g(x, v) v - qG(x, v)),$$

where $|\mathbb{B}|$ is the measure of $\mathbb{B}$ in the sense of $\frac{dx_1}{x_1} \, dx'$, and $o(1) \rightarrow 0$ as $m \rightarrow \infty$. By hypotheses (2) and (3), the last term is finite and the desired conclusion follows. Moreover, since $g$ is odd and $E$ is even, then $E(0) = 0$.

Denote $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ the eigenvalues of $-\Delta_B$ on $\mathbb{B}$ with homogeneous Dirichlet data, and let $\varphi_j$ be the corresponding eigenfunctions.

We claim that for $k_0$ sufficiently large there exist $\rho > 0$, $\alpha > 0$ such that for all $u \in V^+ := \text{span}\{\varphi_k : k \geq k_0\}$ with $\|u\|_{\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B})} = \rho$, and there holds $E(u) \geq \alpha$. Indeed, by hypothesis (2) and cone Sobolev’s embedding $\mathcal{H}^{1, \frac{n}{2}}_2(\mathbb{B}) \hookrightarrow \mathcal{H}^{0, \frac{n}{2}}_2(\mathbb{B})$, and the Hölder inequality, for $u \in V^+$ we have

$$E(u) \geq \frac{1}{2} \int_{\mathbb{B}} |\nabla_B u|^2 \frac{dx_1}{x_1} \, dx' - C \int_{\mathbb{B}} |u|^p \frac{dx_1}{x_1} \, dx' - C$$
\[
\begin{align*}
\frac{1}{2} \| \nabla_B u \|_{H^2(B)}^2 - C \| u \|_{L^2(B)}^2 \| u \|_{H^2(B)}^{p-r} - C \\
\geq \frac{1}{2} \| u \|_{H^{1,2} \cap H^2(B)}^2 - C_1 \| u \|_{H^{1,2} \cap H^2(B)}^2 \| u \|_{H^{1,2} \cap H^2(B)}^{p-r} - C \\
\geq \left( \frac{1}{2} - C_1 \lambda^{-r/2}_{k_0} \| u \|_{H^{1,2} \cap H^2(B)}^{p-r} \right) \| u \|_{H^{1,2} \cap H^2(B)}^2 - C_2
\end{align*}
\]

where \( \frac{q}{2} + \frac{p-r}{2} = 1 \). In particular, \( r = n(1 - \frac{p}{2}) > 0 \), and we may let \( \rho = 2\sqrt{C_2 + 1} \) and choose \( k_0 \in \mathbb{N} \) such that

\[
C_1 \lambda^{-r/2}_{k_0} \rho^{p-2} \leq \frac{1}{4}
\]

to achieve that

\[
E(u) \geq 1 \equiv \alpha, \quad \text{for all } u \in V^+ \text{ with } \| u \|_{H^{1,2} \cap H^2(B)} = \rho.
\]

Now fix \( V^+ \) as above and denote \( V^- = \text{span}\{\varphi_j; j < k_0\} \) as its orthogonal complement.

Finally, from hypothesis (3) there hold

\[
\frac{u}{q} \leq \frac{G(x, u)}{G(x, u)}, \quad \text{whenever } u \geq R_0,
\]

and

\[
\frac{q}{u} \geq \frac{g(x, u)}{G(x, u)}, \quad \text{whenever } u \leq -R_0.
\]

By integrating the above two inequalities with respect to \( u \) on \([R_0, u]\) or \([u, -R_0]\) respectively, one has

\[
q \ln \frac{u}{R_0} \leq \ln \frac{G(x, u)}{G(x, R_0)}, \quad \text{whenever } u \geq R_0,
\]

and

\[
q \frac{R_0}{-u} \geq \ln \frac{G(x, -R_0)}{G(x, u)}, \quad \text{whenever } u \leq -R_0.
\]

That is,

\[
G(x, u) \geq G(x, R_0) \left( \frac{u}{R_0} \right)^q, \quad \text{whenever } u \geq R_0,
\]

and

\[
G(x, u) \geq G(x, -R_0) \left( \frac{-u}{R_0} \right)^q, \quad \text{whenever } u \leq -R_0.
\]
Thus

\[ G(x, u) \geq \beta_0(x)|u|^q, \quad \text{for } |u| \geq R_0, \]

where \( \beta_0(x) = R_0^{-q} \min\{G(x, R_0), G(x, -R_0)\} > 0 \). From the continuity, \( G(x, u) \) is bounded on \( \mathbb{B} \times [-R_0, R_0] \), that is, there exists a constant \( \beta > 0 \) such that

\[ G(x, u) \geq \beta_0|u|^q - \beta, \quad \text{for all } (x, u) \in \mathbb{B} \times \mathbb{R}. \]

For any finite dimensional subspaces \( W \subset \mathcal{H}^{1, \frac{q}{2}}(\mathbb{B}) \), there exist constants \( C_i = C_i(W) > 0 \) such that

\[
\sup_{\|u\|=R} E(u) = \sup_{\|u\|=R} \left[ \frac{1}{2} \int_{\mathbb{B}} |\nabla_B u|^2 \frac{dx_1}{x_1} - \int_{\mathbb{B}} G(x, u) \frac{dx}{x_1} \right] 
\leq C_1 R^2 - C_2 R^q + C_3 \to -\infty
\]

as \( R \to \infty \).

Now, Theorem 3.2 guarantees the existence of an unbounded sequences of critical values

\[ \alpha_k = \inf_{h \in \mathcal{I}_k} \sup_{u \in W_k} E(h(u)), \quad k \geq k_0, \] (3.1)

where \( W_k = \text{span}\{\varphi_j; \ j \leq k\} \) and

\[ \mathcal{I}_k = \{ h \in C^0(\mathcal{H}^{1, \frac{q}{2}}(\mathbb{B}), \mathcal{H}^{1, \frac{q}{2}}(\mathbb{B})); \ h \text{ is odd}, h(u) = u \text{ if } u \in W_j \text{ and } \|u\|_{\mathcal{H}^{1, \frac{q}{2}}(\mathbb{B})} \geq R_j \text{ for } j \leq k \}. \] (3.2)

The proof is complete. \( \square \)

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is divided into four parts.

4.1. Integral estimates for controlled concentrating sequences

For fixed \( p_0 \in (2, 2^*) \), we can choose a sequence \( \{p_m\}_{m \in \mathbb{N}} \) such that \( p_m \to 2^* \). Let \( 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \) be the eigenvalues of \( -\Delta_B \) and let \( \varphi_k(x) \) be the eigenfunction corresponding to \( \lambda_k \). Denote \( E_k := \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\} \). For each \( p_m \)

\[ I^m_k(u) = \frac{1}{2} \|u\|_{\mathcal{H}^{1, \frac{q}{2}}(\mathbb{B})} - \frac{\lambda}{2} \int_{\mathbb{B}} |u|^2 \frac{dx_1}{x_1} dx - \frac{1}{p_m} \int_{\mathbb{B}} |u|^{p_m} \frac{dx_1}{x_1} dx \] (4.1)

for \( u \in \mathcal{H}^{1, \frac{q}{2}}(\mathbb{B}) \).

**Definition 4.1.** Let \( p_1, p_2 \in (2, +\infty) \) be real numbers such that \( p_2 < 2^* < p_1, \alpha > 0 \) and \( \sigma > 0 \). We consider an inequalities system

\[
\begin{cases}
\|u_1\|_{L_{p_1}^{\frac{q}{2}}} \leq \alpha, \\
\|u_2\|_{L_{p_2}^{\frac{q}{2}}} \leq \alpha \sigma^{\frac{n}{p_1^*} - \frac{n}{p_2^*}}.
\end{cases}
\] (4.2)
and set
\[\|u\|_{p_1, p_2, \sigma} = \inf\{\alpha > 0: \exists u_1, u_2 \text{ such that } (4.2) \text{ is satisfied and } |u| \leq u_1 + u_2\}.\]

Occasionally, we briefly denote it by \(\|u\|_{\sigma}\) when \(p_1\) and \(p_2\) are given.

**Remark 4.1.** Let \(p_1, p_2 \in (2, +\infty)\) be real numbers such that \(p_2 < 2^* < p_1\), and \(\sigma > 0\). Then for any function \(u\), we get
\[\|u\|_{\sigma} \leq \|u\|_{L_{p_1}^{\frac{n}{n-2}}} \|u\|_{L_{p_2}^{\frac{n}{n-2}}} \leq \|u\|_{\sigma} \|u\|_{p_2}^{\frac{n}{n-2}}.\]

**Proposition 4.1.** Let \(\{u_m\}_{m \in \mathbb{N}}\) be a controlled concentrating sequence, then for any \(p_1, p_2 \in \left(\frac{2^*}{2}, +\infty\right)\), \(p_2 < 2^* < p_1\) there exists a constant \(C(p_1, p_2)\) such that for any \(m \in \mathbb{N}\)
\[\|u_m\|_{\sigma_n} \leq C.\]

In order to prove the above proposition, we need several lemmas.

**Lemma 4.1.** Let \(u, v \in \mathcal{H}_2^{1, \frac{n}{2}}(\mathbb{R}^n_+)\) and \(a \in L_2^{\frac{n}{n-2}}(\mathbb{R}^n_+)\) be three positive functions such that
\[-\Delta_B u \leq a(x)v.\]

Then for each \(p_1, p_2 \in (2, +\infty)\) there exists a constant \(C(p_1, p_2, n)\) such that for any \(\sigma > 0\)
\[\|u\|_{\sigma} \leq C(p_1, p_2, n)\|a\|_{L_2^{\frac{n}{n-2}}} \|v\|_{\sigma}.\]

**Proof.** Let us fix \(\sigma > 0, \varepsilon > 0\), and \(v \leq v_1 + v_2\) such that \(v_1\) and \(v_2\) satisfy (4.2) for \(\alpha = \|v\|_{p_1, p_2, \sigma} + \varepsilon\). For \(i = 1, 2\), let us consider the solutions \(u_i \in \mathcal{H}_2^{1, \frac{n}{2}}(\mathbb{R}^n_+)\) for \(-\Delta u_i = av_i\), then
\[\|u_i\|_{L_{p_i}^{\frac{n}{n-2}}} \leq C(n, p_i, a)\|v_i\|_{L_{p_i}^{\frac{n}{n-2}}} \|a\|_{L_2^{\frac{n}{n-2}}},\]
and \(-\Delta_B u_1 - \Delta_B u_2 = av_1 + av_2 \geq av \geq -\Delta_B u\). By the maximum principle we have \(u \leq u_1 + u_2\). Since the functions \(u_i\) satisfy (4.2) with \(\alpha = C(n, p_i)\|a\|_{L_2^{\frac{n}{n-2}}} (\|v\|_{\sigma} + \varepsilon)\) and the arbitrariness of \(\varepsilon\), we get the assertion. \(\Box\)

**Lemma 4.2.** Let \(p_1, p_2 \in \left(\frac{n+2}{n-2}, \frac{n+2}{2}\right)\) be such that \(p_2 < 2^* < p_1\) and let \(q_i\) be defined, for \(i = 1, 2\), by
\[\frac{1}{q_i} = \frac{n + 2}{n-2} \frac{1}{p_i} - \frac{2}{n}.\] (4.3)

If \(u\) and \(v\) are two positive functions with supports contained in a bounded set \(\Omega\) and such that
\[-\Delta_B u \leq v^{2^*-1} + A,\]
then there exists a constant \(C(p_1, p_2, n, \Omega)\) such that for any \(\sigma > 0\)
\[\|u\|_{q_1, q_2, \sigma} \leq C(p_1, p_2, n, \Omega)\left(\|v\|_{p_1, p_2, \sigma}\right)^{\frac{n+2}{n-2}} + 1)\]. (4.4)
Proof. By the proceeding in Lemma 4.1, we consider \( v = v_1 + v_2 \) where the functions \( v_1 \) satisfy (4.2) for \( \alpha = \|v\|_{p_1, p_2, \sigma} + \epsilon \) and \( \epsilon \) is a real strictly positive number arbitrarily small. Let \( u_1 \) and \( u_2 \) be two functions in \( H^{1,0}_{2,0}(B) \) such that

\[
-\Delta_B u_1 = 2^{\frac{4}{n-2}} v_1^{\frac{n+2}{n-2}} + A,
-\Delta_B u_2 = 2^{\frac{4}{n-2}} v_2^{\frac{n+2}{n-2}}.
\]

Then we have

\[
-\Delta_B u \leq v^{\frac{n+2}{n-2}} + A \leq 2^{\frac{n+2}{n-2}-1} v_1^{\frac{n+2}{n-2}} + A + 2^{\frac{n+2}{n-2}-1} v_2^{\frac{n+2}{n-2}} = -\Delta_B u_1 - \Delta_B u_2,
\]

by the maximum principle we have \( u \leq u_1 + u_2 \). Hence, we now estimate \( \|u\|_{L_{q_1}}^{\frac{n}{2}} \) and \( \|u_2\|_{L_{q_2}}^{\frac{n}{2}} \). By using (4.3) and \( \frac{n+2}{n-2} < p_i < \frac{n(n+2)}{2(n-2)} \), we get

\[
\|u_1\|_{L_{q_1}}^{\frac{n}{2}} \leq C(n, p_1) \|v_1\|_{L_{p_1}}^{\frac{n+2}{n-2}} + A \|v_1\|_{L_{p_1}}^{\frac{n}{2}},
\]

Then we have

\[
\|u_2\|_{L_{q_2}}^{\frac{n}{2}} \leq C(n, p_2) \|v_2\|_{L_{p_2}}^{\frac{n+2}{n-2}} + A
\]

Analogously, since the equality \( \frac{n}{2} - \frac{n+2}{n-2} = \left( \frac{n}{2} - \frac{n+2}{n-2} \right) \frac{n+2}{n-2} \) holds, we obtain

\[
\|u_2\|_{L_{q_2}}^{\frac{n}{2}} \leq C(n, p_2) \|v_2\|_{L_{p_2}}^{\frac{n+2}{n-2}} + A
\]

Therefore, \( u_1 \) and \( u_2 \) solve (4.2) for \( C = C(n, p_1, p_2, \Omega)(\|v\|_{p_1, p_2, \sigma}^\frac{n+2}{n-2} + 1) \), which concludes the proof by the arbitrary choice of \( \epsilon \). □

Lemma 4.3. Let \( \{u_m\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence, then there exist a constant \( C \) and exponents \( p_1, p_2 \in (\frac{n}{2}, +\infty) \), \( p_2 < 2^* < p_1 \), such that for any \( m \in \mathbb{N} \)

\[
\|u_m\|_{\sigma_m} \leq C.
\]

Proof. This proof will follow a Brezis–Kato type argument (see [2]). For any \( m \in \mathbb{N} \), we consider \( u_m = u_m^0 + u_m^1 + u_m^2 \), where

- \( u_{m}^{0} \) stands for the weak limit \( u_\infty \);
- \( u_{m}^{1} \) stands for the sum of rescaled function \( \varphi_i \), and \( u_{m}^{1} = \sum_{i=1}^{k} \rho_{m}^{i}(\varphi_{i}) \);
- \( u_{m}^{2} = u_{m} - u_{\infty} + u_{m}^{2} \) is an infinitesimal term in \( L_{2^*}^{\frac{2}{n}} \)-norm.
Let \( u \) be one of the terms \( u_m \), for simplicity, we denote \( u_i = u_{i,m} \), \( a_i = \max(1, \frac{\lambda}{(r^n)^{\frac{n-2}{2}}})u_i^{\frac{2}{n-2}} \) for \( i = 0, 1, 2 \) and \( \sigma = \sigma_m \). The infinitesimal character of \( u^0_m \) allows us to consider \( a_0 \) as small as we want in \( L^{\frac{n}{2}} \)-norm. Since

\[
a = u^{2^* - 2} < \max(1, \frac{\lambda}{(r^n)^{\frac{n-2}{2}}})(|u_0|^{\frac{2}{n-2}} + u_1^{\frac{2}{n-2}} + u_2^{\frac{2}{n-2}}),
\]

we consider \( u \) as a solution for \(-\Delta_B u \leq (a_0 + a_1 + a_2) u + A\), then by the monotonicity of the operator \( G := (-\Delta_B)^{-\frac{1}{2}} : H^{2, -\frac{n}{2}}(\B) \to H^{1, \frac{n}{2}}(\B) \) we have

\[
u \leq G(a_0 u) + G(a_1 u + A) + G(a_2 u).
\]

Since \( \B \) is a bounded set and \( a_1 \in L^{\frac{n}{2}}(\B) \), we get that \( G(a_1 u + A) \) is bounded in \( H^{2, -\frac{n}{2}}(\B) \to L^{\frac{n}{2}}(\B) \), for any \( p \) such that \( \frac{1}{p_1} > 1 - n - \frac{2}{2r} \), and (see Remark 4.1)

\[
\|G(a_1 u + A)\|_{\sigma} \leq \|G(a_1 u + A)\|_{p_1} \leq C.
\]

Now let \( 2^{*'} < p_2 < 2^* \) be given. We consider the index \( r \) such that

\[
\frac{1}{p_2} = \frac{1}{r} + \frac{1}{2^*} - \frac{2}{n},
\]

and it is easy to know that \( r > \frac{n}{2} \) by \( p_2 > 2^{*'} \). The decay speed of the solution \( \psi = \varphi_i \) gives us \( a_2 \in L^{\frac{n}{2}}(\B) \), and in order to estimate the \( L^{\frac{n}{2}} \)-norm of \( a_2 \), we just need to take into account the following less concentrated term, namely \( \rho_m(\psi) \), from \( r < \frac{n}{2} \) and \( p_2 < 2^* \). Since \( \frac{2r}{n-2} < p_2 \), it is obvious that

\[
\|a\|_{L^{\frac{n}{2}}} \leq C\|u_2\|_{L^{\frac{n}{2}}} \leq C\sigma^{\frac{n}{2} - \frac{n}{2r}}
\]

which implies

\[
\|G(a_2 u)\|_{p_2} \leq C\|a\|_{L^{\frac{n}{2}}} \|u\|_{L^{\frac{n}{2}}} \leq C\sigma^{\frac{n}{2} - \frac{n}{2r}}
\]

together with the fact that \( 2 - \frac{n}{r} = \frac{n}{2r} - \frac{n}{p_2} \). Therefore, from Remark 4.1, we get

\[
\|G(a_2 u)\|_{\sigma} \leq C\|a_2\|_{L^{\frac{n}{2}}} \|u\|_{L^{\frac{n}{2}}} \|u\|_{\sigma} \leq C.
\]

Now by Lemma 4.1 together with the \( p_1 \) and \( p_2 \) chosen as above, we get

\[
\|G(a_0 u)\|_{\sigma} \leq C\|a_0\|_{L^{\frac{n}{2}}} \|u\|_{\sigma} \leq \frac{1}{2}\|u\|_{\sigma},
\]

under a suitable choice of the bound of the norm of \( a_0 \). Finally we use the triangular inequality to obtain

\[
\|u\|_{\sigma} \leq 2\|G(a_1 u + A)\|_{\sigma} + 2\|G(a_2 u)\|_{\sigma} \leq C,
\]

which gives us the assertion. \( \Box \)
Proof of Proposition 4.1. Now let \( \{u_m\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence. By applying Lemma 4.3, we can find a constant \( C > 0 \) and two exponents \( p_1 \) and \( p_2 \in \left( \frac{n+2}{n-2}, \frac{n+2}{n} \right) \), \( p_2 < 2^* < p_1 \) such that (4.5) holds. Using the bootstrap Lemma 4.2, we can repeatedly enlarge the interval \( (p_2, p_1) \) to \( (q_2, q_1) \), where the exponents \( q_i \) are given by (4.3), obtaining (4.4). This procedure allows us to manage, in a finite number of steps, every exponent \( p_1, p_2 \in \left( \frac{2^*}{2}, +\infty \right) \).

4.2. Local uniform bounds on controlled concentration sequences

In this section we shall establish a local uniform bound on the terms of a controlled concentrating sequence on the safe region \( A^2_m \).

Proposition 4.2. Let \( \{u_m\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence. Then there exists a constant \( C > 0 \) such that for any \( m \in \mathbb{N} \) and for any \( x \in A^2_m \),

\[
\left| u_m(x) \right| \leq C.
\]

We begin with a weaker estimate.

Proposition 4.3. Let \( \{u_m\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence. Then there exists a constant \( C > 0 \) such that for any \( m \in \mathbb{N} \) and for any \( x \in A^2_m \),

\[
\left| u_m(x) \right| \leq C \sigma_m^{-\frac{n-2}{4}}.
\]

Proof. We shall give the argument by the contradiction. Suppose \( \{y_m\}_{m \in \mathbb{N}} = \{(y_m, 1, y'_m)\}_{m \in \mathbb{N}} \) be a sequence such that \( y_m \in A^1_m \) for any \( m \in \mathbb{N} \) and

\[
\lim_{m \to \infty} u_m(y_m) \sigma_m^{\frac{2-n}{2}} = +\infty. \tag{4.6}
\]

Let us scale the functions \( u_m \) in the following way that we move the point \( y_m \) to the “origin” in the sense of (2.11) and normalize the value of the functions. The required scaling sends \( u_m \) to \( \tilde{u}_m \) defined as

\[
\tilde{u}_m(x) = \rho_m^{\frac{n}{2}} u_m(y_m, 1, x_1) \rho_m^{\frac{n-2}{2}} + y'_m,
\]

where

\[
\rho_m = \left( u_m(y_m) \right)^{\frac{2-n}{2}} = \left( u_m(y_m) \right)^{\frac{-\frac{n}{2}}{2}},
\]

such that \( \tilde{u}_m(1, 0) = 1 \). By using (4.6), we have

\[
\lim_{m \to \infty} \frac{\rho_m}{\sigma_m^{-1/2}} = 0.
\]

Since \( y_m \in A^1_m \), there is no concentration point which approximates \( y_m \) at a distance less or equal to \( \sigma_m^{-1/2} \) which is of the order of \( \rho_m \), we can deduce that \( \tilde{u}_m \to \tilde{u} = 0 \). The contradiction will be achieved if we can find the points \( y_m \) such that \( \tilde{u} \neq 0 \).

In fact, if we have

\[
\tilde{u}_m(y) \leq 2, \quad \forall y \in \Omega_{\rho}(x_0), \quad x_0 = (1, 0), \tag{4.7}
\]
for some given \( \rho > 0 \), then we consider an \( \varepsilon \sigma^{-1/2} \)-neighborhood of \( A_m^1 \) such that \( \tilde{u}_m \) still satisfies \((2.6)\). By estimating the variation of the mean value of \( \tilde{u}_m \), for \( 0 < r \leq \rho \), we have

\[
\int_{\partial \Omega_r} \tilde{u}_m \, dS = \tilde{u}_m(x_0) + \int_0^r \frac{1}{n \sigma_t^{n-1}} \left( \int_{\Omega_t} \Delta \tilde{u}_m \, dx \right) \, dt
\]

\[
\geq 1 - C \int_0^r \frac{1}{t^{n-1}} \left( \int_{\Omega_t} (2^{2s-1} + A) \frac{dx}{x_1} \right) \, dt
\]

\[
= 1 - Cr^2 \geq \frac{1}{2}.
\]

where \( b_n \) is the \((n-1)\)-dimensional \( \frac{dx}{x_1} \) measure of unit sphere in \( \mathbb{R}_+^n \), provided we choose \( r \) small enough. Thus, the weak limit \( \tilde{u} \) cannot be 0. Therefore we only have to prove \((4.7)\). To this aim, we fix \( \rho > 0 \) and assume that, for a given \( m \in \mathbb{N} \), \( y_m \) does not satisfy \((4.7)\). Then we look for a better point satisfying \((4.7)\). Since \((4.7)\) is false, we can find \( z_m \in \Omega_{\rho}(1,0) \) such that

\[
\tilde{u}_m(z_m, z_m') = \rho_m^{n-2} u_m(y_m(z_m, z_m', \rho_m z_m' + y_m')) \geq 2. \tag{4.8}
\]

The first candidate to replace \( y_m \) is \( y_m^{(1)} = (y_m(z_m, z_m', \rho_m z_m' + y_m'), \rho_m z_m' + y_m') \) which leads to replacement \( \rho_m \) by

\[
\rho_m^{(1)} = \left[ u_m(y_m^{(1)}) \right]^{\frac{2}{n-2}} \rho_m \leq 2^{\frac{2}{n-2}} \rho_m. \tag{4.9}
\]

It is sure that \( y_m^{(1)} \) is at least as good as \( y_m \) to let \((4.6)\) hold since \((4.8)\) implies that

\[
u_m^{(1)} \leq 2u_m(y_m).
\]

Moreover, since \( z_m \in \Omega_{\rho}(1,0) \), we get

\[
|y_m^{(1)} - y_m| = |(y_m(z_m, z_m', \rho_m z_m' + y_m'), \rho_m z_m' + y_m') - (y_m(z_m, z_m', \rho_m z_m' + y_m'))| \leq \sqrt{(\rho_m \ln y_m, z_m)^2 + \rho_m z_m')^2 = \rho \rho_m. \tag{4.10}
\]

Note that in \((4.10)\) we use the distance defined under the measure \( \frac{dx}{x_1} \). We can define \( \tilde{u}_m \) as before by substituting \( y_m \) and \( \rho_m \) with \( y_m^{(1)} \) and \( \rho_m^{(1)} \) respectively. If this new \( \tilde{u}_m \) could satisfy \((4.7)\) we will not need to look for other choices. Otherwise, we repeat the same argument and then choose the second candidate \( y_m^{(2)} \) in the same way. For any fixed \( m \in \mathbb{N} \), we proceed recursively finding a sequence \( y_m^{(1)}, y_m^{(2)}, \ldots, y_m^{(k)}, \ldots \) until a successful choice achieved, which lets us claim \((4.7)\). It is obvious that this process cannot go on indefinitely. In fact, \((4.9)\) and \((4.10)\) become in the general sense, for \( i > 0 \),

\[
\rho_m^{(i+1)} \leq 2^{\frac{2}{n-2}} \rho_m^{(i)}
\]

and

\[
|y_m^{(i+1)} - y_m^{(i)}| \leq \rho \rho_m^{(i)}.
\]
Then by taking the sum of those sequence, we can get that \( y_m^{(i)} \) converges to a point \( y_m^{(\infty)} \) as \( i \to \infty \). But by construction, we have \( u_m(y_m^{(i)}) \to +\infty \) which contradicts the smoothness of \( u_m \). Finally, for every \( i > 0 \), we have

\[
|y_m^{(i)} - y_m| \leq \rho \rho_m \sum_{j=0}^{\infty} 2^{\frac{2}{n} j} \sigma_m^{-\frac{1}{2}},
\]

for \( n \) large. Then all the points \( y_m^{(i)} \) are in the \( \sigma_m^{-\frac{1}{2}} \)-neighborhood of \( A_m \) and can be used to replace \( y_m \).

**Proposition 4.4.** Let \( \{u_m\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence, then there exists a constant \( C > 0 \) such that for any \( n \in \mathbb{N} \) and for any \( r \in [\overline{C}\sigma_m^{-\frac{1}{2}}, (\overline{C} + 5)\sigma_m^{-\frac{1}{2}}] \),

\[
\int_{\partial \Omega_r(x_m)} u_m dS \leq C.
\]

**Proof.** By continuity, \( \{u_m\}_{m \in \mathbb{N}} \) is bounded in \( L_2^2 \subset L_1^2 \), then we can suppose \( \int_{\Omega_1(x_m)} u_m \frac{dx_1}{x_1} dx' \leq C \) with a constant \( C \) independent of \( m \). For any \( m \in \mathbb{N} \), there exists \( r_m \in [\frac{1}{2}, 1] \), such that

\[
\int_{\partial \Omega_r(x_m)} u_m dS = C.
\]

Take \( p_1 = \frac{n+2}{n-2} \) and \( p_2 = \frac{n+2}{n-2} \), and for any \( m \in \mathbb{N} \), we choose \( u_1 = u_{1,m} \) and \( u_2 = u_{2,m} \) such that (4.2) is satisfied for \( \sigma = \sigma_m \) with a constant \( \alpha \) independent of \( m \). Estimating the spherical mean variation from \( r_m \) to \( r \) and taking into account that \( (\overline{C} + 5)\sigma_m^{-\frac{1}{2}} < \frac{1}{2} \), i.e., \( r < r_m \) for \( m \) large, give us

\[
\int_{\partial \Omega_r(x_m)} u_m dS = C + \int_{r_m}^{r} \frac{d}{dt} \left( \int_{\partial \Omega_r(x_m)} u_m dS \right) dt
\]

\[
= C + \int_{r_m}^{r} \frac{1}{n b_n t^{n-1}} \left( \int_{\Omega_1(x_m)} -\Delta_B u_m \frac{dx_1}{x_1} dx' \right) dt
\]

\[
\leq C + \int_{\overline{C}\sigma_m^{-1/2}}^{1} \frac{1}{n b_n t^{n-1}} \left( \int_{\overline{C}\sigma_m^{-1/2}}^{1} (u_m^{n+2} - 1) \frac{dx_1}{x_1} dx' \right) dt
\]

\[
\leq C + \int_{\overline{C}\sigma_m^{-1/2}}^{1} 2^{\frac{n}{n-2}} \frac{1}{n b_n t^{n-1}} \left( \int_{\overline{C}\sigma_m^{-1/2}}^{1} u_{1,m}^{n+2} \frac{dx_1}{x_1} dx' \right) dt
\]

\[
+ \int_{\overline{C}\sigma_m^{-1/2}}^{1} 2^{\frac{n}{n-2}} \frac{1}{n b_n t^{n-1}} \left( \int_{\overline{C}\sigma_m^{-1/2}}^{1} u_{2,m}^{n+2} \frac{dx_1}{x_1} dx' \right) dt + \frac{A}{n} \int_{0}^{1} t dt
\]

\[
= C + \frac{2^{\frac{n}{n-2}}}{n b_n} (A_1 + A_2) + \frac{A}{2n},
\]
where for \( i = 1, 2, \)
\[
A_i = \int_0^1 \frac{4}{2^{n-2}} \int \frac{1}{t^{n-1}} \left( \int \frac{\sigma^\frac{n+2}{2}}{x_1} dx \right) dt.
\]

Since \( u_{1,m} \in L^n_{\frac{n+2}{2}}, \) by the Hölder inequality, we get
\[
A_1 \leq C \int_0^1 \frac{1}{t^{n-1}} (t^n)^{\frac{1}{n}} \|u_{1,m}\|^{\frac{n+2}{n}}_{L^n_{\frac{n+2}{2}}} dt \leq C \alpha \leq C.
\]

On the other hand, since \( u_{2,m} \in L^n_{\frac{n+2}{2}}, \) i.e., \( u_{2,m} \in L^n_1, \) we have
\[
A_2 \leq \int_{\Sigma_{\sigma_m^{-1/2}}} \frac{1}{t^{n-1}} \left[ \alpha \sigma_m^\frac{2-n}{2} \right]^{\frac{n+2}{n}} dt = \alpha \sigma_m^\frac{2-n}{2} \int_{\Sigma_{\sigma_m^{-1/2}}} \frac{1}{t^{n-1}} dt \leq C,
\]
and this concludes the proof. \( \square \)

From Proposition 4.4 we get, by integrating with respect to \( r, \) that
\[
\int_{A_{m}} u_m \, dS \leq C. \tag{4.11}
\]

Since \( \forall x \in A_{m}, \) \( \Omega_{\sigma_m^{-1/2}}(x) \subset A_{m} \) and the measure of the two sets are of the same order, from (4.11) we deduce that
\[
\forall x \in A_{m}^2: \int_{\Omega_{\sigma_m^{-1/2}}(x)} u_m \, dS \leq C. \tag{4.12}
\]

Since
\[
u_m(x) = \lim_{\rho \to 0} \int_{\Omega_{\rho}(x)} u_m \, dS,
\]
Proposition 4.4 follows from (4.12) if we estimate the variation of
\[
\int_{\Omega_{\rho}(x)} u_m \, dS
\]
for \( 0 \leq \rho \leq \sigma_m^{-1/2}. \)
Proof of Proposition 4.2. Let us fix an index \( m \in \mathbb{N} \) and a point \( x \in A_m^2 \). If the estimate

\[
u_m(x) \leq 2 \int_{\Omega_{\sigma_m^{-1/2}}(x)} u_m \, dS \]

holds then by (4.12) we have done. Otherwise, by setting for any \( \rho > 0 \),

\[
m(\rho) = \int_{\partial \Omega_\rho(x)} u_m \, dS, \quad m(0) = u_m(x),
\]

we deduce that

\[
\exists \bar{\rho} \leq \sigma_m^{-1/2} \text{ such that } m(\bar{\rho}) \leq \frac{1}{2} m(0) = \frac{1}{2} u_m(x).
\]

Then we take \( \rho_1 \) and \( \rho_2 \) from \([0, \bar{\rho}]\) such that \( m(\rho) \) attains its maximum in \( \rho_1 \) and \( \rho_2 \) is the least value of \( \rho \geq \rho_1 \) such that \( m(\rho) \leq \frac{1}{2} m(\rho_1) \).

Since \( \{u_m\}_{m \in \mathbb{N}} \) is a solution of (2.6), and \( \Omega_{\rho_2}(x) \subset A_m^1 \), thus on such a set, by Proposition 4.3, \( u_m \leq \sigma_m \), we have the following estimate for \( m \) sufficiently large,

\[
\frac{1}{2} m(\rho_1) = \int_{\rho_1}^{\rho_2} \left( \frac{d}{d\rho} \int_{\partial \Omega_\rho(x)} u_m \, dS \right) \, d\rho = \int_{\rho_1}^{\rho_2} \frac{1}{n b_n \rho^{n-1}} \left( \int_{\Omega_\rho(x)} -\Delta u_m \frac{dx_1}{x_1} \, dx' \right) \, d\rho
\]

\[
\leq \int_{\rho_1}^{\rho_2} \frac{1}{nb_n} \rho^{n-1} \left( \int_{\Omega_\rho(x)} u_m^4 + A d\rho^n \right) d\rho
\]

\[
\leq C \int_{\rho_1}^{\rho_2} \frac{1}{\rho^{n-1}} \left( \sigma_m \int_{\Omega_\rho(x)} dx_1 + A \rho^n \right) d\rho
\]

\[
\leq C (m(\rho_1) \sigma_m + A) \int_{\rho_1}^{\rho_2} \rho d\rho \leq C m(\rho_1) \sigma_m (\rho_2^2 - \rho_1^2),
\]

therefore \( \rho_2^2 - \rho_1^2 > C \sigma_m^{-1} \) and \( \rho_2 - \rho_1 > C \sigma_m^{-1/2} \) hold. Denoting by \( \mathcal{A} \) the annulus centered in \( x \) of radius \( \rho_1 \) and \( \rho_2 \) we have the measure of \( \mathcal{A} \) being of the order of \( \sigma^{-\frac{1}{2}} \), i.e., of the same order as \( A_m^1 \) and as in (4.3), we have

\[
\int \mathcal{A} u_m \, dS \leq C.
\]
On the other hand,
\[ \int_{A} u_{m} \, dS \geq m(\rho_{2}) = \frac{1}{2} m(\rho_{1}) , \]
that means
\[ u_{n}(x) = m(0) \leq m(\rho_{1}) \leq C . \]

**Proposition 4.5.** Let \( \{u_{m}\}_{m \in \mathbb{N}} \) be a controlled concentrating sequence. Then there exists a constant \( C > 0 \) such that for any \( m \in \mathbb{N} \):

\[ \int_{A_{m}^{3}} |\nabla u_{m}|^{2} \frac{dx_{1}}{x_{1}} \, dx' \leq C \sigma_{m}^{2-n} . \]

**Proof.** Let us fix \( m \in \mathbb{N} \) and consider \( \varphi_{m} : \mathbb{R}^{n}_{+} \to [0,1] \) a smooth positive mollifier such that

1. \( \varphi_{m} = 1 \) on \( A_{m}^{3} \);
2. \( \varphi_{m} = 0 \) out of \( A_{m}^{2} \);
3. \( \Delta_{\mathbb{R}} \varphi_{m} \leq C_{\sigma_{m}} \).

By (2) we have \( \varphi_{m} = 0 \) and \( \nabla \varphi_{m} = 0 \) on \( \partial A_{m}^{2} \). From (1), we apply integrating by parts to get

\[ \int_{\mathbb{R}^{n}_{+}} -\Delta_{\mathbb{R}} u_{m} \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' = \int_{A_{m}^{2}} |\nabla_{\mathbb{R}} u_{m}|^{2} \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' + \int_{A_{m}^{2}} \nabla_{\mathbb{R}} u_{m} \cdot \nabla \varphi_{m} u_{m} \frac{dx_{1}}{x_{1}} \, dx' \]
\[ \geq \int_{A_{m}^{2}} |\nabla_{\mathbb{R}} u_{m}|^{2} \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' + \int_{A_{m}^{2}} \nabla \left( \frac{1}{2} u_{m}^{2} \right) \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' \]
\[ = \int_{A_{m}^{2}} |\nabla_{\mathbb{R}} u_{m}|^{2} \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' - \frac{1}{2} \int_{A_{m}^{2}} \Delta_{\mathbb{R}} \varphi_{m} u_{m}^{2} \frac{dx_{1}}{x_{1}} \, dx' . \]

Since \( u_{m} \) is a solution to (2.6), by Proposition 4.2 and (3) we have

\[ \int_{A_{m}^{2}} |\nabla_{\mathbb{R}} u_{m}|^{2} \frac{dx_{1}}{x_{1}} \, dx' \leq \int_{A_{m}^{2}} (|u_{m}|^{2} + A u_{m}) \varphi_{m} \frac{dx_{1}}{x_{1}} \, dx' + \frac{1}{2} \int_{A_{m}^{2}} \Delta_{\mathbb{R}} \varphi_{m} u_{m}^{2} \frac{dx_{1}}{x_{1}} \, dx' \]
\[ \leq C(1 + \sigma_{m}) |A_{m}^{2}| \leq C \sigma_{m}^{\frac{2-n}{2}} . \]

Here we use the fact that \( \sigma_{m} \geq 1 \) for \( m \) large. □

**Corollary 4.1.** For any \( m \in \mathbb{N} \) there exists \( t_{m} \in [\bar{C} + 2, \bar{C} + 3] \) such that

\[ \int_{\partial \Omega_{m}} |\nabla_{\mathbb{R}} u_{m}|^{2} \frac{dx_{1}}{x_{1}} \, dx' \leq C \sigma_{m}^{\frac{3-n}{2}} . \]

where \( \Omega_{m} := \Omega_{m} \sigma_{m}^{-1/2}(x_{m}) \) and \( C \) is the same constant as in Proposition 4.5.
4.3. Local Pohozaev identity

Now we fix a general open smooth set $B$ in $\mathbb{R}^n_+$ and consider a semilinear elliptic equation of the form

$$-\Delta_B u = g(u).$$

Let $u$ be a smooth solution to (4.13) on a smooth domain $B$. Multiplying by $u$ and integrating by parts give us

$$\int_B |\nabla_B u|^2 \frac{dx_1}{x_1} dx' = \int_B g(u)u \frac{dx_1}{x_1} dx' + \int_{\partial B} (\nabla_B u \cdot \tilde{v}) u dS,$$

(4.14)

where $\tilde{v}$ is the outward normal to $\partial B$. Multiplying (4.13) by $\nabla_B u \cdot (\ln x_1, x')$ from both sides, since

$$\nabla_B \cdot \left( (\nabla_B u \cdot (\ln x_1, x')) \nabla_B u \right) = \Delta_B u (\nabla_B u \cdot (\ln x_1, x')) + (\nabla_B (\nabla_B \cdot (\ln x_1, x'))) \cdot \nabla_B u,$$

using the Divergence Theorem and integrating by parts we can get

$$\int_B -\Delta_B u (\nabla_B u \cdot (\ln x_1, x')) \frac{dx_1}{x_1} dx' = -\int_{\partial B} (\nabla_B u \cdot (\ln x_1, x')) (\nabla_B u \cdot \tilde{v}) dS$$

$$+ \int_B \nabla_B u \cdot (\nabla_B^2 u \cdot (\ln x_1, x') + I \cdot \nabla_B u) \frac{dx_1}{x_1} dx'$$

$$= -\int_{\partial B} (\nabla_B u \cdot (\ln x_1, x')) (\nabla_B u \cdot \tilde{v}) dS$$

$$+ \int_B \nabla_B \left( \frac{1}{2} |\nabla_B u|^2 \right) \cdot (\ln x_1, x') \frac{dx_1}{x_1} dx' + \int_B |\nabla_B u|^2 \frac{dx_1}{x_1} dx'$$

$$= -\int_{\partial B} (\nabla_B u \cdot (\ln x_1, x')) (\nabla_B u \cdot \tilde{v}) dS$$

$$+ \frac{1}{2} \int_{\partial B} |\nabla_B u|^2 ((\ln x_1, x') \cdot \tilde{v}) dS + \frac{2 - n}{2} \int_B |\nabla_B u|^2 \frac{dx_1}{x_1} dx'.$$

(4.15)

On the other hand, if we denote by $G(u)$ a primitive of the function $g(u)$, then by integrating by parts we have

$$\int_B g(u) (\nabla_B u \cdot (\ln x_1, x')) \frac{dx_1}{x_1} dx' = \int_B \nabla_B G(u) \cdot (\ln x_1, x') \frac{dx_1}{x_1} dx'$$

$$= \int_{\partial B} G(u) ((\ln x_1, x') \cdot \tilde{v}) dS - n \int_B G(u) \frac{dx_1}{x_1} dx'.$$

(4.16)

Combing (4.15) and (4.16), we obtain
\[
\frac{n}{2^*} \int_B |\nabla_B u|^2 \frac{dx_1}{x_1} \, dx = n \int_B G(u) \frac{dx_1}{x_1} \, dx' - \int_{\partial B} G(u)(\ln x_1, x') \cdot \bar{v}) \, dS
\]

\[
- \int_{\partial B} (\nabla_B u \cdot (\ln x_1, x'))(\nabla_B u \cdot \bar{v}) \, dS + \frac{1}{2} \int_{\partial B} |\nabla_B u|^2 (\ln x_1, x') \cdot \bar{v}) \, dS.
\]

(4.17)

Multiplying (4.14) by \( \frac{n}{2^*} \) and together with (4.17), we get

\[
n \int_B G(u) \frac{dx_1}{x_1} \, dx' - \frac{n}{2^*} \int_B g(u) \frac{dx_1}{x_1} \, dx' = \int_{\partial B} G(u)((\ln x_1, x') \cdot \bar{v}) \, dS
\]

\[
+ \int_{\partial B} (\nabla_B u \cdot (\ln x_1, x'))(\nabla_B u \cdot \bar{v}) \, dS
\]

\[
- \frac{1}{2} \int_{\partial B} |\nabla_B u|^2 (\ln x_1, x') \cdot \bar{v}) \, dS + \frac{n}{2^*} \int_{\partial B} (\nabla_B u \cdot \bar{v}) u \, dS.
\]

(4.18)

In our case, taking \( g(u) = \lambda u + |u|^{p-2} \, u \), the above equality becomes

\[
\left( \frac{n}{p} - \frac{n}{2^*} \right) \int_B |u|^p \frac{dx_1}{x_1} \, dx' + \lambda \int_B |u|^2 \frac{dx_1}{x_1} \, dx'
\]

\[
= \frac{1}{p} \int_{\partial B} |u|^p (\ln x_1, x') \cdot \bar{v}) \, dS + \frac{\lambda}{2} \int_{\partial B} |u|^2 (\ln x_1, x') \cdot \bar{v}) \, dS + \int_{\partial B} (\nabla_B u \cdot (\ln x_1, x'))(\nabla_B u \cdot \bar{v}) \, dS
\]

\[
- \frac{1}{2} \int_{\partial B} |\nabla_B u|^2 (\ln x_1, x') \cdot \bar{v}) \, dS + \frac{n}{2^*} \int_{\partial B} (\nabla_B u \cdot \bar{v}) u \, dS.
\]

Since we can move (1, 0) to any point \( x_0 = (x_{0,1}, x'_0) \in \mathbb{R}^n \) by a translation and \( p < 2^* \), we get the following "Pohozaev-type" inequality

\[
\lambda \int_B |u|^2 \frac{dx_1}{x_1} \, dx' \leq \frac{1}{p} \int_{\partial B} |u|^p \left( \left( \ln \left( \frac{x_1}{x_{0,1}} \right), x' - x'_0 \right) \cdot \bar{v} \right) \, dS
\]

\[
+ \frac{\lambda}{2} \int_{\partial B} |u|^2 \left( \left( \ln \left( \frac{x_1}{x_{0,1}} \right), x' - x'_0 \right) \cdot \bar{v} \right) \, dS
\]

\[
+ \int_{\partial B} (\nabla_B u \cdot (\ln \left( \frac{x_1}{x_{0,1}} \right), x' - x'_0) \right)(\nabla_B u \cdot \bar{v}) \, dS
\]

\[
- \frac{1}{2} \int_{\partial B} |\nabla_B u|^2 \left( \left( \ln \left( \frac{x_1}{x_{0,1}} \right), x' - x'_0 \right) \cdot \bar{v} \right) \, dS + \frac{n}{2^*} \int_{\partial B} (\nabla_B u \cdot \bar{v}) u \, dS. \quad (4.19)
\]

Now we shall use the local "Pohozaev" Identity to prove that concentrations are not possible for balanced sequences in dimension \( n \geq 7 \).
Lemma 4.4. If $n \geq 7$, no concentrating sequence can be balanced.

Proof. Let a concentrating sequence $\{u_m\}_{m \in \mathbb{N}}$ be given and assume by contradiction that it is balanced. Fix $m \in \mathbb{N}$, we use (4.19) on $\Omega_m := \Omega_{\sigma_m^{-1/2}}(x_m) \cap \mathbb{B}$, where $\Omega_m$ is the same as in Corollary 4.1, then we split $\partial \Omega_m = \partial_{e} \Omega_m \cup \partial_{s} \Omega_m$, where $\partial_{e} \Omega_m = \partial \mathbb{B} \cap \overline{\Omega}_m$ and it is empty in the case that the concentration point $x_m$ of the basic rescaled function $\phi$ is sufficiently far from $\partial \mathbb{B}$. When $\partial_{e} \Omega_m = \emptyset$, to the aim of applying (4.19), we shall take $x_0$ equal to the concentration point $x_m$. Otherwise, we will take $x_0$ out of $\mathbb{B}$ such that $d(x_0, x_m) \leq 2t_m \sigma_m^{-1/2}$ and

$$\forall x \in \partial_{e} \Omega_m: \quad \vec{v} \cdot \left(\ln \left(\frac{x_1}{x_{0,1}}\right), x' - x_0\right) < 0,$$

where $\vec{v}$ is the outward normal to $\partial \Omega_m$. We want to show that (4.19) cannot be valid, in contradiction to the assumption that the sequence is balanced. To this aim, we have to show that the left-hand side of (4.19) has a lower bound and the right-hand side of (4.19) has a smaller upper bound. In the first case, we restrict the integral on the “ball” $\Omega_m' = \Omega_{\sigma_m^{-1}}(x_m)$, which is contained in $\mathbb{B}$ for $m$ large, and we shall make use of the decomposition $u_m = u_m^0 + u_m^1 + u_m^2$ as in the proof of Lemma 4.3. Then we have

$$\int_{\Omega_m' \cap \mathbb{B}} (u_m)^2 \frac{dx_1}{x_1} \, dx' \geq \int_{\Omega_m'} u_m^2 \frac{dx_1}{x_1} \, dx' \geq \frac{1}{2} \int_{\Omega_m'} (u_m^2) \frac{dx_1}{x_1} \, dx' - 2 \int_{\Omega_m'} (u_m^1) \frac{dx_1}{x_1} \, dx'. \quad (4.20)$$

Now $\int_{\Omega_m'} (u_m^2) \frac{dx_1}{x_1} \, dx'$ is of the same order as $\int_{\Omega_m'} (\rho_m(\phi))^2 \frac{dx_1}{x_1} \, dx'$, namely of the order of $\sigma_m^{-2}$ because $\phi$ corresponds to the less concentrated global solution. Moreover, the following estimates hold

$$\int_{\Omega_m} (u_m^1)^2 \frac{dx_1}{x_1} \, dx' \leq \| u_\infty \| \Omega_m' \| \leq C \sigma_m^{-n},$$

and

$$\int_{\Omega_m'} (u_m^0)^2 \frac{dx_1}{x_1} \, dx' \leq \| (u_m^0)^2 \|_{L^2_{x'}} \| \Omega_m' \|^{1-\frac{2}{n}} \leq \| u_m^0 \|^2_{L^2_{x'}} \sigma_m^{-2}.$$

Since $\| u_m^0 \|^2_{L^2_{x'}} \to 0$ and (4.20), we obtain that the left-hand side of (4.19) has a lower bound of the form $C \sigma_m^{-2}$, for a suitable constant $C$. Passing to the right-hand side, we first evaluate the possible contributions of $\partial_{s} \Omega_m$. Since $u_m = 0$ on $\partial_{s} \Omega_m \subset \partial \mathbb{B}$, and $\nabla_{\mathbb{B}} u_m$ has the same direction as $\vec{v}$, the whole sum in (4.19) can be written as

$$\frac{1}{2} \int_{\partial_{e} \Omega_m} |\nabla_{\mathbb{B}} u_m|^2 \left(\ln \left(\frac{x_1}{x_{0,1}}\right), x' - x_0\right) \cdot \vec{v} \, dS \leq 0.$$

Now we focus on the integrals over $\partial_{e} \Omega_m$. From Proposition 4.1, we get
\[
\frac{\lambda}{2} \int_{\partial B} |u|^2 \left( \ln \left( \frac{x_1}{x_{0,1}}, x' - x_0 \right) \cdot \bar{v} \right) dS + \frac{1}{p} \int_{\partial B} |u|^p \left( \left( \ln \left( \frac{x_1}{x_{0,1}}, x' - x_0 \right) \cdot \bar{v} \right) dS \right.
\]
\[
\leq C \int_{\partial_\delta \Omega_m} \left( \ln \left( \frac{x_1}{x_{0,1}}, x' - x_0 \right) \bar{v} \right) dS \leq C\sigma_m^{-\frac{n}{2}},
\]
and Corollary 4.1 and our choice of \( \Omega_m \), give us
\[
\int_{\partial_\delta \Omega_m} |\nabla_B u_m|^2 \left( \ln \left( \frac{x_1}{x_{0,1}}, x' - x_0 \right) \right) dS \leq C\sigma_m^{-\frac{2-n}{2}}.
\]
Finally, from Proposition 4.1, Corollary 4.1, and the Hölder inequality, we have
\[
\int_{\partial_\delta \Omega_m} (\nabla_B u_m \cdot \bar{v}) u_m dS \leq \left( \int_{\partial_\delta \Omega_m} |\nabla_B u_m|^2 dS \right)^{1/2} \left( \int_{\partial_\delta \Omega_m} |u_m|^2 dS \right)^{1/2}.
\]

Combining these estimates, we see that the right-hand side of (4.19) is bounded by \( C\sigma_m^{-\frac{2-n}{2}} \). Then we get the inequality
\[
\lambda \sigma_m^{-2} \leq C\sigma_m^{-\frac{2-n}{2}},
\]
for \( m \) large, i.e., \( n \leq 6 \), which contradicts the fact \( n \geq 7 \). \( \square \)

The next lemma shows that from a noncompact balanced sequence \( \{u_m\}_{m \in \mathbb{N}} \) we can always extract a concentrating sequence, even if \( \{u_m\}_{m \in \mathbb{N}} \) is not a (PS) sequence.

Let \( \{u_m\}_{m \in \mathbb{N}} \) be a given bounded sequence of functions in \( \mathcal{H}^{1,\frac{2}{n}}(\mathbb{B}) \). If there exists a sequence \( \{\rho_m\}_{m \in \mathbb{N}} \) of rescalings such that the rescaled functions \( \{\rho_m(u_m)\}_{m \in \mathbb{N}} \) have a nonzero weak limit point in \( L^{\frac{2}{n}}(\mathbb{B}) \), this weak limit point is called a restored scaled limit of sequence \( \{u_m\}_{m \in \mathbb{N}} \).

**Lemma 4.5.** Let \( \{u_m\}_{m \in \mathbb{N}} \) be a noncompact bounded balanced sequence in \( \mathcal{H}^{1,\frac{2}{n}}(\mathbb{B}) \). Then from \( \{u_m\}_{m \in \mathbb{N}} \) we can extract a concentrating subsequence.

**Proof.** Assume that \( \{u_m\}_{m \in \mathbb{N}} \) has no converging subsequence. Under a null extension of \( u_m \) to the whole \( \mathbb{R}^{n}_+ \), we can use the analogous structure theorem for bounded sequence (see [25], where we can modify the result for our Sobolev spaces), according to which every term of the sequence can be approximated by a sum in \( \mathcal{H}^{1,\frac{2}{n}}_+(\mathbb{R}^n_+) \) of the scaled “restored scale limits” of the sequence itself. Furthermore, it is also needed to know how to quantify the number of such limits and to quantify them as multiplicities of global solutions. To this aim, we will prove that

1. the weak limit \( u_\infty \) of the sequence solves (1.4);
2. any restored scale limit \( \varphi_i \) of the sequence is a solution to the limit critical problem on \( \mathbb{R}^n_+ \) multiplied by a constant \( \alpha_i \).

For any \( m \in \mathbb{N} \), we call \( p_m \) the exponent such that \( u_m \) is solution to (1.5). Now, since \( \{u_m\}_{m \in \mathbb{N}} \) is a bounded sequence, by reflexivity of \( \mathcal{H}^{1,\frac{2}{n}}_+(\mathbb{B}) \), we can pass to a subsequence such that

- \( p_m \to \bar{p} \leq 2^*; \)
\[ u_m \rightharpoonup u_\infty \text{ weakly in } H^1_{1,0}(\mathbb{B}); \]

\[ u_m \to u_\infty \text{ a.e. in } \mathbb{B}. \]

Therefore,

\[ |u_m|^{p_m - 2} u_m + \lambda u_m \to |u_\infty|^{\bar{p} - 2} u_\infty + \lambda u_\infty, \quad \text{a.e. in } \mathbb{B}; \tag{4.21} \]

moreover, by linearity of \( \Delta \mathbb{B} \) we have \(-\Delta \mathbb{B} u_m \to -\Delta \mathbb{B} u_\infty \). Since \( u_m \) is a solution to (1.5) the two limits must coincide, i.e.,

\[ -\Delta \mathbb{B} u_\infty = |u_\infty|^{\bar{p} - 2} u_\infty + \lambda u_\infty. \]

Here \( \bar{p} = 2^* \), otherwise we would use the compact Sobolev embedding \( H^1_{1,0}(\mathbb{B}) \hookrightarrow L^2_{\mathbb{B}}(\mathbb{B}) \) obtaining a strongly converging subsequence and it is contradiction to our hypotheses; therefore (1) is proved.

Let \( \varphi = \lim_{m \to \infty} \rho_m(u_m) \) in the weak \( H^1_{1,0} \)-topology be any restored scale limit, where \( \{\rho_m\}_{m \in \mathbb{N}} \) is any diverging sequence of scalings each of modulus \( \nu_m > 0 \). By an easy calculation, since \( u_m \) is a solution to (1.4) we get that

\[ -\Delta \mathbb{B} \rho_m(u_m) = -\Delta \mathbb{B} \left( (v_m)\frac{p_m}{p_m - 2} u_m \left( \frac{x_1}{\nu_m}, x' + v_m(x' - \bar{x}) \right) \right) \]

\[ = -v_m^{\frac{2}{p_m} + 2} \Delta \mathbb{B} u_m = (v_m)^{2 - \frac{n}{p_m}(p_m - 2)} |\rho_m(u_m)|^{p_m - 2} \rho_m(u_m) + \lambda \nu_m^2 \rho_m(u_m). \tag{4.22} \]

Now, it is obvious that, since \( \mathbb{B} \) is a fixed bounded domain, the only way to get nonzero weak limits is to have \( \{\rho_m\}_{m \in \mathbb{N}} \) diverging by vanishing, i.e., \( \lim_{m \to \infty} \nu_m = 0 \). Passing to a subsequence, we can assume that

\[ v_m^{2 - \frac{n}{p_m}(p_m - 2)} = \nu_m^{n \left(1 - \frac{p_m}{p_m} \right)} \to \mu \leq 1 \]

and \( \mu > 1 \), because \( \varphi \neq 0 \). Using the linearity of the operator \( \Delta \mathbb{B} \) and Rellich theorem we can pass to the limit in (4.22) obtaining \(-\Delta \mathbb{B} \varphi = \mu |\varphi|^{2^* - 2} \varphi \). Therefore, \( \mu \frac{n}{p_m} \varphi \) solves (1.4) in \( H^1_{1,0}(\mathbb{R}^n_+) \) for \( \lambda = 0 \), and since \( \mu \frac{n}{p_m} \leq 1 \), \( \varphi \) is as required by the definition of concentrating sequence (see Theorem 2.8). Statement (2) is proved. It also implies that \( \|\varphi\|_{H^1_{1,0}(\mathbb{R}^n_+)} \geq S^{n/2} \) and the bound on \( k \) required by the definition of concentration sequence. \( \square \)

**Proof of Theorem 1.3.** Let us suppose, by contradiction, that there exists a bounded balance sequence \( \{u_m\}_{m \in \mathbb{N}} \) such that

\[ \sup_{m \in \mathbb{N}} \sup_{x \in \mathbb{B}} |u_m(x)| = +\infty. \]

A standard regularity argument shows that \( u_m \) cannot be compact in \( H^1_{1,0}(\mathbb{R}^n_+) \), and by Lemma 4.5 it has a balanced concentrating subsequence and this excluded by Lemma 4.4. \( \square \)

### 4.4. Multiple solutions to the critical problem

In this part we will give the proof of Theorem 1.2. Let us choose a sequence \( \{p_m\}_{m \in \mathbb{N}} \) such that \( p_m \to 2^* \) and the functionals

\[ I_m^\lambda(v) = \frac{1}{2} \int_{\mathbb{B}} |\nabla v|^2 \frac{dx_1}{x_1} \, dx' - \frac{\lambda}{2} \int_{\mathbb{B}} |v|^2 \frac{dx_1}{x_1} \, dx' - \frac{1}{p_m} \int_{\mathbb{B}} |v|^{p_m} \frac{dx_1}{x_1} \, dx', \]
whose critical points are solutions to (1.5) for $p = p_m$. Applying Theorem 1.1 for $g(x, u) = \lambda u + |u|^{p_m - 2}u$, we find an infinite number of critical levels $c_m^{(k)}$ of the subcritical functionals $I_m^{(\lambda)}$ obtained on a $k$-dimensional min–max class of compact sets $I_k$ which does not depend on $m$. For fixed $m, k \in \mathbb{N}$, we set
\[
\bar{c}_k = \inf_{A \in I_k} \sup_{v \in A} I_{\lambda}(v), \quad \bar{c}_m^{(k)} = \inf_{A \in I_k} \sup_{v \in A} I_m^{(\lambda)}(v).
\]
Set $V = \{u \in H^{1,2}_{2,0}(\mathbb{R}) : \int_{\mathbb{R}} |\nabla u|^2 \frac{dx}{x^1} - \lambda \int_{\mathbb{R}} |u|^2 \frac{dx}{x_1} dx' = 1\}$, then it is easy to show that given $\tilde{u}_m^{(k)} \in V$ such that $I_m^{(\lambda)}(\tilde{u}_m^{(k)}) = \bar{c}_m^{(k)}$, then $\alpha_m^{(k)}\tilde{u}_m^{(k)}$ with $\alpha_m^{(k)} = \left(\frac{1}{2} - \frac{1}{\bar{c}_m^{(k)}}\right) p_m$ is a solution to (1.5) at level
\[
c_m^{(k)} = \left[\left(\frac{1}{2} - \bar{c}_m^{(k)}\right) p_m\right]^{\frac{1}{2-p_m}} \left(\frac{1}{2} - \frac{1}{p_m}\right).
\]
(4.23)

Analogously we shall call
\[
c_k = \left[\left(\frac{1}{2} - \bar{c}_k\right) 2^*\right]^{\frac{1}{2*}} \left(\frac{1}{2} - \frac{1}{2^*}\right).
\]
(4.24)

The proof of Theorem 1.2 will follow from the following two lemmas.

**Lemma 4.6.** $\lim_{m \to \infty} c_m^{(k)} = c_k$ for any $k \in \mathbb{N}$.

**Lemma 4.7.** $\lim_{k \to \infty} c_k = +\infty$.

**Proof of Theorem 1.2.** Fixed $k \in \mathbb{N}$, we take, for any $m \in \mathbb{N}$, $u_m = u_m^{(k)}$ a critical point at level $c_m^{(k)}$ for the functional $I_m^{(\lambda)}$. By Theorem 1.1 and Lemma 4.6, we know that any (PS) sequence $\{u_m^{(k)}\}_{m \in \mathbb{N}}$ for $I_m^{(\lambda)}$ is bounded in $H^{1,2}_{2,0}(\mathbb{R})$. Then by Theorem 1.3, the sequence $\{u_m\}_{m \in \mathbb{N}}$ is uniformly bounded, hence by standard compactness arguments we can find a convergent subsequence to a solution $u^{(k)}$ to (1.4) at level $c_k$, as mentioned in Lemma 4.6. By Lemma 4.7, we have infinitely many distinct values of $c_k$ for $k \in \mathbb{N}$ and so the proof is complete. \(\square\)

Now we give the proofs of Lemmas 4.6 and 4.7.

**Proof of Lemma 4.6.** Let us fix $k \in \mathbb{N}$ and $A \in I_k$, then for any $u \in A$, we have $I_m^{(\lambda)}(u) \to I_{\lambda}(u)$. Since $A$ is compact and the functionals are equicontinuous,
\[
\sup_{u \in A} I_m^{(\lambda)}(u) \to \sup_{u \in A} I_{\lambda}(u).
\]
Then $\lim\sup_{m \to \infty} c_m^{(k)} \leq \sup_{u \in A} I_{\lambda}(u)$ and since $A$ is an arbitrary set in $I_k$, we get $\lim\sup_{m \to \infty} c_m^{(k)} \leq \bar{c}_k$. By (4.23) and (4.24) we get $\lim\sup_{m \to \infty} c_m^{(k)} \leq c_k$.

Since for $s > 0$ the function $f(s) = \frac{1}{p_m} s^{p_m} - \frac{1}{2^*} s^{2^*}$ gets its maximum value in $s = 1$ we have $f(s) \leq \frac{1}{p_m} - \frac{1}{2^*}$ for all $s > 0$. Therefore, for every $u \in H^{1,2}_{2,0}(\mathbb{R})$, we have
\[
I_{\lambda}(u) \leq I_m^{(\lambda)}(u) + \left(\frac{1}{p_m} - \frac{1}{2^*}\right) |\mathbb{R}|,
\]
which implies

$$\bar{c}_k \leq \liminf_{m \to \infty} c^{(k)}_m.$$  

By (4.23) and (4.24) we obtain

$$c_k \leq \liminf_{m \to \infty} c^{(k)}_m,$$

and then we get the assertion. \(\square\)

**Proof of Lemma 4.7.** Let us suppose, by contradiction, that the sequence \(\{c_k\}_{k \in \mathbb{N}}\) is bounded, hence it converges to a real number \(c\). For any \(k \in \mathbb{N}\) by Lemma 4.6 there exists \(m_k > k\) such that \(|c^{(k)}_{m_k} - c_k| < \frac{1}{k}\), hence

$$\lim_{k \to \infty} c^{(k)}_{m_k} = \lim_{k \to \infty} c_k = c$$  \hspace{1cm} (4.25)

and the sequence \(\{m_k\}_{k \in \mathbb{N}}\) is diverging, i.e.,

$$\lim_{k \to \infty} n_k = +\infty.$$  

Let \(u_{n_k}\) be a solution of (1.5) at level \(c^{(k)}_{m_k}\). Using the Morse Index estimates on min–max points (see [3]), we can select the sequence \(\{u_{m_k}\}_{k \in \mathbb{N}}\) such that every \(u_{m_k}\) has an augmented Morse index greater or equal to \(n_k\). By our assumption, we can claim that the sequence \(\{u_{m_k}\}_{k \in \mathbb{N}}\) is bounded in \(H^{1, \frac{2}{n}}(\mathbb{S})\). Indeed, since \(u_{m_k}\) is a solution of (1.5) we have

$$I^{m_k}_{\lambda}(u_{m_k}) = \left(\frac{1}{2} - \frac{1}{p_{m_k}}\right) \int_{\mathbb{S}} |u_{m_k}|^{p_{m_k}} \frac{dx_1}{x_1} \to c,$$

which gives the boundedness of \(-\Delta u_{m_k} - \lambda u_{m_k}\) in \(H^{1, -\frac{2}{n}}(\mathbb{S})\) and in turn, the boundedness of \(u_{m_k}\) on \(H^{1, \frac{2}{n}}(\mathbb{S})\). So the sequence \(\{u_{m_k}\}_{k \in \mathbb{N}}\) is uniformly bounded by Theorem 1.3 and therefore the Morse index of \(u_{m_k}\) must keep bounded in contradiction to our construction. \(\square\)

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