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Large isoperimetric regions in the product of a compact manifold with Euclidean space ☆

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Given a compact Riemannian manifold M without boundary, we show that large isoperimetric regions in $M \times \mathbb{R}^k$ are tubular neighborhoods of $M \times \{x\}$, with $x \in \mathbb{R}^k$.

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1. Introduction

We consider the *isoperimetric problem* of minimizing perimeter under a given volume constraint inside $N = M \times \mathbb{R}^k$, where $\mathbb{R}^k$ is the k-dimensional Euclidean space and M is an m-dimensional compact Riemannian manifold without boundary. Our main result is the following:

Theorem 1.1. *Let M be a compact Riemannian manifold. There exists a constant $v_0 > 0$ such that any isoperimetric region in $M \times \mathbb{R}^k$ of volume $v \geqslant v_0$ is a tubular neighborhood of $M \times \{x\}$, with $x \in \mathbb{R}^k$.*

This result, in case $k = 1$, was first proven by Duzaar and Steffen [4, Prop. 2.11]. As observed by Morgan, an alternative proof for $k = 1$ can be given using the monotonicity formula and properties of the isoperimetric profile of $M \times \mathbb{R}$ (see [20, Cor. 4.12] for a proof when M is a convex body). Gonzalo considered the general problem in his Ph.D. Thesis [9]. In $\mathbb{S}^1 \times \mathbb{R}^k$, the result follows from the classification of isoperimetric regions by Pedrosa and Ritoré [19]. Large isoperimetric regions in asymptotically flat manifolds have been recently characterized by Eichmair and Metzger [5]. It is worth mentioning that W.-T. Hsiang and W.-Y. Hsiang [12] completely solved the isoperimetric problem in products of Euclidean and hyperbolic spaces. Morgan [16], after Barthé [1], using results by Ros [22], provides a lower bound of the isoperimetric profile of a Riemannian product in terms of concave lower bounds of the isoperimetric profiles of the factors.

In our proof we use symmetrization and show in Corollary 2.2 that anisotropic scaling of symmetrized isoperimetric regions of large volume L^1-converge to a tubular neighborhood of $M \times \{0\}$. This convergence is improved in Lemma 2.4 to Hausdorff convergence of the boundaries using the density estimates on tubes from Lemma 2.3, similar to the ones obtained by Ritoré and Vernadakis [21]. Results of White [23] and Grosse-Brauckmann [11] on stable submanifolds then imply that the scaled boundaries are cylinders, see Theorem 3.2. For small dimensions, it is also possible to use a result by Morgan and Ros [18] to get the same conclusion only using L^1-convergence. Once it is shown that the symmetrized set is a tube, it is not difficult to prove that the original isoperimetric region is also a tube.

After the distribution of this manuscript, Gonzalo informed us that he had obtained a proof of Theorem 1.1 in [10]. His techniques are different from ours and similar to the ones used in [9].

Given a measurable set $E \subset N$, their *perimeter* and *volume* will be denoted by $P(E)$ and $|E|$, respectively. We refer the reader to Maggi's book [14] for background on finite perimeter sets. The r-dimensional Hausdorff measure of a set E will be denoted by $H^r(E)$.

On $M \times \mathbb{R}^k$ we shall consider the *anisotropic dilation* of ratio $t > 0$ defined by

$$\varphi_t(p, x) = (p, tx), \qquad (p, x) \in M \times \mathbb{R}^k.$$

Since the Jacobian of the map φ_t is t^k, we have

$$|\varphi_t(E)| = t^k |E|, \quad \text{for any measurable set } E \subset M \times \mathbb{R}^k. \tag{1.1}$$

Let $\Sigma \subset M \times \mathbb{R}^k$ be an $(n-1)$-rectifiable set, where $n = m + k$ is the dimension of N. At a regular point $p \in \Sigma$, the unit normal ξ can be decomposed as $\xi = av + bw$, with $a^2 + b^2 = 1$, v tangent to M and w tangent to $\mathbb{R}^k$. Then the Jacobian of $\varphi_t|\Sigma$ is equal to $t^{k-1}(t^2a^2 + b^2)^{1/2}$. For $t \geqslant 1$ we get

$$t^k H^{n-1}(\Sigma) \geqslant H^{n-1}(\varphi_t(\Sigma)) \geqslant t^{k-1} H^{n-1}(\Sigma), \tag{1.2}$$

and the reversed inequalities when $t \leqslant 1$. Similar properties hold for the perimeter. Equality holds in the right hand side of (1.2) if and only if $a = 0$, or equivalently if and only if ξ is tangent to $\mathbb{R}^k$.

An open ball in $\mathbb{R}^k$ of radius $r > 0$ and center x will be denoted by $D(x, r)$. If it is centered at the origin, we set $D(r) = D(0, r)$. We shall also denote by $T(x, r)$ the set $M \times D(x, r)$, and by $T(r)$ the set $M \times D(r)$. Observe that $\varphi_t(T(x, r)) = T(tx, tr)$ and that $T(x, r)$ is the tubular neighborhood of radius $r > 0$ of $M \times \{x\}$.

Given any set $E \subset N$ of finite perimeter, we can replace it by a *normalized* set $\operatorname{sym} E$ by requiring $\operatorname{sym} E \cap (\{p\} \times \mathbb{R}^k) = \{p\} \times D(r(p))$, where $H^k(D(r(p))$ is equal to the H^k-measure of $E \cap (\{p\} \times \mathbb{R}^k)$. For such a set we get

Theorem 1.2.

1. $|\operatorname{sym} E| = |E|$,
2. $P(\operatorname{sym} E) \leqslant P(E)$.

The proof of Theorem 1.2 is similar to the one of symmetrization in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k$ with respect to one of the factors, see Burago and Zalgaller [2, § 9] (or Maggi [14] for the case $m = 1$). The main ingredients are a corresponding inequality for the Minkowski content and approximation of finite perimeter sets by sets with smooth boundary.

Given $E \subset N$, we denote by E^* its orthogonal projection onto M. If E is normalized, and $u : E^* \to \mathbb{R}^+$ measures the radius of the disk obtained projecting $E \cap (\{p\} \times \mathbb{R}^k)$ to $\mathbb{R}^k$, we get, assuming enough regularity on u, that

$$|E| = \omega_k \int_{E^*} u^k \, dH^m,$$

$$H^{n-1}(\partial E) = k\omega_k \int_{E^*} u^{k-1} \sqrt{1 + |\nabla u|^2} \, dH^m,$$

where $\omega_k = H^k(D(1))$, and $k\omega_k = H^{k-1}(\mathbb{S}^{k-1})$. The above formulas imply

$$|T(r)| = \omega_k r^k H^m(M),$$
$$P(T(r)) = k\omega_k r^{k-1} H^m(M),$$

so that

$$P(T(r)) = k\left(\omega_k H^m(M)\right)^{1/k} |T(r)|^{(k-1)/k}. \tag{1.3}$$

The *isoperimetric profile* of $M \times \mathbb{R}^k$ is the function $I : (0, +\infty) \to [0, +\infty)$ defined by

$$I(v) = \inf\{P(E); |E| = v\}.$$

An *isoperimetric region* is a set $E \subset M \times \mathbb{R}^k$ satisfying $I(|E|) = P(E)$. Existence of isoperimetric regions in $M \times \mathbb{R}^k$ is guaranteed by a result of Morgan [17, p. 129], since the quotient of $M \times \mathbb{R}^k$ by its isometry group is compact. From his arguments, it also follows that isoperimetric regions are bounded in $M \times \mathbb{R}^k$ (see also [7]). From (1.3) we get

$$I(v) \leqslant k\left(\omega_k H^m(M)\right)^{1/k} v^{(k-1)/k}, \tag{1.4}$$

for any $v > 0$. The regularity of isoperimetric regions in Riemannian manifolds is well-known, see Morgan [15] and Gonzalez–Massari–Tamanini [8]. The boundary is regular except for a singular set of vanishing H^{n-7} measure. The following properties of the isoperimetric profile hold

Proposition 1.3. *The isoperimetric profile I of $M \times \mathbb{R}^k$ is non-decreasing and continuous.*

Proof. Let $v_1 < v_2$, and $E \subset N$ an isoperimetric region of volume v_2. Let $0 < t < 1$ so that $|\varphi_t(E)| = v_1$. By (1.2) we have

$$I(v_1) \leqslant P(\varphi_t(E)) \leqslant P(E) = I(v_2).$$

This shows that I is non-decreasing.

Let us prove now the right-continuity of I at v. Consider an isoperimetric region E of volume v. Take a smooth vector field Z with support in the regular part of the boundary of E such that $\int_E \operatorname{div} Z \neq 0$. The flow $\{\varphi_t\}_{t\in\mathbb{R}}$ of Z satisfies $(d/dt)|_{t=0}|\varphi_t(E)| \neq 0$. Using the Inverse Function Theorem we obtain a smooth family $\{E_w\}$, for w near v, with $|E_w| = w$ and $E_v = E$. The function $f(w) = P(E_w)$ satisfies $f \geqslant I$ and $I(v) = f(v)$. This implies that I is right-continuous at v since, for $v_i \downarrow v$, we have

$$I(v) = f(v) = \lim_{i\to\infty} f(v_i) \geqslant \lim_{i\to\infty} I(v_i) \geqslant I(v),$$

by the monotonicity of I.

To prove the left-continuity of I at v we take a sequence of isoperimetric regions E_i with $v_i = |E_i| \uparrow v$ and we consider balls B_i disjoint from E_i so that $|E_i \cup B_i| = |E_i| + |B_i|$. Then $I(v) \leqslant P(E_i \cup B_i) = I(v_i) + P(B_i) \leqslant I(v) + P(B_i)$ by the monotonicity of I, and the left-continuity follows by taking limits since $\lim_{i\to\infty} P(B_i) = 0$. $\square$

We shall also use the following well-known isoperimetric inequalities in M and $M \times \mathbb{R}^k$

Lemma 1.4 *([4]). Given $0 < v_0 < H^m(M)$, there exists a constant $a(v_0) > 0$ such that*

$$H^{m-1}(\partial E) \geqslant a(v_0)\, H^m(E)$$

for any set $E \subset M$ satisfying $0 < H^m(E) < v_0$.

Lemma 1.5. *Given $v_0 > 0$, there exists a constant $c(v_0) > 0$ so that*

$$I(v) \geqslant c(v_0)\, v^{(n-1)/n} \tag{1.5}$$

for any $v \in (0, v_0)$.

Lemma 1.5 follows from the facts that $I(v)$ is strictly positive for $v > 0$ and asymptotic to the Euclidean isoperimetric profile when v approaches 0.

2. Large isoperimetric regions in $M \times \mathbb{R}^k$

In this Section we shall prove that normalized isoperimetric regions of large volume, when scaled down to have constant volume v_0, have their boundaries uniformly close to the boundary of the normalized tube of volume v_0.

If $E \subset N$ is any finite perimeter set and $T(E)$ is the tube with the same volume as E, we define

$$E^- = E \cap T(E), \quad E^+ = E \setminus T(E).$$

Let $t > 0$, and $\Omega = \varphi_t(E)$. Since $\varphi_t(E^+) = \Omega^+$, (1.1) implies

$$\frac{|E^+|}{|E|} = \frac{|\Omega^+|}{|\Omega|}. \tag{2.1}$$

A similar equality holds replacing E^+ by E^-.

Proposition 2.1. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized sets with volumes $|E_i| \to \infty$. Let $v_0 > 0$ and $0 < t_i < 1$ so that $|\varphi_{t_i}(E_i)| = v_0$ for all $i \in \mathbb{N}$, and let T be the tube of volume v_0 around M_0.*

If $\varphi_{t_i}(E_i)$ does not converge to T in the L^1-topology, then there is a constant $c > 0$, only depending on $\{E_i\}_{i\in\mathbb{N}}$, so that, passing to a subsequence, there holds

$$H^{n-1}(\partial E_i) \geqslant c|E_i|. \tag{2.2}$$

Proof. Assume $T = M \times D(r)$, and set $\Omega_i = \varphi_{t_i}(E_i)$. As $|\Omega_i| = |T|$, we get $2\,|\Omega_i^+| = |\Omega_i \triangle T|$ and, since $|\Omega_i \triangle T|$ does not converge to 0, the sequence $|\Omega_i^+|$ does not converge to 0 either. Let $c_1 > 0$ be a constant so that $\limsup_{i\to\infty}(|\Omega_i^+|/|\Omega_i|) > c_1$. From (2.1) we obtain

$$\limsup_{i\to\infty} \frac{|E_i^+|}{|E_i|} > c_1. \tag{2.3}$$

Now we claim that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < H^m(M). \tag{2.4}$$

To prove (2.4) we argue by contradiction. Assume that $\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) = H^m(M)$. As Ω_i is normalized, we have $(\Omega_i \cap \partial T)^* \subset (\Omega_i \cap T)^*$ and so $(T \setminus \Omega_i) \subset (M \setminus (\Omega_i \cap \partial T)^*) \times D(r)$. This implies $\limsup_{i\to\infty} |T \setminus \Omega_i| = 0$. Since $|\Omega_i| = |T|$, we get $\lim_{i\to\infty} |\Omega_i \triangle T| = 2\,\lim_{i\to\infty} |T \setminus \Omega_i| = 0$, a contradiction that proves the claim.

Hence there exists $w \in (0, H^m(M))$ so that

$$\liminf_{i\to\infty} H^m((\Omega_i \cap \partial T)^*) < w. \tag{2.5}$$

Let $T(r_i)$ be the normalized tube with $|T(r_i)| = |E_i|$. As $\Omega_i \cap T = \varphi_{t_i}(E_i \cap T(r_i))$, we have $(E_i \cap \partial T(r_i))^* = (\Omega_i \cap \partial T)^*$; from (2.5) we get $\liminf_{i\to\infty} H^m((E_i \cap \partial T(r_i))^*) < w$, and we obtain

$$\liminf_{i\to\infty} H^m((E_i \cap \partial T(s))^*) < w, \qquad \forall s \geqslant r_i. \tag{2.6}$$

This last step to go from the particular r_i to every $s \geqslant r_i$ is easy to check as, for any normalized set $E = \bigcup_{p\in E^*}(\{p\} \times D(r(p)))$, we have $(E \cap \partial T(s))^* = \{p \in M : r(p) \geqslant s\}$, therefore $(E \cap \partial T(s))^* \subset (E \cap \partial T(r))^*$ whenever $s \geqslant r$.

The above arguments imply, replacing the original sequence by a subsequence, that

$$|E_i^+| > c_1\,|E_i|, \qquad H^m((E_i \cap \partial T(s))^*) < w, \qquad i \in \mathbb{N},\ s \geqslant r_i. \tag{2.7}$$

Let $a = a(w)$ be the constant in Lemma 1.4. For the elements of the subsequence satisfying (2.7) we have

$$\begin{aligned} H^{n-1}(\partial E_i) &\geqslant H^{n-1}(\partial E_i \cap (N \setminus T(r_i))) \\ &\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial E_i \cap \partial T(s))\, ds \end{aligned}$$

$$
\begin{aligned}
&\geqslant \int_{r_i}^{\infty} H^{n-2}(\partial(E_i \cap \partial T(s)))\,ds \\
&= \int_{r_i}^{\infty} H^{m-1}(\partial(E_i \cap \partial T(s))^*)\,H^{k-1}(\partial D(s))\,ds \\
&\geqslant \int_{r_i}^{\infty} a\,H^{m}((E_i \cap \partial T(s))^*)\,H^{k-1}(\partial D(s))\,ds \\
&= a\int_{r_i}^{\infty} H^{n-1}(E_i \cap \partial T(s))\,ds = a\,|E_i^+| > a\,c_1|E_i|,
\end{aligned}
$$

thus proving the result. In the previous inequalities we have used the coarea formula for the distance function to $M \times \{0\}$; that $\partial(E_i \cap \partial T(s)) \subset \partial E_i \cap \partial T(s)$, where the first ∂ denotes the boundary operator in $\partial T(s)$; the fact that for an $O(k)$-invariant set F we have $F \cap \partial T(s) = (F \cap \partial T(s))^* \times \partial D(s)$, and so $H^{r+k-1}(F \cap \partial T(s)) = H^r((F \cap \partial T(s))^*)\,H^{k-1}(\partial D(s))$; that $(\partial(E_i \cap \partial T(s)))^* = \partial(E_i \cap \partial T(s))^*$; and the isoperimetric inequality on M given in Lemma 1.4. □

Corollary 2.2. *Let $\{E_i\}_{i\in\mathbb{N}}$ be a sequence of normalized isoperimetric sets with volumes $\lim_{i\to\infty}|E_i| = \infty$. Let $v_0 > 0$ and $0 < t_i < 1$ such that $\Omega_i = \varphi_{t_i}(E_i)$ has volume v_0 for all $i \in \mathbb{N}$. Then $\Omega_i \to T$ in the L^1-topology, where T is the tube of volume v_0.*

Proof. Regularity results for isoperimetric regions imply that $P(E_i) = H^{n-1}(\partial E_i)$, choosing as representative of every isoperimetric set the closure of the set of density one points. If Ω_i does not converge to T in the L^1-topology then, using (2.2) in Proposition 2.1 and (1.4), we get

$$c\,|E_i| \leqslant P(E_i) \leqslant k\left(\omega_k H^m(M)\right)^{1/k} |E_i|^{(k-1)/k}$$

for a subsequence, thus yielding a contradiction by letting $i \to \infty$ since $|E_i| \to \infty$. □

Using density estimates, we shall show now that the L^1 convergence of the scaled isoperimetric regions can be improved to Hausdorff convergence.

In a similar way to Leonardi and Rigot [13, p. 18] (see also [21] and David and Semmes [3]), given $E \subset N$, we define a function $h : \mathbb{R}^k \times (0, +\infty) \to \mathbb{R}^+$ by

$$h(x,R) = \frac{\min\left\{|E \cap T(x,R)|, |T(x,R) \setminus E|\right\}}{R^n},$$

for $x \in \mathbb{R}^k$ and $R > 0$. We remark that the quantity $h(x,R)$ is not homogeneous in the sense of being invariant by scaling since $h(x,R) \leqslant \frac{1}{2}(k\omega_k H^m(M))\,R^{k-n}$, which goes to infinity when R goes to 0. When the set E should be explicitly mentioned, we shall write

$$h(E, x, R) = h(x, R).$$

Lemma 2.3. *Let $E \subset N$ be an isoperimetric region of volume $v > v_0$. Let $\tau > 1$ such that $\Omega = \varphi_\tau^{-1}(E)$ has volume v_0. Choose ε so that*

$$0 < \varepsilon < \min\left\{v_0, \left(\frac{c(v_0)\, v_0^{1/k}}{2k\omega_k H^m(M)}\right)^n, \left(\frac{c(v_0)}{8n}\right)^n\right\}, \tag{2.8}$$

where $c(v_0)$ is as in (1.5).

Then, for any $x \in \mathbb{R}^k$ and $R \leqslant 1$ so that $h(\Omega, x, R) \leqslant \varepsilon$, we get

$$h(\Omega, x, R/2) = 0.$$

Moreover, in case $h(\Omega, x, R) = |\Omega \cap T(x, R))|\, R^{-n}$, we get $|\Omega \cap T(x, R/2)| = 0$ and, in case $h(\Omega, x, R) = |T(x, R) \setminus \Omega|\, R^{-n}$, we have $|T(x, R/2) \setminus \Omega| = 0$.

Proof. Using Lemma 1.5 we get a positive constant $c(v_0)$ so that (1.5) is satisfied (i.e., $I(w) \geqslant c(v_0)\, w^{(n-1)/n}$, for all $0 \leqslant w \leqslant v_0$).

Assume first that

$$h(x, R) = h(\Omega, x, r) = \frac{|\Omega \cap T(x, R)|}{R^n}.$$

Define

$$m(r) = |\Omega \cap T(x, r)|, \quad 0 < r \leqslant R.$$

The function $m(r)$ is non-decreasing and, for $r \leqslant R \leqslant 1$, we get

$$m(r) \leqslant m(R) \leqslant |\Omega \cap T(x, R)| \leqslant \varepsilon\, R^n \leqslant \varepsilon < v_0 \tag{2.9}$$

by (2.8). Hence $v_0 - m(r) > 0$ for $0 < r \leqslant R$.

By the coarea formula, when $m'(r)$ exists, we get

$$m'(r) = \frac{d}{dr} \int_0^r H^{n-1}(\Omega \cap \partial T(x, s))\, ds = H^{n-1}(\Omega \cap \partial T(x, r)).$$

Now define

$$\lambda(r) = \frac{v_0^{1/k}}{(v_0 - m(r))^{1/k}} = \frac{v^{1/k}}{|E \setminus T(\tau x, \tau r)|^{1/k}} \geqslant 1,$$

and

$$\Omega(r) = \varphi_{\lambda(r)}(\Omega \setminus T(x, r)),$$

so that $|\Omega(r)| = |\Omega|$. Then

$$E(r) = \varphi_\tau(\Omega(r)) = \varphi_{\lambda(r)}(E \setminus T(\tau x, \tau r)),$$

and $|E(r)| = |E|$. Then, using (1.2) for $\lambda(r) \geqslant 1$ and standard properties of finite perimeter sets [14, Lemmas 12.22 and 15.12], we have

$$\begin{aligned} I(v) &\leqslant P(E(r)) \leqslant \lambda(r)^k \left(P(E \setminus T(\tau x, \tau r))\right) \\ &\leqslant \frac{v_0}{v_0 - m(r)} \left(P(E) - P(E \cap T(\tau x, \tau r)) + 2H^{n-1}(E \cap \partial T(\tau x, \tau r))\right). \end{aligned} \tag{2.10}$$

Since $\tau \geqslant 1$ and $E \cap \partial T(\tau x, \tau r)$ is part of a cylinder, using (1.2) again we get

$$\begin{aligned} P(E \cap T(\tau x, \tau r) &\geqslant \tau^{k-1} P(\Omega \cap T(x, r)) \geqslant \tau^{k-1} c(v_0)\, m(r)^{(n-1)/n}, \\ H^{n-1}(E \cap \partial T(\tau x, \tau r)) &= \tau^{k-1} H^{n-1}(\Omega \cap \partial T(x, r)) = \tau^{k-1} m'(r). \end{aligned}$$

Replacing these expressions in (2.10), since $P(E) = I(v)$ and $\tau^k v_0 = v$, we have

$$\begin{aligned} 2m'(r) &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{m(r)^{1/n}}{\tau^{k-1} v_0}\, I(v)\right) \\ &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{m(r)^{1/n}}{v_0^{1/k}}\, \frac{I(v)}{v^{(k-1)/k}}\right) \\ &\geqslant m(r)^{(n-1)/n} \left(c(v_0) - \frac{\varepsilon^{1/n}}{v_0^{1/k}}\, (k\omega_k H^m(M))\right) \\ &\geqslant \frac{c(v_0)}{2}\, m(r)^{(n-1)/n}, \end{aligned} \tag{2.11}$$

where we have also used $m(r) \leqslant \varepsilon$, (1.4), and (2.8)

If there is $r \in [R/2, R]$ such that $m(r) = 0$ then, by the monotonicity of the function $m(r)$, we would conclude $m(R/2) = 0$ as well. So we assume $m(r) > 0$ in $[R/2, R]$. Then by (2.11), we get

$$\frac{c(v_0)}{4} \leqslant \frac{m'(t)}{m(t)^{(n-1)/n}}, \qquad H^1\text{-a.e.}$$

By (2.9) we get $m(R) \leqslant \varepsilon R^n$. Integrating between $R/2$ and R,

$$c(v_0)\, R/8 \leqslant n\, (m(R)^{1/n} - m(R/2)^{1/n}) \leqslant n\, m(R)^{1/n} \leqslant n\, \varepsilon^{1/n} R.$$

This is a contradiction, since $\varepsilon < (c(v_0)/8n)^n$ by (2.8). So the proof in case $h(x, R) = |\Omega \cap T(x, R)|\, R^{-n}$ is completed.

Now we deal with the case $h(x, R) = |T(x, R) \setminus \Omega|\, R^{-n}$. Define

$$m(r) = |T(x, r) \setminus \Omega|.$$

Then $m(r)$ is a non-decreasing function and

$$m'(r) = H^{n-1}(\Omega^c \cap \partial T(x, r)) = \frac{1}{\tau^{k-1}} H^{n-1}(E^c \cap \partial T(\tau x, \tau r)), \tag{2.12}$$

since $E^c \cap \partial T(\tau x, \tau r)$ is part of a tube. We also have $m(r) \leqslant m(R) \leqslant \varepsilon R^n \leqslant \varepsilon < v_0$ by (2.8). Observe that

$$P(E \cup T(\tau x, \tau r) \leqslant P(E) - P(T(\tau x, \tau r) \setminus E) + 2H^{n-1}(E^c \cap \partial E(\tau x, \tau r)). \tag{2.13}$$

Since $\varphi_\tau(T(x, r) \setminus \Omega) = T(\tau x, \tau r) \setminus E$ and $\tau \geqslant 1$, we get

$$\begin{aligned} P(T(\tau x, \tau r) \setminus E) &= P(\varphi_\tau(T(x, r) \setminus \Omega)) \\ &\geqslant \tau^{k-1} P(T(x, r) \setminus \Omega) \geqslant \tau^{k-1}\, c(v_0)\, m(r)^{(n-1)/n}. \end{aligned} \tag{2.14}$$

Now, using that I is a non-decreasing function we easily obtain $P(E) = I(v) \leqslant I(|E \cup T(\tau x, \tau r)|) \leqslant P(E \cup T(\tau x, \tau r))$. We estimate $P(E \cup T(\tau x, \tau r))$ from (2.13). Using (2.14) and (2.12), we get

$$\begin{aligned} I(v) = P(E) &\leqslant P(E \cup T(\tau x, \tau r)) \\ &\leqslant I(v) - \tau^{k-1} c(v_0)\, m(r)^{(k-1)/k} + 2\tau^{k-1} m'(r), \end{aligned} \tag{2.15}$$

and so

$$\frac{c(v_0)}{2} \leqslant \frac{m'(r)}{m(r)^{(n-1)/n}}, \qquad H^1\text{-a.e.}$$

By (2.9) we get $m(R) \leqslant \varepsilon R^n$. Integrating between $R/2$ and R,

$$c(v_0)\, R/4 \leqslant n\, (m(R)^{1/n} - m(R/2)^{1/n}) \leqslant n\, m(R)^{1/n} \leqslant n\, \varepsilon^{1/n} R,$$

and we get a contradiction since by (2.8) we have $\varepsilon < (c(v_0)/(8n))^n < (c(v_0)/(4n))^n$. This concludes the proof. □

Let $F \subset N$, then we define $F_r = \{x \in N : d(x, F) \leqslant r\}$. We improve now the L^1-convergence of normalized isoperimetric regions obtained in Corollary 2.2 to Hausdorff convergence of their boundaries

Lemma 2.4. *Let $\{E_i\}_{i \in \mathbb{N}}$ be a sequence of isoperimetric sets in N with $\lim_{i\to\infty} |E_i| = \infty$. Let $v_0 > 0$ and $\{t_i\}_{i \in \mathbb{N}}$ such that $\lim_{i\to\infty} t_i = 0$ and $|\Omega_i| = v_0$ for all $i \in \mathbb{N}$, where $\Omega_i = \varphi_{t_i}(E_i)$. Then for every $r > 0$, $\partial\Omega_i \subset (\partial T)_r$, for large enough $i \in \mathbb{N}$, where T is the tube of volume v_0.*

Proof. Since $|\Omega_i| = v_0$, using (2.8) we can choose a uniform $\varepsilon > 0$ so that Lemma 2.3 holds with this ε for all Ω_i, $i \in \mathbb{N}$. This means that, for any $x \in N$ and $0 < r \leqslant 1$, whenever $h(\Omega_i, x, r) \leqslant \varepsilon$ we get $h(\Omega_i, x, r/2) = 0$.

As $\Omega_i \to T$ in $L^1(N)$ by Corollary 2.2, we can choose a sequence $r_i \to 0$ so that

$$|\Omega_i \triangle T| < r_i^{n+1}. \tag{2.16}$$

Now fix some $0 < r < 1$. We reason by contradiction assuming that, for some subsequence, there exist

$$x_i \in \partial\Omega_i \setminus (\partial T)_r. \tag{2.17}$$

We distinguish two cases.

First case: $x_i \in N \setminus T$, for a subsequence. Choosing i large enough, (2.17) implies $T(x_i, r_i) \cap T = \emptyset$ and (2.16) yields

$$|\Omega_i \cap T(x_i, r_i)| \leqslant |\Omega_i \setminus T| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|\Omega_i \cap T(x_i, r_i)|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 2.3, we conclude that $|\Omega_i \cap T(x_i, r_i/2)| = 0$, a contradiction.

Second case: $x_i \in T$. Choosing i large enough, (2.17) implies $T(x_i, r_i) \subset T$ and so

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i|, \quad \text{for every } r_i < r.$$

Then, by (2.16), we get

$$|T(x_i, r_i) \setminus \Omega_i| \leqslant |T \setminus \Omega_i| \leqslant |\Omega_i \triangle T| < r_i^{n+1}.$$

So, for i large enough, we get

$$h(\Omega_i, x_i, r_i) = \frac{|T(x_i, r_i) \setminus \Omega_i|}{r_i^n} < r_i \leqslant \varepsilon.$$

By Lemma 2.3, we conclude that $|T(x_i, r_i/2) \setminus \Omega_i| = 0$, and we get again contradiction that proves the Lemma. $\square$

3. Strict $O(k)$-stability of tubes with large radius

In this Section we consider the orthogonal group $O(k)$ acting on the product $M \times \mathbb{R}^k$ through the second factor.

Let $\Sigma \subset M \times \mathbb{R}^k$ be a compact hypersurface with constant mean curvature. It is well-known that Σ is a critical point of the area functional under volume-preserving deformations, and that Σ is a second order minimum of the area under volume-preserving variations if and only if

$$\int_\Sigma \big(|\nabla u|^2 - q\,u^2\big)\,d\Sigma \geqslant 0, \tag{3.1}$$

for any smooth function $u : \Sigma \to \mathbb{R}$ with mean zero on Σ. In the above formula ∇ is the gradient on Σ and q is the function $\mathrm{Ric}(\xi,\xi) + |\sigma|^2$, where $|\sigma|^2$ is the sum of the squared principal curvatures in Σ, ξ is a unit vector field normal to Σ, and Ric is the Ricci curvature on N.

A hypersurface satisfying (3.1) is usually called *stable* and condition (3.1) is referred to as *stability condition*. In case Σ is $O(k)$-invariant we can consider an equivariant stability condition: we shall say that Σ is *strictly* $O(k)$*-stable* if there exists a positive constant $\lambda > 0$ such that

$$\int_\Sigma \big(|\nabla u|^2 - q\,u^2\big)\,d\Sigma \geqslant \lambda \int_\Sigma u^2\,d\Sigma$$

for any $O(k)$-invariant function $u : \Sigma \to \mathbb{R}$ with mean zero.

We consider now the tube $T(r) = M \times D(r)$. The boundary of $T(r)$ is the $O(k)$-invariant cylinder $\Sigma(r) = M \times \partial D(r)$, with $(k-1)$ principal curvatures equal to $1/r$. Hence its mean curvature is equal to $(k-1)/r$ and the squared norm of the second fundamental form satisfies $|\sigma|^2 = (k-1)/r^2$. The inner unit normal to $\Sigma(r)$ is the normal to $\partial D(r)$ in $\mathbb{R}^k$ (it is tangent to the factor $\mathbb{R}^k$). This implies $\mathrm{Ric}(\xi,\xi) = 0$.

We have the following result

Lemma 3.1. *The cylinder* $\Sigma(r)$ *is strictly* $O(k)$*-stable if and only if*

$$r^2 > \frac{k-1}{\lambda_1(M)},$$

where $\lambda_1(M)$ *is the first positive eigenvalue of the Laplacian in* M*.*

Proof. Let $\Sigma = \Sigma(r) = M \times D(r)$. Observe that an $O(k)$-invariant function with mean zero on Σ is determined by a function $u : M \to \mathbb{R}$ with $\int_M u\,dM = 0$. Hence

$$\begin{aligned}\int_\Sigma \big(|\nabla u|^2 - q\,u^2\big)\,d\Sigma &= k\omega_k r^{k-1} \int_M \Big(|\nabla_M u|^2 - \frac{k-1}{r^2}\,u^2\Big)\,dM \\ &\geqslant k\omega_k r^{k-1}\Big(\lambda_1(M) - \frac{k-1}{r^2}\Big) \int_M u^2\,dM\end{aligned}$$

$$= \left(\lambda_1(M) - \frac{k-1}{r^2}\right) \int\limits_{\Sigma} u^2\, d\Sigma.$$

This proves the Lemma. $\square$

Using results by White [23] and Grosse-Brauckmann [11] we get

Theorem 3.2. *Let T be a normalized tube so that $\Sigma = \partial T$ is a strictly $O(k)$-stable cylinder. Then there exists $r > 0$ so that any $O(k)$-invariant finite perimeter set E with $|E| = |T|$ and $\partial E \subset T_r$ has larger perimeter than T unless $E = T$.*

Proof. Since Σ is strictly $O(k)$-stable, Grosse-Brauckmann [11, Lemma 5] implies that, for some $C > 0$, Σ has strictly positive second variation for the functional

$$F_C = \text{area} + H\,\text{vol} + \frac{C}{2}\,(\text{vol} - \text{vol}(T))^2,$$

in the sense that the second variation of F_C in the normal direction of a function u satisfies

$$\delta_u^2 F_C = \int\limits_{\Sigma} \left(|\nabla u|^2 - q\,u^2\right) d\Sigma + C\left(\int\limits_{\Sigma} u\, d\Sigma\right)^2 \geqslant \lambda \int\limits_{\Sigma} u^2\, d\Sigma,$$

for any smooth $O(k)$-invariant function u (see the discussion in the proof of Theorem 2 in Morgan and Ros [18]). In White's proof of Theorem 3 in [23] it is observed that a sequence of minimizers of F_C in tubular neighborhoods of radius $1/i$ of Σ are *almost minimizing*, and hence $C^{1,\alpha}$ submanifolds that converge Hölder differentiably to Σ, contradicting the positivity of the second variation of Σ. Theorem 1.2 implies that the symmetrization of these minimizers are again minimizers. Thus we get a family of $O(k)$-minimizers of F_C converging Hölder differentiably to Σ, thus contradicting the strict $O(k)$-stability of Σ. $\square$

4. Proof of Theorem 1.1

First we claim that there exists $v_0 > 0$ such that, for any isoperimetric region E of volume $|E| \geqslant v_0$, the set $\operatorname{sym} E$ is a tube.

To prove this, consider a sequence of isoperimetric regions $\{E_i\}_{i\in\mathbb{N}}$ with $\lim_{i\to\infty} |E_i| = \infty$. We know that $\{\operatorname{sym} E_i\}_{i\in\mathbb{N}}$ are also isoperimetric regions. Let $T = M \times D$ be a strictly $O(k)$-stable tube, that exists by Lemma 3.1. For large i, we scale down the sets $\operatorname{sym} E_i$ so that $\Omega_i = \varphi_{t_i}^{-1}(\operatorname{sym} E_i)$ has the same volume as T. As $\operatorname{sym} E_i$ is isoperimetric and $t_i > 1$, we get from (1.4) and (1.2) that $P(\Omega_i) \leqslant P(T)$. By Corollary 2.2, the sets $\{\partial\Omega_i\}_{i\in\mathbb{N}}$ converge to ∂T in Hausdorff distance. By Theorem 3.2, $\Omega_i = T$ for large i and so $\operatorname{sym} E_i$ is a tube. This proves the claim. In particular, $H^m(E \cap (\{p\} \times \mathbb{R}^k)) = H^m(D)$ for any $p \in M$.

Hence the isoperimetric profile satisfies $I(v) = C\, v^{(k-1)/k}$ for the constant C in (1.3) and any $v \geqslant v_0$. We conclude that

$$I(t^k v) = t^{k-1} I(v), \quad \text{whenever } t^k v \geqslant v_0. \tag{4.1}$$

Let E be an isoperimetric region with volume $|E| > v_0$, and $t < 1$ so that $t^k|E| = v_0$. Then

$$I(t^k|E|) \leqslant P(\varphi_t(E)) \leqslant t^{k-1} P(E) = t^{k-1} I(|E|)$$

by the inequality corresponding to (1.2) when $t \leqslant 1$. By (4.1), equality holds and the unit normal ξ to $\operatorname{reg}(\partial E)$, the regular part of ∂E, is tangent to the $\mathbb{R}^k$ factor. This implies that the m-Jacobian of the restriction f of the projection $\pi_1 : M \times \mathbb{R}^k \to M$ to the regular part of ∂E is equal to 1. By Federer's coarea formula for rectifiable sets [6, 3.2.22] we get

$$H^{n-1}(\partial E) = \int_M H^{k-1}(f^{-1}(p))\, dH^m(p).$$

Assume that $\operatorname{sym} E$ is the tube $T(E) = M \times D$. The Euclidean isoperimetric inequality implies $H^{k-1}(f^{-1}(p)) \geqslant H^{k-1}(\{p\} \times \partial D)$ and so $H^{n-1}(\partial E) \geqslant H^{n-1}(\partial T(E))$, again by the coarea formula. As $P(E) = P(\operatorname{sym} E) = P(T(E))$, we get $H^{k-1}(f^{-1}(p)) = H^{k-1}(\partial D)$ for H^m-a.e. $p \in M$ and so $\pi_1^{-1}(p)$ is equal to a disc $\{p\} \times D_p$ for H^m-a.e. $p \in M$.

The fact that ξ is tangent to $\mathbb{R}^k$ in $\operatorname{reg}(\partial E)$ implies that $\operatorname{reg}(\partial E)$ is locally a cylinder of the form $U \times S$, where $U \subset M$ is an open set and $S \subset \mathbb{R}^k$ is a smooth hypersurface. Hence the discs D_p are centered at the same point (i.e., E is the translation of a normalized tube). This concludes the proof of the theorem.

Remark 4.1. The equivariant version of Theorem 2 in Morgan and Ros [18], together with Corollary 2.2, can be used to prove Theorem 1.1 for small dimensions.

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