Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 217 (2008) 2141–2158
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Noncommutative spectral decomposition with quasideterminant

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Received 9 April 2007; accepted 1 September 2007

Available online 8 November 2007

Communicated by David Ben-Zvi

Abstract

We develop a noncommutative analogue of the spectral decomposition with the quasideterminant defined by I. Gelfand and V. Retakh. In this theory, by introducing a noncommutative Lagrange interpolating polynomial and combining a noncommutative Cayley–Hamilton's theorem and an identity given by a Vandermonde-like quasideterminant, we can systematically calculate a function of a matrix even if it has noncommutative entries. As examples, the noncommutative spectral decomposition and the exponential matrices of a quaternionic matrix and of a matrix with entries being harmonic oscillators are given.

Keywords: Quasideterminant; Spectral decomposition; Noncommutative

1. Introduction

The theory of spectral decomposition of a square matrix over a commutative field is well known in linear algebra and is used for calculation of a function of the matrix, especially the exponential matrix. However, for a matrix with noncommutative entries, the determinant or the characteristic polynomial are not defined because of the ordering problem. Therefore, "eigenvalues" used in the spectral decomposition are undefined and we have no systematic method for calculation of function of a matrix with noncommutative entries until now.

Under these circumstances, we studied the exponential of a matrix with entries being harmonic osillators for a model in quantum optics and developed "the quantum diagonalization

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2007.09.011

method" for a special type of matrices derived from the representation theory [3]. Moreover, we had a chance to encounter with the quasideterminant defined by I. Gelfand and V. Retakh. By using the quasideterminant, "noncommutative determinants" such as quaternionic determinants [1], superdeterminant, quantum determinant, Capelli determinant, etc. are expressed in the unified form [8]. In the theory of the noncommutative integrable system, quasideterminants are very useful to express the solution of the noncommutative integrable equations [2,6,7,10]. Furthermore, various noncommutative analogues of theories using determinants are developed, for example, noncommutative analogue of Cramer's formula, the Vandermonde determinant, symmetric functions, Plücker coordinates, and so on (see [4,5,9] and references therein).

In particular, in [5], they investigated a noncommutative Cayley–Hamilton's theorem. In their theory, a different characteristic polynomial for each row was introduced and the trace or determinant were of the form of diagonal matrices. Moreover, we knew through the study of the quantum diagonalization method that eigenvalues should be generalized as "eigen-diagonalmatrix" due to the noncommutativity of entries of the matrix. That is why we find that a noncommutative Cayley–Hamilton's theorem in [5] is suitable to a noncommutative analogue of the spectral decomposition.

In this paper, we define a noncommutative analogue of the Lagrange interpolating polynomial and develop a noncommutative analogue of the spectral decomposition by using the noncommutative Cayley–Hamilton's theorem with the quasideterminant. An identity given by a Vandermonde-like quasideterminant plays an essential role. As examples, we explicitly calculate the noncommutative spectral decomposition and the exponential matrices of a quaternionic matrix and of a matrix with entries being harmonic oscillators.

The contents of this paper are as follows. In Section 2, we give a brief review of the spectral decomposition in linear algebra. In Section 3, we introduce the quasideterminant defined by I. Gelfand and V. Retakh and describe some important properties used in our theory. In Section 4, we review the noncommutative Cayley–Hamilton's theorem in [5] shortly. In Section 5, we develop a noncommutative analogue of the spectral decomposition with the quasideterminant. In Section 6, we apply our method to a quaternionic matrix and a matrix with entries being harmonic oscillators. Section 7 is devoted to discussion.

2. Brief review of the spectral decomposition

Firstly, we give a brief review of the spectral decomposition in linear algebra.

Let A be an $(n \times n)$-matrix with commutative entries. For simplicity, we suppose that all the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are distinct. For $j = 1, \dots, n$, we set

$$P_j = \prod_{1 \leqslant i \leqslant n,\, i \neq j} \frac{(A - \lambda_i I)}{(\lambda_j - \lambda_i)}.$$

The polynomial of right-hand side is called the Lagrange interpolating polynomial. Then we have the spectral decomposition of A:

$$A = \lambda_1 P_1 + \cdots + \lambda_n P_n.$$

Moreover, if *the Cayley–Hamilton's theorem holds*, then $P_1, \dots, P_n$ are projection matrices i.e.

$$P_i^2 = P_i, \quad P_i P_j = O \ (i \neq j), \quad P_1 + \cdots + P_n = I.$$

Therefore, we can calculate $\exp A$ explicitly:

$$\exp A = e^{\lambda_1} P_1 + \cdots + e^{\lambda_n} P_n.$$

Remark 1. Lagrange interpolating polynomials $f_j(z) = \prod_{1 \leqslant i \leqslant n,\, i \neq j} \frac{(z-x_i)}{(x_j-x_i)}$ $(j = 1, \ldots, n)$ satisfy the following relations:

(1) $x_1^j f_1(z) + x_2^j f_2(z) + \cdots + x_n^j f_n(z) = z^j$ $(j = 0, 1, \ldots, n-1)$.
(2) $f_i(x_j) = \delta_{ij}$.

We note that from (1), if $x_1, \ldots, x_n$ are all distinct, then we have

$$\begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} z^{n-1} \\ \vdots \\ 1 \end{pmatrix}.$$

3. Quasideterminant

In this section, we introduce the quasideterminant defined by I. Gelfand and V. Retakh and describe some important properties used in our theory.

3.1. Definition

Let R be a (not necessary commutative) associative algebra. For a position (i, j) in a square matrix $A = (a_{rs})_{1 \leqslant r,s \leqslant n} \in M(n, R)$, let A^{ij} denote the $(n-1) \times (n-1)$-matrix obtained from A by deleting the ith row and the jth column. Let also $\mathrm{r}_i{}^j = (a_{i1}, \ldots, \hat{a}_{ij}, \ldots, a_{in})$ and $\mathrm{c}_j{}^i = (a_{1j}, \ldots, \hat{a}_{ij}, \ldots, a_{nj})^T$.

Definition 1. We assume that A^{ij} is invertible over R. The (i, j)-quasideterminant of A is defined by

$$|A|_{ij} = a_{ij} - \mathrm{r}_i{}^j \cdot \left(A^{ij}\right)^{-1} \cdot \mathrm{c}_j{}^i. \tag{1}$$

Example 2. For $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$,

$$|A|_{11} = a_{11} - a_{12} a_{22}^{-1} a_{21}, \qquad |A|_{12} = a_{12} - a_{11} a_{21}^{-1} a_{22},$$
$$|A|_{21} = a_{21} - a_{22} a_{12}^{-1} a_{11}, \qquad |A|_{22} = a_{22} - a_{21} a_{11}^{-1} a_{12}.$$

It is sometimes convenient to adopt the following more explicit notation

$$|A|_{11} = \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} - a_{12} a_{22}^{-1} a_{21}.$$

Remark 3. If the elements a_{ij} of the matrix A commute, then

$$|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}.$$

3.2. Inverse matrix and quasideterminant

Proposition 2. *If all $|A|_{ij}^{-1}$ exist, A^{-1} is given by*

$$A^{-1} = \left(|A|_{ji}^{-1}\right)_{1 \leqslant i,j \leqslant n}.$$

Example 4. For a quaternionic matrix $A = \left(\begin{smallmatrix} 1 & i \\ j & k \end{smallmatrix}\right)$,

$$|A|_{11}^{-1} = \left(1 - i \cdot k^{-1} j\right)^{-1} = (1 + ikj)^{-1} = \frac{1}{2},$$
$$|A|_{21}^{-1} = \left(j - k \cdot i^{-1} 1\right)^{-1} = (j + ki)^{-1} = (2j)^{-1} = -\frac{j}{2},$$
$$|A|_{12}^{-1} = \left(i - 1 \cdot j^{-1} k\right)^{-1} = (i + jk)^{-1} = (2i)^{-1} = -\frac{i}{2},$$
$$|A|_{22}^{-1} = \left(k - j \cdot 1^{-1} i\right)^{-1} = (k - ji)^{-1} = (2k)^{-1} = -\frac{k}{2}.$$

Therefore

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -j \\ -i & -k \end{pmatrix}.$$

Example 5. We can calculate quasideterminants inductively:

$$\begin{vmatrix} \boxed{a_{11}} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} - (a_{12}\ a_{13}) \begin{pmatrix} |A^{11}|_{22}^{-1} & |A^{11}|_{32}^{-1} \\ |A^{11}|_{23}^{-1} & |A^{11}|_{33}^{-1} \end{pmatrix} \begin{pmatrix} a_{21} \\ a_{31} \end{pmatrix}$$
$$= a_{11} - a_{12}\left(a_{22} - a_{23}a_{33}^{-1}a_{32}\right)^{-1} a_{21} - a_{12}\left(a_{32} - a_{33}a_{23}^{-1}a_{22}\right)^{-1} a_{31}$$
$$- a_{13}\left(a_{23} - a_{22}a_{32}^{-1}a_{33}\right)^{-1} a_{21} - a_{13}\left(a_{33} - a_{32}a_{22}^{-1}a_{23}\right)^{-1} a_{31}.$$

3.3. Homological relations

For $A = (a_{ij}) \in M(n, R)$, n^2 quasideterminants are defined. They are related by the so-called homological relations. For example,

$$\begin{vmatrix} a_{11} & a_{12} \\ \boxed{a_{21}} & a_{22} \end{vmatrix} = -a_{22}\, a_{12}^{-1} \begin{vmatrix} \boxed{a_{11}} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

In general, we have important identities as follows:

Proposition 3.

1. *Row homological relations*:

$$-|A|_{ij} \cdot \left|A^{il}\right|_{sj}^{-1} = |A|_{il} \cdot \left|A^{ij}\right|_{sl}^{-1}, \quad s \neq i.$$

2. *Column homological relations*:

$$-\left|A^{kj}\right|_{it}^{-1} \cdot |A|_{ij} = \left|A^{ij}\right|_{kt}^{-1} \cdot |A|_{kj}, \quad t \neq j.$$

3.4. The multiplication of rows and columns

Let B be the matrix obtained from the matrix A by multiplying the ith row by $\lambda \in R$, then

$$|B|_{kj} = \begin{cases} \lambda |A|_{ij} & \text{if } k = i, \\ |A|_{kj} & \text{if } k \neq i. \end{cases}$$

Let C be the matrix obtained from the matrix A by multiplying the jth column by $\mu \in R$C then

$$|C|_{il} = \begin{cases} |A|_{ij}\mu & \text{if } l = j, \\ |A|_{il} & \text{if } l \neq j. \end{cases} \tag{2}$$

Example 6.

$$\begin{vmatrix} a_{11} & \boxed{a_{12}\mu} \\ a_{21} & a_{22}\mu \end{vmatrix} = a_{12}\mu - a_{11}a_{21}^{-1}a_{22}\mu = |A|_{12}\mu,$$

$$\begin{vmatrix} a_{11}\mu & \boxed{a_{12}} \\ a_{21}\mu & a_{22} \end{vmatrix} = a_{12} - a_{11}\mu(a_{21}\mu)^{-1}a_{22} = |A|_{12}.$$

3.5. Sylvester's identity

Let $A = (a_{ij}) \in M(n, R)$ be a matrix and $A_0 = (a_{ij})$, $i, j = 1, \dots, k$, a submatrix of A that is invertible over R. For $p, q = k + 1, \dots, n$, set

$$c_{pq} = \begin{vmatrix} & & & a_{1q} \\ & A_0 & & \vdots \\ & & & a_{kq} \\ a_{p1} & \cdots & a_{pk} & \boxed{a_{pq}} \end{vmatrix}. \tag{3}$$

These quasideterminants are defined because matrix A_0 is invertible.

Consider the $(n-k) \times (n-k)$ matrix

$$C = (c_{pq}), \quad p, q = k + 1, \dots, n.$$

The submatrix A_0 is called the *pivot* for the matrix C.

Theorem 4 *(Sylvester's identity). For $i, j = k + 1, \dots, n$,*

$$|A|_{ij} = |C|_{ij}.$$

Example 7.

$$\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}_{ij} \quad (i, j = 2, 3).$$

Applying Theorem 4 with the $(1,1)$-entry 1 as a pivot, we put

$$c_{pq} = \begin{vmatrix} 1 & a_{1q} \\ 0 & a_{pq} \end{vmatrix}_{pq} = a_{pq} \quad (p, q = 2, 3)$$

and

$$\begin{vmatrix} 1 & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{vmatrix}_{ij} = |C|_{ij} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}_{ij} \quad (i, j = 2, 3).$$

4. Noncommutative version of the characteristic polynomial and the Cayley–Hamilton's theorem

In this section, we review the noncommutative Cayley–Hamilton's theorem in [5] shortly. We use notations $\Phi_i(\lambda)$, $C_{(i)j}$ instead of $Q_i(t)$, $L_j^{(i)}(A)$ in it. For $A = \left(\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{smallmatrix}\right)$, we denote $\Phi_1(\lambda), \Phi_2(\lambda)$ as two polynomials given by

$$\begin{aligned} \Phi_1(\lambda) &= \lambda^2 - \left(a_{11} + a_{12}a_{22}a_{12}^{-1}\right)\lambda + \left(a_{12}a_{22}a_{12}^{-1}a_{11} - a_{12}a_{21}\right) \\ &\equiv \lambda^2 - \mathrm{tr}_1(A)\lambda + \det{}_1(A), \\ \Phi_2(\lambda) &= \lambda^2 - \left(a_{22} + a_{21}a_{11}a_{21}^{-1}\right)\lambda + \left(a_{21}a_{11}a_{21}^{-1}a_{22} - a_{21}a_{12}\right) \\ &\equiv \lambda^2 - \mathrm{tr}_2(A)\lambda + \det{}_2(A). \end{aligned}$$

Then we can check the noncommutative Cayley–Hamilton's theorem for the generic matrix of order 2:

$$A^2 - \begin{pmatrix} \mathrm{tr}_1(A) & 0 \\ 0 & \mathrm{tr}_2(A) \end{pmatrix} A + \begin{pmatrix} \det{}_1(A) & 0 \\ 0 & \det{}_2(A) \end{pmatrix} = O.$$

The general result is as follows. We also give a simple proof.

Theorem 5. *(See [5].) For $A = (a_{ij}) \in M(n, R)$, we define a "noncommutative characteristic polynomial for the ith row" as follows*:

$$\begin{aligned} \Phi_i(\lambda) &= \begin{vmatrix} a_{i1}^{(n)} & a_{i2}^{(n)} & \cdots & a_{in}^{(n)} & \boxed{\lambda^n} \\ a_{i1}^{(n-1)} & a_{i2}^{(n-1)} & \cdots & a_{in}^{(n-1)} & \lambda^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ a_{i1}^{(1)} & a_{i2}^{(1)} & \cdots & a_{in}^{(1)} & \lambda \\ a_{i1}^{(0)} & a_{i2}^{(0)} & \cdots & a_{in}^{(0)} & 1 \end{vmatrix} \\ &\equiv \lambda^n - \sum_{k=1}^{n} C_{(i)k}\lambda^{n-k}, \end{aligned} \tag{4}$$

where $A^k = (a_{ij}^{(k)})$. *Then we have a noncommutative version of the Cayley–Hamilton theorem*

$$A^n - \sum_{k=1}^{n} \begin{pmatrix} C_{(1)k} & & & \\ & C_{(2)k} & & \\ & & \ddots & \\ & & & C_{(n)k} \end{pmatrix} A^{n-k} = O. \tag{5}$$

Proof. For unknown $C_{(i)k}$ $(i, k = 1, \dots, n)$, consider Eq. (5). Then the (i, j)-entry of (5) is

$$a_{ij}^{(n)} - \sum_{k=1}^{n} C_{(i)k} a_{ij}^{(n-k)} = 0 \quad (i, j = 1, \dots, n), \tag{6}$$

namely

$$(C_{(i)1}, \dots, C_{(i)n}) \begin{pmatrix} a_{i1}^{(n-1)} & \cdots & a_{in}^{(n-1)} \\ \vdots & & \vdots \\ a_{i1}^{(0)} & \cdots & a_{in}^{(0)} \end{pmatrix} = \left(a_{i1}^{(n)}, \dots, a_{in}^{(n)}\right). \tag{7}$$

Therefore we obtain $C_{(i)k}$ by solving the linear equations (7). Moreover, by using (6), the noncommutative characteristic polynomial for the ith row is written as

$$\begin{aligned} \Phi_i(\lambda) &= \begin{vmatrix} \sum_{k=1}^{n} C_{(i)k} a_{i1}^{(n-k)} & \cdots & \sum_{k=1}^{n} C_{(i)k} a_{in}^{(n-k)} & \boxed{\lambda^n} \\ a_{i1}^{(n-1)} & \cdots & a_{in}^{(n-1)} & \lambda^{n-1} \\ \vdots & & \vdots & \vdots \\ a_{i1}^{(0)} & \cdots & a_{in}^{(0)} & 1 \end{vmatrix} \\ &= \begin{vmatrix} 0 & \cdots & 0 & \boxed{\lambda^n - \sum_{k=1}^{n} C_{(i)k} \lambda^{n-k}} \\ a_{i1}^{(n-1)} & \cdots & a_{in}^{(n-1)} & \lambda^{n-1} \\ \vdots & & \vdots & \vdots \\ a_{i1}^{(0)} & \cdots & a_{in}^{(0)} & 1 \end{vmatrix} \\ &= \lambda^n - \sum_{k=1}^{n} C_{(i)k} \lambda^{n-k}. \qquad \square \end{aligned}$$

By this proof, we obtain an important corollary.

Corollary 6. *If an identity* (5) *holds, the noncommutative characteristic polynomials* $\Phi_i(\lambda)$ *defined by* (4) *are equal to* $\lambda^n - \sum_{k=1}^{n} C_{(i)k} \lambda^{n-k}$. *Especially, the* (*usual*) *Cayley–Hamilton theorem for* A (*i.e.* $C_{(i)k} = C_k$ *for all* i) *holds, then* $\Phi_i(\lambda)$ $(i = 1, \dots, n)$ *coincide with the usual characteristic polynomial* $\Phi(\lambda)$ *of* A.

Moreover, as a contraposition, we have the following:

Corollary 7. *For a given matrix A, if the noncommutative characteristic polynomials $\Phi_i(\lambda)$ are different for each i, then no commutative-Cayley–Hamilton-theorem type of identity with respect to A exists.*

Example 8. Let A be a matrix $A_q = (a_{ij})$ of the generators of the quantum group $GL_q(n)$, the noncommutative Cayley–Hamilton theorem (the quantum Cayley–Hamilton theorem) holds [5]. For example, $n = 2$, by using relations

$$\begin{aligned}
&a_{11}a_{22} - a_{22}a_{11} = \left(q^{-1} - q\right)a_{12}a_{21}, \quad a_{12}a_{22} = q^{-1}a_{22}a_{12},\\
&a_{11}a_{21} = q^{-1}a_{21}a_{11}, \quad a_{12}a_{21} = a_{21}a_{12},
\end{aligned}$$

we have

$$A_q^2 - \left(q^{1/2}a_{11} + q^{-1/2}a_{22}\right)\begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix} A_q + \left(a_{11}a_{22} - q^{-1}a_{12}a_{21}\right)\begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} = O.$$

However, the noncommutative characteristic polynomials for each row do not coincide each other:

$$\begin{aligned}
&\Phi_1(\lambda) = \lambda^2 - \left(a_{11} + q^{-1}a_{22}\right)\lambda + q^{-1}a_{11}a_{22} - q^{-2}a_{12}a_{21},\\
&\Phi_2(\lambda) = \lambda^2 - (qa_{11} + a_{22})\lambda + qa_{11}a_{22} - a_{12}a_{21}.
\end{aligned}$$

Therefore, there is no identity for A of commutative-Cayley–Hamilton-theorem type.

5. Noncommutative spectral decomposition

In this section, we develop a noncommutative analogue of the spectral decomposition with the quasideterminant. First, we review the Vandermonde quasideterminant and define a noncommutative analogue of the Lagrange interpolating polynomial. Next, we present the main theorem and our method of a noncommutative spectral decomposition. We also give a proof of the theorem by using properties of the quasideterminant prepared in Section 3.

5.1. Vandermonde quasideterminant

First, for $x_1, x_2, \ldots, x_k \in R$, the Vandermonde quasideterminant [4,5] is defined by

$$V(x_1, \ldots, x_k) = \begin{vmatrix} x_1^{k-1} & \cdots & \boxed{x_k^{k-1}} \\ & \cdots & \\ x_1 & \cdots & x_k \\ 1 & \cdots & 1 \end{vmatrix}.$$

Example 9.

$$V(x_1, x_2, z) = \begin{vmatrix} x_1^2 & x_2^2 & \boxed{z^2} \\ x_1 & x_2 & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$
\begin{aligned}
&= z^2 - \begin{pmatrix} x_1^2 & x_2^2 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z \\ 1 \end{pmatrix} \\
&= z^2 - \begin{pmatrix} x_1^2 & x_2^2 \end{pmatrix} \begin{pmatrix} (x_1 - x_2)^{-1} & (1 - x_2^{-1} x_1)^{-1} \\ (x_2 - x_1)^{-1} & (1 - x_1^{-1} x_2)^{-1} \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \\
&= z^2 + \begin{pmatrix} -x_1 - (x_2 - x_1) x_2 (x_2 - x_1)^{-1} & (x_2 - x_1) x_2 (x_2 - x_1)^{-1} x_1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \\
&= z^2 + \begin{pmatrix} -(y_1 + y_2) & y_2 y_1 \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} \\
&= z^2 - (y_1 + y_2) z + y_2 y_1
\end{aligned}
$$

where we put $y_1 = x_1$, $y_2 = (x_2 - x_1) x_2 (x_2 - x_1)^{-1}$. This is the noncommutative version of the relationship between solutions and coefficients for a (left) algebraic equation of degree 2 [4].

Remark 10. If $z = A = (a_{ij}) \in M(2, R)$ and $x_j = \begin{pmatrix} x_{(1)j} & \\ & x_{(2)j} \end{pmatrix}$ $(j = 1, 2)$, y_j $(j = 1, 2)$ are also diagonal matrices. Moreover, comparing the equation $V(x_1, x_2, A) = A^2 - (y_1 + y_2) A + y_2 y_1 = O$ with the noncommutative Cayley–Hamilton's theorem

$$
A^2 - \begin{pmatrix} C_{(1)1} & \\ & C_{(2)1} \end{pmatrix} A - \begin{pmatrix} C_{(1)2} & \\ & C_{(2)2} \end{pmatrix} = O,
$$

if $y_1 + y_2 = \begin{pmatrix} C_{(1)1} & \\ & C_{(2)1} \end{pmatrix}$ and $y_2 y_1 = -\begin{pmatrix} C_{(1)2} & \\ & C_{(2)2} \end{pmatrix}$, by the relationship between solutions and coefficients again, $x_{(i)1}, x_{(i)2}$ are the solutions of the noncommutative characteristic equation of A for the ith row.

For a given $z = A = (a_{ij}) \in M(n, R)$ and the equation $V(x_1, \ldots, x_n, A) = O$, diagonal components of diagonal matrices x_j are the solutions of the noncommutative characteristic equations of A in the same way.

5.2. Noncommutative Lagrange interpolating polynomial

For $x_1, \ldots, x_n \in R$, suppose that the inverse of the Vandermonde matrix $\begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}^{-1}$ exists. Then we define polynomials $f_i(z)$ $(i = 1, \ldots, n)$ with respect to $z \in R$ as follows.

Definition 8.

$$
\begin{pmatrix} f_1(z) \\ \vdots \\ f_n(z) \end{pmatrix} = \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} z^{n-1} \\ \vdots \\ 1 \end{pmatrix}.
$$

We call them noncommutative Lagrange interpolating polynomials.

Example 11. For $n=2$,

$$\begin{aligned}
f_1(z) &= \begin{vmatrix} \boxed{x_1} & x_2 \\ 1 & 1 \end{vmatrix}^{-1} z + \begin{vmatrix} x_1 & x_2 \\ \boxed{1} & 1 \end{vmatrix}^{-1} 1 \\
&= (x_1 - x_2)^{-1} z + \left(1 - x_2^{-1} x_1\right)^{-1} \\
&= (x_1 - x_2)^{-1} z + (x_2 - x_1)^{-1} x_2 \\
&= (x_1 - x_2)^{-1} (z - x_2), \\
f_2(z) &= \begin{vmatrix} x_1 & \boxed{x_2} \\ 1 & 1 \end{vmatrix}^{-1} z + \begin{vmatrix} x_1 & x_2 \\ 1 & \boxed{1} \end{vmatrix}^{-1} 1 \\
&= (x_2 - x_1)^{-1} z + \left(1 - x_1^{-1} x_2\right)^{-1} \\
&= (x_2 - x_1)^{-1} (z - x_1).
\end{aligned}$$

By the definition above, we obtain the following theorem.

Theorem 9. *For $x_1, \ldots, x_n,\ z \in R$, we have*

(1) $x_1^j f_1(z) + x_2^j f_2(z) + \cdots + x_n^j f_n(z) = z^j \ (j = 0, 1, \ldots, n-1)$,
(2) $f_i(x_j) = \delta_{ij}$.

5.3. Our method of noncommutative spectral decomposition

Theorem 10 *(Main Theorem). If given $z, x_1, \ldots, x_n \in R$ satisfy the equation $V(x_1, \ldots, x_n, z) = 0$, then we have the following identities*

$$V_m \equiv \begin{vmatrix} x_1^m & \cdots & x_n^m & \boxed{z^m} \\ x_1^{n-1} & \cdots & x_n^{n-1} & z^{n-1} \\ & \cdots & & \\ 1 & \cdots & 1 & 1 \end{vmatrix} = 0 \quad (m = 0, \ldots, n, n+1, \ldots). \tag{8}$$

Rewriting (8), by the definition of noncommutative Lagrange interpolating polynomials

$$\begin{aligned}
z^m &= \begin{pmatrix} x_1^m & \cdots & x_n^m \end{pmatrix} \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}^{-1} \begin{pmatrix} z^{n-1} \\ \vdots \\ 1 \end{pmatrix} \\
&= x_1^m f_1(z) + \cdots + x_n^m f_n(z),
\end{aligned}$$

then we have the noncommutative spectral decomposition of z

$$z^m = x_1^m f_1(z) + \cdots + x_n^m f_n(z) \quad (m = 0, 1, \ldots).$$

In particular, if z is a matrix $A = (a_{ij}) \in M(n, R)$, put $x_1, \ldots, x_n$ as unknown diagonal matrices and solve the equation

$$V(x_1, \ldots, x_n, A) = O.$$

By Remark 10, this equation is nothing but the noncommutative Cayley–Hamilton's theorem and the diagonal components of diagonal matrices x_j are the solutions of the noncommutative characteristic equations of A. Therefore, by using the solutions of them, we obtain the noncommutative spectral decomposition of A

$$A^m = x_1^m f_1(A) + \cdots + x_n^m f_n(A) \quad (m = 0, 1, \ldots).$$

5.4. A proof of Main Theorem 10

Proof. In case of $m = 0, 1, \ldots, n-1$, the identity (8) is trivial. If $m = n$, (8) is nothing but $V(x_1, \ldots, x_n, z) = 0$. In the following, we suppose $m = n+1, n+2, \ldots$.

Consider a matrix A and the submatrix A_0 defined by

$$A = \left(\begin{array}{ccc|cc} x_1^m & \cdots & x_n^m & 0 & z^m \\ x_1^n & \cdots & x_n^n & 0 & z^n \\ \hline x_1^{n-1} & \cdots & x_n^{n-1} & 0 & z^{n-1} \\ & \cdots & & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 \end{array}\right), \qquad A_0 = \begin{pmatrix} x_1^{n-1} & \cdots & x_n^{n-1} \\ & \cdots & \\ 1 & \cdots & 1 \end{pmatrix}.$$

For $p = 1, 2$, $q = n+1, n+2$, we put a matrix $C = (c_{pq})$ which entries are quasideterminants with A_0 as a pivot like (3) (note that quasideterminants are unchanged under permutations of rows or columns) and we remark

$$c_{1,n+2} = \begin{vmatrix} x_1^m & \cdots & x_n^m & \boxed{z^m} \\ & & & z^{n-1} \\ & A_0 & & \vdots \\ & & & 1 \end{vmatrix} = V_m,$$

$$c_{2,n+2} = \begin{vmatrix} x_1^n & \cdots & x_n^n & \boxed{z^n} \\ & & & z^{n-1} \\ & A_0 & & \vdots \\ & & & 1 \end{vmatrix} = V(x_1, \ldots, x_n, z) = V_n.$$

Then by the Sylvester's identity (Theorem 4), we have

$$\begin{aligned} |A|_{1,n+2} = |C|_{1,n+2} &= \begin{vmatrix} c_{1,n+1} & \boxed{c_{1,n+2}} \\ c_{2,n+1} & c_{2,n+2} \end{vmatrix} \\ &= c_{1,n+2} - c_{1,n+1} c_{2,n+1}^{-1} c_{2,n+2} \\ &= V_m - c_{1,n+1} c_{2,n+1}^{-1} V_n. \end{aligned}$$

On the other hand, since

$$|A|_{1,n+2} = \left|\begin{array}{ccc|cc} x_1^m & \cdots & x_n^m & 0 & \boxed{z^m} \\ x_1^n & \cdots & x_n^n & 0 & z^n \\ \hline x_1^{n-1} & \cdots & x_n^{n-1} & 0 & z^{n-1} \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 1 \end{array}\right| = \left|\begin{array}{cccc|c} x_1^m & \cdots & x_n^m & \boxed{z^m} & 0 \\ x_1^n & \cdots & x_n^n & z^n & 0 \\ x_1^{n-1} & \cdots & x_n^{n-1} & z^{n-1} & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ \hline 1 & \cdots & 1 & 1 & 1 \end{array}\right|$$

$$= \begin{vmatrix} x_1^m & \cdots & x_n^m & \boxed{z^m} \\ x_1^n & \cdots & x_n^n & z^n \\ x_1^{n-1} & \cdots & x_n^{n-1} & z^{n-1} \\ \vdots & \cdots & \vdots & \vdots \\ x_1 & \cdots & x_n & z \end{vmatrix} \quad \left(\begin{array}{c}\text{by the Sylvester's identity with} \\ (n+2, n+2)\text{-entry 1 as a pivot}\end{array}\right)$$

$$= \begin{vmatrix} x_1^{m-1} & \cdots & x_n^{m-1} & \boxed{z^{m-1}} \\ x_1^{n-1} & \cdots & x_n^{n-1} & z^{n-1} \\ x_1^{n-2} & \cdots & x_n^{n-2} & z^{n-2} \\ \vdots & \cdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 \end{vmatrix} z \quad \bigl(\text{by the property (2)}\bigr)$$

$$= V_{m-1} z,$$

we obtain the important identities

$$V_{m-1} z = V_m - c_{1,n+1} c_{2,n+1}^{-1} V_n. \tag{9}$$

Therefore $V_n = 0$ implies $V_m = 0$ $(m = n+1, n+2, \ldots)$ by the mathematical induction. □

Remark 12. In particular, for $n = 2$,

$$V_m = z^m - \begin{pmatrix} x_1^m & x_2^m \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} z \\ 1 \end{pmatrix} = z^m - \bigl(x_1^m f_1(z) + x_2^m f_2(z)\bigr).$$

Then the identity (9) is

$$\begin{aligned} &\bigl\{z^{m-1} - \bigl(x_1^{m-1} f_1(z) + x_2^{m-1} f_2(z)\bigr)\bigr\} z \\ &\quad = z^m - \bigl(x_1^m f_1(z) + x_2^m f_2(z)\bigr) \\ &\qquad - \bigl(x_2^{m-1} - x_1^{m-1}\bigr)(x_2 - x_1)^{-1} \bigl\{z^2 - \bigl(x_1^2 f_1(z) + x_2^2 f_2(z)\bigr)\bigr\} \quad (m = 2, 3, \ldots). \end{aligned} \tag{10}$$

6. Examples of noncommutative spectral decomposition and the exponential matrices

In this section, we apply our method to a quaternionic matrix and a matrix with entries being harmonic oscillators. As a result, we obtain the noncommutative spectral decomposition and the exponential matrices of them explicitly.

6.1. Quaternionic matrices

As a quaternionic matrix, we consider an element A of Lie algebra $sp(2)$:

$$A = \begin{pmatrix} i & j \\ j & -i \end{pmatrix}.$$

We apply our method to A and calculate the spectral decomposition and the exponential matrix $\exp tA$ explicitly. First, from

$$A^2 = \begin{pmatrix} -2 & 2k \\ -2k & -2 \end{pmatrix},$$

noncommutative characteristic equations for each row are

$$\Phi_1(\lambda) = \begin{vmatrix} -2 & 2k & \boxed{\lambda^2} \\ i & j & \lambda \\ 1 & 0 & 1 \end{vmatrix} = \lambda^2 - 2i\lambda = 0,$$

then $\lambda = 0, 2i$, and

$$\Phi_2(\lambda) = \begin{vmatrix} -2k & -2 & \boxed{\lambda^2} \\ j & -i & \lambda \\ 0 & 1 & 1 \end{vmatrix} = \lambda^2 + 2i\lambda = 0,$$

then $\lambda = 0, -2i$.

Next, in the noncommutative Lagrange interpolating polynomials

$$f_1(z) = (x_1 - x_2)^{-1}(z - x_2), \qquad f_2(z) = (x_2 - x_1)^{-1}(z - x_1),$$

we put $z = A$, $x_1 = \left(\begin{smallmatrix} 2i & \\ & -2i \end{smallmatrix}\right)$, $x_2 = \left(\begin{smallmatrix} 0 & \\ & 0 \end{smallmatrix}\right)$, then

$$P_1 = f_1(A) = \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}^{-1} A = \frac{1}{2}\begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix},$$

$$P_2 = f_2(A) = \left\{\begin{pmatrix} 2i & \\ & -2i \end{pmatrix}\right\}^{-1} \left\{A - \begin{pmatrix} 2i & \\ & -2i \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix}.$$

We can check $P_i^2 = P_i$, $P_i P_j = 0$ $(i \neq j)$ easily and we obtain

$$\begin{aligned} \exp tA &= (\exp tx_1)P_1 + (\exp tx_2)P_2 \\ &= \begin{pmatrix} e^{2it} & \\ & e^{-2it} \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & -k \\ k & 1 \end{pmatrix} + \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1 & k \\ -k & 1 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} e^{2it} + 1 & -e^{2it}k + k \\ e^{-2it}k - k & e^{-2it} + 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{it}\cos t & e^{it}\sin tj \\ e^{-it}\sin tj & e^{-it}\cos t \end{pmatrix} \in Sp(2). \end{aligned}$$

Remark 13. If we put

$$x_1 = \begin{pmatrix} 2i & \\ & 0 \end{pmatrix}, \qquad x_2 = \begin{pmatrix} 0 & \\ & -2i \end{pmatrix},$$

we have $[A, x_1] \neq O$, $[A, x_2] \neq O$ and $f_1(A)$, $f_2(A)$ are *not* projection matrices. Nevertheless, by Theorem 10, we have $A^m = x_1^m f_1(A) + x_2^m f_2(A)$ and we can calculate $\exp tA$ explicitly. This result is derived from the fact that Theorem 10 not depends on the ordering of solutions for noncommutative characteristic equations for each row.

6.2. *A matrix with entries being harmonic oscillators*

Let $a, a^\dagger$ be the generator of the harmonic oscillator. The relation is $[a, a^\dagger] = 1$. We also denote N as the number operator $N = a^\dagger a$.

We consider a matrix $A = \sqrt{2}\begin{pmatrix} 0 & a & 0 \\ a^\dagger & 0 & a \\ 0 & a^\dagger & 0 \end{pmatrix}$. This matrix is related to a Hamiltonian of a model in quantum optics [3]. So, it is important to calculate the exponential of A as the time-evolution operator of the Hamiltonian.

From

$$A^2 = 2\begin{pmatrix} N+1 & 0 & a^2 \\ 0 & 2N+1 & 0 \\ (a^\dagger)^2 & 0 & N \end{pmatrix}, \qquad A^3 = 2\sqrt{2}\begin{pmatrix} 0 & (2N+3)a & 0 \\ (2N+1)a^\dagger & 0 & (2N+1)a \\ 0 & (2N-1)a^\dagger & 0 \end{pmatrix},$$

the noncommutative characteristic equations for each row are

$$\Phi_1(\lambda) = \begin{vmatrix} 0 & 2\sqrt{2}(2N+3)a & 0 & \boxed{\lambda^3} \\ 2(N+1) & 0 & 2a^2 & \lambda^2 \\ 0 & \sqrt{2}a & 0 & \lambda \\ 1 & 0 & 0 & 1 \end{vmatrix} = \lambda^3 - 2(2N+3)\lambda = 0,$$

then $\lambda = \pm\sqrt{2(2N+3)}, 0$, and

$$\Phi_3(\lambda) = \begin{vmatrix} 0 & 2\sqrt{2}(2N-1)a^\dagger & 0 & \boxed{\lambda^3} \\ 2(a^\dagger)^2 & 0 & 2N & \lambda^2 \\ 0 & \sqrt{2}a^\dagger & 0 & \lambda \\ 0 & 0 & 1 & 1 \end{vmatrix} = \lambda^3 - 2(2N-1)\lambda = 0,$$

then $\lambda = \pm\sqrt{2(2N-1)}, 0$.

Remark 14. For the second row, the quasideterminant

$$\Phi_2(\lambda) = \begin{vmatrix} 2\sqrt{2}(2N+1)a^\dagger & 0 & 2\sqrt{2}(2N+1)a & \boxed{\lambda^3} \\ 0 & 2(2N+1) & 0 & \lambda^2 \\ \sqrt{2}a^\dagger & 0 & \sqrt{2}a & \lambda \\ 0 & 1 & 0 & 1 \end{vmatrix}$$

is not defined because "rank" of the matrix $\begin{pmatrix} 0 & 2(2N+1) & 0 \\ \sqrt{2}a^\dagger & 0 & \sqrt{2}a \\ 0 & 1 & 0 \end{pmatrix}$ is 2 (on "rank of $A \in M(n, R)$," see [4]). Then, we put

$$A^3 + UA^2 + VA + W = 0, \quad U, V, W \text{ are diagonal matrices}$$

and simplify them, then the second row is

$$\lambda^3 + u\lambda^2 - 2(2N+1)\lambda - 2u(2N+1) = 0 \quad (\text{for arbitrary } u).$$

Therefore, if we put $u = 0$, we obtain $\lambda = \pm\sqrt{2(2N+1)}, 0$.

Next, we calculate the noncommutative Lagrange interpolating polynomials

$$\begin{aligned} f_1(z) &= \begin{vmatrix} \boxed{x_1^2} & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}^{-1} z^2 + \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ \boxed{x_1} & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}^{-1} z + \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ \boxed{1} & 1 & 1 \end{vmatrix}^{-1} 1 \\ &= \big(x_1^2 - x_2^2(x_2 - x_3)^{-1}x_1 - x_3^2(x_3 - x_2)^{-1}x_1 \\ &\quad - x_2^2(x_3 - x_2)^{-1}x_3 - x_3^2(x_2 - x_3)^{-1}x_2\big)^{-1} z^2 \\ &\quad + \big(x_1 - x_2\big(x_2^2 - x_3^2\big)^{-1}x_1^2 - x_3\big(x_3^2 - x_2^2\big)^{-1}x_1^2 \\ &\quad - x_2\big(x_3^2 - x_2^2\big)^{-1}x_3^2 - x_3\big(x_2^2 - x_3^2\big)^{-1}x_2^2\big)^{-1} z \\ &\quad + \big(1 - \big(x_2^2 - x_3x_2\big)^{-1}x_1^2 - \big(x_3^2 - x_2x_3\big)^{-1}x_1^2 \\ &\quad - \big(x_2 - x_3^{-1}x_2^2\big)^{-1}x_1 - \big(x_3 - x_2^{-1}x_3^2\big)^{-1}x_1\big)^{-1}. \end{aligned}$$

In particular, in the case of $x_1 = x$, $x_2 = 0$, $x_3 = -x$,

$$\begin{aligned} f_1(z) &= \big(x^2 - (-x)^2(-x)^{-1}x\big)^{-1} z^2 + \big(x - (-x)(-x)^{-2}x^2\big)^{-1} z + 0 \\ &= \big(2x^2\big)^{-1} z^2 + (2x)^{-1} z, \end{aligned}$$

where the last term of $f_1(z)$ is calculated by using the homological relation as follows:

$$\begin{aligned} \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ x_1 & x_2 & x_3 \\ \boxed{1} & 1 & 1 \end{vmatrix}^{-1} &= -\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ \boxed{x_1} & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}^{-1} \begin{vmatrix} x_2^2 & x_3^2 \\ \boxed{x_2} & x_3 \end{vmatrix} \begin{vmatrix} x_2^2 & x_3^2 \\ \boxed{1} & 1 \end{vmatrix}^{-1} \\ &= -\begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ \boxed{x_1} & x_2 & x_3 \\ 1 & 1 & 1 \end{vmatrix}^{-1} \big(x_2 - x_3^{-1}x_2^2\big)\big(1 - x_3^{-2}x_2^2\big)^{-1} \\ &\to -(2x)^{-1} \cdot 0 \cdot 1 = 0 \quad (x_1 \to x,\ x_2 \to 0,\ x_3 \to -x). \end{aligned}$$

Then we put

$$z=A,\quad x=\begin{pmatrix}\lambda(N)&&\\&\lambda(N-1)&\\&&\lambda(N-2)\end{pmatrix},\quad \lambda(N)=\sqrt{2(2N+3)},$$

we have

$$\begin{aligned}
P_1&=f_1(A)\\
&=\bigl(2x^2\bigr)^{-1}A^2+(2x)^{-1}A\\
&=\begin{pmatrix}(2\lambda(N))^{-2}&&\\&(2\lambda(N-1))^{-2}&\\&&(2\lambda(N-2))^{-2}\end{pmatrix}\cdot 2\begin{pmatrix}N+1&0&a^2\\0&2N+1&0\\(a^\dagger)^2&0&N\end{pmatrix}\\
&\quad+\begin{pmatrix}(2\lambda(N))^{-1}&&\\&(2\lambda(N-1))^{-1}&\\&&(2\lambda(N-2))^{-1}\end{pmatrix}\cdot\sqrt{2}\begin{pmatrix}0&a&0\\a^\dagger&0&a\\0&a^\dagger&0\end{pmatrix}\\
&=\begin{pmatrix}\frac{1}{2(2N+3)}&&\\&\frac{1}{2(2N+1)}&\\&&\frac{1}{2(2N-1)}\end{pmatrix}\begin{pmatrix}N+1&0&a^2\\0&2N+1&0\\(a^\dagger)^2&0&N\end{pmatrix}\\
&\quad+\begin{pmatrix}\frac{1}{2\sqrt{2N+3}}&&\\&\frac{1}{2\sqrt{2N+1}}&\\&&\frac{1}{2\sqrt{2N-1}}\end{pmatrix}\begin{pmatrix}0&a&0\\a^\dagger&0&a\\0&a^\dagger&0\end{pmatrix}\\
&=\begin{pmatrix}\frac{N+1}{2(2N+3)}&\frac{1}{2\sqrt{2N+3}}a&\frac{1}{2(2N+3)}a^2\\\frac{1}{2\sqrt{2N+1}}a^\dagger&\frac{1}{2}&\frac{1}{2\sqrt{2N+1}}a\\\frac{1}{2(2N-1)}(a^\dagger)^2&\frac{1}{2\sqrt{2N-1}}a^\dagger&\frac{N}{2(2N-1)}\end{pmatrix}.
\end{aligned}$$

In the same manner, if we put $z=A$, $x_1=x$, $x_2=0$, $x_3=-x$ in $f_2(z)$, $f_3(z)$, then we have

$$\begin{aligned}
P_2&=f_2(A)\\
&=\bigl(-x^2\bigr)^{-1}A^2+I_3\\
&=-\begin{pmatrix}\frac{1}{2(2N+3)}&&\\&\frac{1}{2(2N+1)}&\\&&\frac{1}{2(2N-1)}\end{pmatrix}\cdot 2\begin{pmatrix}N+1&0&a^2\\0&2N+1&0\\(a^\dagger)^2&0&N\end{pmatrix}+\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}\\
&=\begin{pmatrix}\frac{N+2}{2N+3}&0&-\frac{1}{2N+3}a^2\\0&0&0\\-\frac{1}{2N-1}(a^\dagger)^2&0&\frac{N-1}{2N-1}\end{pmatrix},\\
P_3&=f_3(A)=\bigl(2x^2\bigr)^{-1}A^2-(2x)^{-1}A\\
&=\begin{pmatrix}\frac{1}{2(2N+3)}&&\\&\frac{1}{2(2N+1)}&\\&&\frac{1}{2(2N-1)}\end{pmatrix}\begin{pmatrix}N+1&0&a^2\\0&2N+1&0\\(a^\dagger)^2&0&N\end{pmatrix}
\end{aligned}$$

$$
-\begin{pmatrix} \frac{1}{2\sqrt{2N+3}} & & \\ & \frac{1}{2\sqrt{2N+1}} & \\ & & \frac{1}{2\sqrt{2N-1}} \end{pmatrix}\begin{pmatrix} 0 & a & 0 \\ a^\dagger & 0 & a \\ 0 & a^\dagger & 0 \end{pmatrix}
$$

$$
=\begin{pmatrix} \frac{N+1}{2(2N+3)} & -\frac{1}{2\sqrt{2N+3}}a & \frac{1}{2(2N+3)}a^2 \\ -\frac{1}{2\sqrt{2N+1}}a^\dagger & \frac{1}{2} & -\frac{1}{2\sqrt{2N+1}}a \\ \frac{1}{2(2N-1)}(a^\dagger)^2 & -\frac{1}{2\sqrt{2N-1}}a^\dagger & \frac{N}{2(2N-1)} \end{pmatrix}.
$$

We can check $P_i^2 = P_i$, $P_i P_j = 0$ $(i \neq j)$ and for a constant g,

$$
\begin{aligned}
&\exp(-itgA) \\
&\quad = \exp(-itgx)P_1 + (\exp 0)P_2 + \exp(itgx)P_3 \\
&\quad = \begin{pmatrix} \frac{N+2+(N+1)\cos(tg\lambda(N))}{2N+3} & -i\frac{1}{\sqrt{2N+3}}\sin(tg\lambda(N))a & \frac{1}{2N+3}(-1+\cos(tg\lambda(N)))a^2 \\ -i\frac{1}{\sqrt{2N+1}}\sin(tg\lambda(N-1))a^\dagger & \cos(tg\lambda(N-1)) & -i\frac{1}{\sqrt{2N+1}}\sin(tg\lambda(N-1))a \\ \frac{1}{2N-1}(-1+\cos(tg\lambda(N-2)))(a^\dagger)^2 & -i\frac{1}{\sqrt{2N-1}}\sin(tg\lambda(N-2))a^\dagger & \frac{N-1+N\cos(tg\lambda(N-2))}{2N-1} \end{pmatrix}.
\end{aligned}
$$

Remark 15. For this A, by using “the quantum diagonalization method” [3],

$$
\begin{aligned}
&\begin{pmatrix} 1 & & \\ & a\frac{1}{\sqrt{N}} & \\ & & a^2\frac{1}{\sqrt{N(N-1)}} \end{pmatrix} A \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{N}}a^\dagger & \\ & & \frac{1}{\sqrt{N(N-1)}}(a^\dagger)^2 \end{pmatrix} \\
&\quad = \sqrt{2}\begin{pmatrix} 0 & \sqrt{N+1} & 0 \\ \sqrt{N+1} & 0 & \sqrt{N+2} \\ 0 & \sqrt{N+2} & 0 \end{pmatrix}.
\end{aligned}
$$

Since the matrix on the right-hand side has only commutative entries, we calculate the characteristic equation as usual, then

$$
\lambda^3 - 2(2N+3)\lambda = 0, \quad \lambda = 0, \pm\sqrt{2(2N+3)}.
$$

We remark that the result of $\exp(-itgA)$ in [3] and the explicit form of it with our noncommutative spectral decomposition described in this section coincide.

7. Discussion

In this paper, we developed a noncommutative version of the spectral decomposition with the quasideterminant and calculated some interesting examples. In particular, we defined a noncommutative analogue of the Lagrange interpolating polynomials and applied to the systematic method for constructing projection matrices with noncommutative entries.

Our method is very powerful to calculate a function of a matrix with noncommutative entries and is expected to apply for the theory of noncommutative geometry, quantum physics, and so on. A study of other applications with our theory is in progress.

Acknowledgments

The author is very grateful to Kazuyuki Fujii for helpful comments on an earlier draft on this paper and to Masashi Hamanaka and Hideyuki Ishi for helpful suggestion.

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