

Congruence ABC implies ABC

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ABSTRACT

The ABC conjecture of Masser and Oesterlé states that if (a, b, c) are coprime integers with $a + b + c = 0$, then $\sup(|a|, |b|, |c|) < c_\epsilon (\text{rad}(abc))^{1+\epsilon}$ for any $\epsilon > 0$. In [2], Oesterlé observes that if the ABC conjecture holds for all (a, b, c) with $16|abc$, then the full ABC conjecture holds. We extend that result to show that, for every integer N , the ‘congruence ABC conjecture’ that ABC holds for all (a, b, c) with $N|abc$ implies the full ABC conjecture.

1. INTRODUCTION

The ABC conjecture was introduced by Masser and Oesterlé in 1985, and has since been shown to be related to many other conjectures, especially conjectures regarding the arithmetic of elliptic curves [1].

For our purposes, an *ABC-solution* \mathbf{s} is a triple (a, b, c) of distinct relatively prime integers satisfying $a + b + c = 0$, and such that a and b are negative. (The requirement that the integers be distinct is included only to simplify the exposition below.) If $n > 0$ is an integer, the radical $\text{rad}(n)$ is defined to be the product of all primes dividing n .

For any $\epsilon > 0$, we define a function on ABC-solutions by

$$f(\mathbf{s}, \epsilon) = \log(c) - (1 + \epsilon) \log \text{rad}(abc).$$

Then the ABC conjecture can be phrased as follows:

Conjecture 1 (ABC conjecture). For each $\epsilon > 0$, there exists a constant C_ϵ such that

$$f(\mathbf{s}, \epsilon) < C_\epsilon$$

for all \mathbf{s} .

In [2], Oesterlé showed that the ABC conjecture is equivalent to a conjecture of Szpiro on elliptic curves ([2, Conj. 4]). In the proof, he observes that if the ABC conjecture is known to hold for all (a, b, c) with $16|abc$, then the full ABC conjecture can be shown to hold. This suggests considering a family of weaker conjectures indexed by integers N , as follows:

Conjecture 2 (Congruence ABC conjecture for N). For each $\epsilon > 0$, there exists a constant C_ϵ such that

$$f(\mathbf{s}, \epsilon) < C_\epsilon$$

for all \mathbf{s} such that $N|abc$.

It has long been known to experts that the congruence ABC conjecture for any N is equivalent to the full ABC conjecture. However, a proof has never to our knowledge appeared in the literature, and we take the opportunity to provide one in this note.

2. CONGRUENCE ABC IMPLIES ABC

Theorem 3. *The congruence ABC conjecture for N implies the ABC conjecture.*

Proof. For each positive even integer n , we define an operation Θ_n on ABC-solutions as follows. Let $\mathbf{s} = (a, b, c)$ be an ABC-solution. Then

$$\Theta_n(\mathbf{s}) = (-2^{-m}(a-b)^n, -2^{-m}[c^n - (a-b)^n], 2^{-m}c^n)$$

where $m = n$ if c is even, and $m = 0$ otherwise. Then $\Theta_n(\mathbf{s})$ is again an ABC-solution.

Lemma 4. *There exist constants $c_{n,\epsilon} > 0$ and $c'_{n,\epsilon}$ such that*

$$f(\Theta_n(\mathbf{s}), \epsilon/[n + (n-1)\epsilon]) \geq c_{n,\epsilon} f(\mathbf{s}, \epsilon) + c'_{n,\epsilon}.$$

Proof. Let $A = -2^{-m}(a-b)^n$, $B = 2^{-m}(-c^n + (a-b)^n)$, $C = 2^{-m}c^n$. Then

$$\log \text{rad}(ABC) \leq \log |a-b| + \log \text{rad}(abc) + \log \text{rad}(B/ab).$$

Now

$$\begin{aligned} B/ab &= \frac{(a+b)^n - (a-b)^n}{ab} \\ &= 4[(a+b)^{n-2} + (a+b)^{n-4}(a-b)^2 + \dots + (a-b)^{n-2}] \\ &\leq 2n(a+b)^{n-2}. \end{aligned}$$

So

$$\begin{aligned}
 \log \text{rad}(ABC) &\leq \log |a - b| + \log \text{rad}(abc) + (n - 2) \log c + \log 2n \\
 &\leq (n - 1) \log c + \log \text{rad}(abc) + \log 2n \\
 &= (n - 1) \log c + (1 + \epsilon)^{-1} (\log c - f(\mathbf{s}, \epsilon)) + \log 2n \\
 &= n \log c - \epsilon (1 + \epsilon)^{-1} \log c - (1 + \epsilon)^{-1} f(\mathbf{s}, \epsilon) + \log 2n \\
 &= (1 - \epsilon [n(1 + \epsilon)]^{-1}) (\log C + m \log 2) \\
 &\quad - (1 + \epsilon)^{-1} f(\mathbf{s}, \epsilon) + \log 2n.
 \end{aligned}$$

It follows that

$$\log C \geq \frac{n(1 + \epsilon)}{n + n\epsilon - \epsilon} (\log \text{rad}(ABC) + (1 + \epsilon)^{-1} f(\mathbf{s}, \epsilon) - \log 2n) - n \log 2.$$

Thus

$$f\left(\Theta_n(\mathbf{s}), \frac{\epsilon}{n + n\epsilon - \epsilon}\right) \geq c_{n,\epsilon} f(\mathbf{s}, \epsilon) + c'_{n,\epsilon}$$

where

$$c_{n,\epsilon} = \frac{n}{n + n\epsilon - \epsilon}$$

and

$$c'_{n,\epsilon} = \frac{-n(1 + \epsilon)(\log 2n)}{n + n\epsilon - \epsilon} - n \log 2. \quad \square$$

We now proceed with the proof of Theorem 3. Assume that the congruence ABC conjecture for N is true. Then there exists C_ϵ such that

$$f(\mathbf{s}, \epsilon) < C_\epsilon$$

for all \mathbf{s} with $N|abc$.

Let $n = \phi(N)$. If $N = 2$, Theorem 3 is trivial. We may therefore assume that n is even.

Lemma 5. *Let $(A, B, C) = \Theta_n(\mathbf{s})$. Then $N|ABC$.*

Proof. Suppose p is an odd prime dividing N , and let p^ν be the largest power of p dividing N . Then $(p - 1)p^{\nu-1}|n$. In particular, $\nu < n$. If p divides c or $a - b$, then $p^\nu|p^n|ABC$. If p divides neither c nor $a - b$, then $-A = 2^{-m}(a - b)^n$ and $C = 2^{-m}c^n$ are congruent mod p^ν ; therefore, $p^\nu|B$.

Now let 2^ν be the largest power of 2 dividing N . If c is even, then so is $a - b$, and exactly one of c and $a - b$ is a multiple of 4. Thus, one of $(a - b)^n$ and c^n is a multiple of 4^n . Since, in this case, $A = 2^{-n}(a - b)^n$ and $C = 2^{-n}c^n$, one of A and C is a multiple of 2^n , whence also of 2^ν . If, on the other hand, c is odd, then so is $a - b$. Then $(a - b)^n$ and c^n are both congruent to 1 mod 2^ν , so $2^\nu|B$. \square

For any $\epsilon > 0$, it now follows that

$$f(\mathbf{s}, \epsilon) \leq c_{n,\epsilon}^{-1} \left(f \left(\Theta_n(\mathbf{s}), \frac{\epsilon}{n + n\epsilon - \epsilon} \right) - c'_{n,\epsilon} \right) \leq c_{n,\epsilon}^{-1} (C_{\epsilon/|n+n\epsilon-\epsilon|} - c'_{n,\epsilon})$$

for any ABC-solution \mathbf{s} . Since the right hand side depends only on ϵ and N , this proves the full ABC conjecture. \square

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REFERENCES

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