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Rings of square stable range one

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ABSTRACT

A ring S is said to have square stable range one (written $\mathrm{ssr}(S) = 1$) if $aS + bS = S$ implies that $a^2 + bx$ is a unit for some $x \in S$. In the commutative case, this extends the class of rings of stable range 1, and allows many new examples such as rings of real-valued continuous functions, and real holomorphy rings. On the other hand, $\mathrm{ssr}(S) = 1$ sometimes forces S to have stable range 1. For instance, this is the case for exchange rings S, for which $\mathrm{ssr}(S) = 1$ is characterized by $S/\mathrm{rad}\, S$ being reduced (or abelian, or right quasi-duo). We also characterize rings S whose (von Neumann) regular elements are strongly regular, by using an element-wise notion of square stable range one. Extending a result of Estes and Ohm, we show that a possibly noncommutative infinite domain with stable range one or square stable range one must have a non-artinian group of units.

1. Introduction

After Bass's study [Ba] of the (injective and surjective) stability properties of the functor K_1, the notion of the stable range of rings has attracted a lot of attention. Early papers written on this topic include, for instance, [EO,Va$_1$,Va$_2$]. The case of stable range one turned out to be especially popular, perhaps because it is a Morita invariant property, and has various applications to the problem of cancellation and substitution of modules. The relevant literature here includes [GM,Ca,Yu$_1$,CY,Ar,Ch$_1$, Ch$_2$,KL], among many others; for a partial survey, see [La$_5$].

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Our goal in this paper is to study certain algebraic variants of the notion of stable range 1. Recall that a ring S is said to have stable range 1 (written $\mathrm{sr}(S) = 1$) if $aS + bS = S$ implies that, for some $x \in S$, $a + bx \in \mathrm{U}(S)$ (the unit group of S). It is well known (albeit nontrivial) that this property is left–right symmetric. Our goal in this paper is to study a variant of the stable range 1 property, which we call "square stable range 1". A bit surprisingly, the path leading to the discovery of this notion emerged in linear algebra (instead of ring theory or K-theory). In considering the problem of factoring the matrix $\left(\begin{smallmatrix} b & 0 \\ a & 0 \end{smallmatrix}\right)$ into a product of two Toeplitz matrices, Khurana, Lam, and Shomron were led to ask for units of the form $a^2 + bx$, given that $aS + bS = S$. This prompted the following definition.

Definition 1.1. A ring S is said to have (right) square stable range 1 (written $\mathrm{ssr}(S) = 1$) if, for any $a, b \in S$, $aS + bS = S$ implies that $a^2 + bx \in \mathrm{U}(S)$ for some $x \in S$.

The square stable range 1 property has some, but not all, of the well-known features of stable range 1. For instance, both properties force the ring S in question to be Dedekind-finite, and are inherited by corner rings, factor rings, and power series rings. These facts are developed in Section 2 of this paper. For commutative rings S, $\mathrm{sr}(S) = 1 \Rightarrow \mathrm{ssr}(S) = 1$, but not conversely in general. Indeed, one thing that makes the notion $\mathrm{ssr}(S) = 1$ interesting and worthwhile is that it allows for many new examples, including rings of continuous real-valued functions on topological spaces, and real holomorphy rings in formally real fields. The latter class leads to many examples of commutative finite-dimensional Prüfer domains S with $\mathrm{ssr}(S) = 1 \neq \mathrm{sr}(S)$, as we'll see in Section 3. Indeed, many of these new examples have the stronger property that $\sum_i a_i S = S \Rightarrow \sum_i a_i^2 \in \mathrm{U}(S)$. This "Artin–Schreier Property" (named in honor of the authors of the seminal paper [AS]) turns out to be left–right symmetric, according to Theorem 4.2.

Section 4 begins with the key result that a ring S with $\mathrm{ssr}(S) = 1$ must be right quasi-duo; that is, maximal right ideals of S are always ideals. This fact implies that a matrix ring $\mathbb{M}_n(S_0)$ cannot have square stable range 1 (if $n > 1$ and $S_0 \neq 0$). For a (von Neumann) regular ring S, the same fact shows that $\mathrm{ssr}(S) = 1$ iff S is strongly regular,[2] while it is classically well known that $\mathrm{sr}(S) = 1$ iff S is unit-regular (see [Go$_2$, (4.12)]). Thus, for regular rings S, $\mathrm{ssr}(S) = 1$ turned out to be *stronger than* $\mathrm{sr}(S) = 1$. This prompted us to consider the more general case of exchange rings. In Theorem 4.4, we show that, for any exchange ring S, $\mathrm{ssr}(S) = 1$ iff S is right quasi-duo, iff $S/\mathrm{rad}\, S$ is a reduced ring, and these properties imply that $\mathrm{sr}(S) = 1$. Moreover, any abelian[3] exchange ring S has $\mathrm{ssr}(S) = 1$: this generalizes earlier results of Chen [Ch$_1$] and Yu [Yu$_1$] on such rings.

In Section 5, we reexamine the idea of square stable range 1 from an "element-wise" point of view, defining an element $a \in S$ to have $\mathrm{ssr}(a) = 1$ if $aS + bS = S$ (for any $b \in S$) implies that $a^2 + bx \in \mathrm{U}(S)$ for some $x \in S$. This is shown to be closely related to the notion of *strongly regular elements* in S (that is, elements $a \in S$ such that $a \in a^2 S \cap Sa^2$). Prompted by the work in [KL], we show in Theorem 5.4 that regular elements in S are strongly regular iff $\mathrm{ssr}(a) = 1$ for all regular $a \in S$. For *exchange rings* S, the former property turns out to be another characterization for $\mathrm{ssr}(S) = 1$. This provides an interesting parallel to an earlier result of Camillo and Yu [CY], which states that $\mathrm{sr}(S) = 1$ iff regular elements in S are unit-regular.

In the last two sections of the paper, we study the unit groups of rings with (square) stable range 1. This was first done by Estes and Ohm in the commutative case for $\mathrm{sr}(S) = 1$. Generalizing their work in [EO], we show in Theorem 6.8 that, if S is a possibly noncommutative infinite domain with $\mathrm{sr}(S) = 1$ or $\mathrm{ssr}(S) = 1$, then the group of units $\mathrm{U}(S)$ is *non-artinian* (and hence infinite). The proof of this ultimately rests on an analysis of $D^* = \mathrm{U}(D)$ for an infinite division ring D. In this case, a combination of methods from number theory and field theory shows (in (7.4)) that D^* actually contains an infinite direct sum of nontrivial cyclic groups. In the preparation work for proving the above results, we also obtain a new characterization for strongly π-regular rings: a ring S is strongly

[2] A ring S is called *strongly regular* if $a \in a^2 S$ for all $a \in S$. This notion came from the work of Arens and Kaplansky on the topological representation of algebras; see [AK].

[3] A ring is called *abelian* if its idempotents are all central.

π-regular iff $\mathrm{sr}(S) = 1$ and all chains of the form $K(aS) \supseteq K(a^2S) \supseteq \cdots$ stabilize, where for any $x \in S$, $K(xS) := \mathrm{U}(S) \cap (1 + xS) \subseteq \mathrm{U}(S)$.

Throughout this paper, S denotes an arbitrary ring with unity, and $\mathrm{rad}(S)$ denotes the Jacobson radical of S. The notions and notations introduced earlier in this section will be used freely in the rest of the paper. Standard facts in ring theory used without mention in the text can be found in [Go$_2$, La$_2$,La$_4$]. In Section 3, the *level* of a commutative ring S is defined to be the smallest integer n such that -1 is a sum of n squares in S. If no such n exists, S is said to be a *semireal ring* (following [La$_7$]), and its level is taken to be the symbol ∞. (A semireal field is classically known as a *formally real* field, after Artin and Schreier [AS]; see [La$_6$, Chapter 8].)

2. Basic properties of square stable range one

In studying the problem of decomposing a 2×2 matrix over a ring S into the product of two Toeplitz matrices, Khurana, Lam, and Shomron [KLS] came across the following three equivalent conditions on the ring S:

(A) $aS + bS = S \Rightarrow a^2 + bx \in \mathrm{U}(S)$ *for some* $x \in S$.
(B) $aS + bS = S \Rightarrow a^2 + (a + b)y \in \mathrm{U}(S)$ *for some* $y \in S$.
(C) $aS + bS = S \Rightarrow ab + (a + b)z \in \mathrm{U}(S)$ *for some* $z \in S$.

The equivalence proof (given in [KLS, (3.4)]) is easy, and works for any (possibly noncommutative) ring S. In view of the resemblance of condition (A) to the standard stable range one condition for S (denoted in Section 1 by $\mathrm{sr}(S) = 1$), we introduce the following central definition for this paper.

Definition 2.0. If the condition (A) above is satisfied, we say that S *has* (*right*) *square stable range* 1, and write $\mathrm{ssr}(S) = 1$.

While it is well known that $\mathrm{sr}(S) = 1$ is a left–right symmetric notion (see, e.g. [Va$_2$]), we have not been able to prove the same result for $\mathrm{ssr}(S) = 1$. Thus, the notation $\mathrm{ssr}(S) = 1$ in the rest of the paper should be taken to mean strictly "right square stable range 1" in the sense of Definition 2.0. However, for certain classes of rings (e.g. exchange rings), we'll be able to show that "left" and "right" square stable range one are equivalent.

We can easily express $\mathrm{ssr}(S) = 1$ in a first-order statement on S as follows.

Proposition 2.1. *For any ring S, $\mathrm{ssr}(S) = 1$ iff, for any $a, s \in S$, there exists $x \in S$ such that $a^2 + (1 - as)x \in \mathrm{U}(S)$.* (E.g., choosing $a = 2$, $s = 4$ shows that $\mathrm{ssr}(\mathbb{Z}) \neq 1$.)

Proof. First suppose $\mathrm{ssr}(S) = 1$, and let $a, s \in S$. We have obviously $aS + (1 - as)S = S$, so there exists $x \in S$ such that $a^2 + (1 - as)x \in \mathrm{U}(S)$. Conversely, suppose the condition in the proposition is satisfied, and consider any equation $aS + bS = S$. Then there exist $r, s \in S$ such that $as + br = 1$. By assumption, there exists $x \in S$ such that $a^2 + (1 - as)x \in \mathrm{U}(S)$. This means that $a^2 + b(rx) \in \mathrm{U}(S)$, checking that $\mathrm{ssr}(S) = 1$. (Of course, a similar result holds for the condition $\mathrm{sr}(S) = 1$.) $\square$

Remark 2.2. Professor K. Goodearl pointed out to us that, *for commutative rings S*, $\mathrm{ssr}(S) = 1$ is related to the notion of power-substitution studied in [Go$_1$]. Indeed, for such S, the condition $\mathrm{ssr}(S) = 1$ (interpreted in the light of Proposition 2.1) is easily seen to be equivalent to the following matrix-theoretic property (where $a, b, s \in S$):

$$(\dagger) \qquad as + b = 1 \quad \Rightarrow \quad aI_2 + bX \in \mathrm{GL}_2(S) \quad \text{for some } X \in \mathbb{M}_2(S).$$

(See [Go$_1$, (3.2)].) Thus, by [Go$_1$, (2.1)], $\mathrm{ssr}(S) = 1$ (for S commutative) is equivalent to the "power-substitution property for $n = 2$" for right modules A (over any ring k) with endomorphism ring S. In particular, if $\mathrm{ssr}(S) = 1$, then A has the following "power-cancellation property": if $A \oplus B \cong A \oplus C$

for right k-modules B, C, then $B \oplus B \cong C \oplus C$. For a much sharper version of this remark, see the Epilogue (after Section 7).

The first result connecting $\mathrm{ssr}(S) = 1$ to $\mathrm{sr}(S) = 1$ is through the use of the notion of right quasi-duo rings.

Theorem 2.3. *If S is a right quasi-duo ring (that is, every maximal right ideal in S is an ideal), and* $\mathrm{sr}(S) = 1$, *then* $\mathrm{ssr}(S) = 1$.

Proof. Given any equation $aS + bS = S$, we claim that $a^2S + bS = S$. Indeed, if this is not the case, then there exists a maximal right ideal $\mathfrak{m} \supseteq a^2S + bS$. Since S is right quasi-duo, $\mathfrak{m}$ is an ideal, and $S/\mathfrak{m}$ is a division ring. But then $a^2 \in \mathfrak{m}$ implies that $a \in \mathfrak{m}$. Thus, $\mathfrak{m}$ contains $aS + bS = S$, a contradiction. Having proved that $a^2S + bS = S$, the assumption that $\mathrm{sr}(S) = 1$ implies that $a^2 + bx \in \mathrm{U}(S)$ for some $x \in S$. This shows that $\mathrm{ssr}(S) = 1$. □

We will show later that Theorem 2.3 has actually a partial converse: namely, *if* $\mathrm{ssr}(S) = 1$, *then S must be right quasi-duo.* (In particular, for rings of stable range 1, $\mathrm{ssr}(S) = 1$ would be equivalent to S being right quasi-duo.) This converse part will appear later in Theorem 4.1(1). At the moment, from Theorem 2.3, we have the following corollary giving the first good supply of classes of rings S with $\mathrm{ssr}(S) = 1$. (As was recalled in the Introduction, a ring S is called *strongly regular* if $b \in b^2S$ for all $b \in S$.)

Corollary 2.4. *If S is a commutative ring with* $\mathrm{sr}(S) = 1$, *or a local ring (e.g. a division ring), or a strongly regular ring (e.g. a reduced algebraic algebra over a field k), then* $\mathrm{ssr}(S) = 1$.

Proof. All rings listed above are right quasi-duo, and they have stable range 1, so Theorem 2.3 applies. (The fact that strongly regular rings have these properties is proved in [La$_4$]; see Exercises (12.6C), (20.10D), and (22.4B). The fact that reduced algebraic k-algebras are strongly regular is due to Jacobson, Arens, and Kaplansky: a proof of this can be found in [La$_4$, Exercise 12.6B].) □

For general rings S, however, the two properties $\mathrm{sr}(S) = 1$ and $\mathrm{ssr}(S) = 1$ turn out to be logically independent. This important point will be illustrated by the three key Examples 2.5, 2.6 and 2.7 below.

Example 2.5. Let S be a ring with $aS + bS = S$ holding for some $a, b \in S$ such that $a^2 = 0$ and b is not right-invertible. Then $\mathrm{ssr}(S) \neq 1$, for otherwise there would exist a unit of the form $a^2 + bx = bx$ (for some $x \in S$), which would have implied that b is right-invertible. For a concrete example, let $S = \mathbb{M}_2(F)$ (where F is any field), and let a, b be the matrix units E_{12} and E_{21} respectively. The desired properties on a, b above are easily seen to be all satisfied, so we must have $\mathrm{ssr}(S) \neq 1$. However, S is a semilocal ring, so $\mathrm{sr}(S) = 1$ by [La$_2$, (20.9)]. The idea of this construction can be sharpened into the formulation of some strong *necessary conditions* for rings of square stable range 1: this will be done in Theorem 4.1 below.

Example 2.6. The ring used in the above example is not reduced. However, even if S is a (noncommutative) domain, it is possible that $\mathrm{sr}(S) = 1$, but $\mathrm{ssr}(S) \neq 1$. To see this, we can utilize an example of Fuller and Shutters. In [FS], these authors have constructed a noncommutative semilocal domain S such that $S/\mathrm{rad}\, S \cong \mathbb{M}_2(F)$ where F is a field. Since $\mathrm{sr}(\mathbb{M}_2(F)) = 1$, we have also $\mathrm{sr}(S) = 1$. However, we have shown in Example 2.5 above that $\mathrm{ssr}(\mathbb{M}_2(F)) \neq 1$. From this, we see easily that $\mathrm{ssr}(S) \neq 1$. For more details on this, see Proposition 2.10(1) below.

Example 2.7. Let $S = C[0, 3]$ be the ring of real-valued continuous functions on the interval $[0, 3]$. In Section 3, we'll show that $\mathrm{ssr}(S) = 1$. However, $\mathrm{sr}(S) \neq 1$. To see this, recall that a (possibly noncommutative) ring S of stable range 1 always has the (right) "unique-generator property"; that is, whenever $fS = gS$, we have $f = gu$ for some $u \in \mathrm{U}(S)$ (see [Ca, (4.5)], or [KL, (6.3)]). For $S = C[0, 3]$, Kaplansky has noted in [Ka, p. 466] that S *does not* have the unique-generator property. Thus, $\mathrm{sr}(S) \neq 1$. Another perhaps more direct way to see this is to work instead with $S = C[-1, 1]$. If we use Vaserstein's unimodular row $(x, 1 - x^2)$ (where the two coordinates are viewed as polynomial functions on $[-1, 1]$), then for any $h \in C[-1, 1]$, the function $H := x + h(1 - x^2)$ has values 1 at $x = 1$, and -1 at $x = -1$. Thus, H has a zero value somewhere on $[-1, 1]$, and so $H \notin \mathrm{U}(S)$. This shows once more that $\mathrm{sr}(S) \neq 1$.

In the next few results, we'll develop a number of basic properties of rings of square stable range 1. These are analogues of some of the best known properties of rings of stable range 1, though in general it would be a mistake to think that every such property has a valid analogue. To begin with, the very desirable Dedekind-finite property of rings of stable range 1 does carry over to the case of $\mathrm{ssr}(S) = 1$. Proceeding somewhat gingerly, we'll formulate this idea via Theorem 2.8 and its Corollary 2.9 below.

Theorem 2.8. *Suppose that, for all $a, b \in S$, $aS + bS = S \Rightarrow (a^2 + bx)S = S$ for some $x \in S$. Then S is Dedekind-finite.*

Proof. Suppose $ua = 1 \in S$. We would like to prove that $a \in \mathrm{U}(S)$. As in the proof of Proposition 2.1, we have $aS + (1 - au)S = S$, so by assumption, $v := a^2 + (1 - au)x$ is right-invertible (i.e. it has a right inverse) for some $x \in S$. Then

$$uv = ua^2 + u(1 - au)x = a + ux - ux = a,$$

so $u^2 v = ua = 1$. This shows that v is also left-invertible. Thus, $v \in \mathrm{U}(S)$, and so $u^2 = v^{-1} \in \mathrm{U}(S)$. It follows that $u \in \mathrm{U}(S)$, and hence also $a \in \mathrm{U}(S)$, as desired. $\square$

Corollary 2.9. *To show that* $\mathrm{ssr}(S) = 1$ *for a ring S, it suffices to check that the condition in the above theorem holds.*

Proof. Suppose that condition holds. If $aS + bS = S$, then for some x, $a^2 + bx$ is right-invertible. By Theorem 2.8 above, $a^2 + bx \in \mathrm{U}(S)$. This checks that $\mathrm{ssr}(S) = 1$. $\square$

Proposition 2.10.

(1) *If* $\mathrm{ssr}(S) = 1$, *then for any ideal $J \subseteq S$,* $\mathrm{ssr}(S/J) = 1$.
(2) *If $S = \prod_i S_i$, then* $\mathrm{ssr}(S) = 1$ *iff* $\mathrm{ssr}(S_i) = 1$ *for all i.*

Proof. (2) is clear. For (1), let $\bar{S} = S/J$, and assume that $\bar{a}\bar{S} + \bar{b}\bar{S} = \bar{S}$. For some $r, s \in S$, we have $as + br = 1 + j$ where $j \in J$. Since $\mathrm{ssr}(S) = 1$, $aS + (br - j)S = S$ implies that $a^2 + (br - j)x \in \mathrm{U}(S)$ for some $x \in S$. Passing to $\bar{S}$ yields $\bar{a}^2 + \bar{b}(\bar{r}\bar{x}) \in \mathrm{U}(\bar{S})$. This checks that $\mathrm{ssr}(\bar{S}) = 1$, as desired. $\square$

Theorem 2.11.

(1) *For any ideal $J \subseteq \mathrm{rad}\, S$,* $\mathrm{ssr}(S) = 1$ *iff* $\mathrm{ssr}(S/J) = 1$.
(2) *If $R = S[[x]]$, then* $\mathrm{ssr}(R) = 1$ *iff* $\mathrm{ssr}(S) = 1$.
(3) *If $E = \mathrm{End}(M_k)$ where M_k is a uniserial module over any ring k, then* $\mathrm{ssr}(E) = 1$.

Proof. (1) follows easily from the observation that $u \in \mathrm{U}(S) \Leftrightarrow \bar{u} \in \mathrm{U}(S/J)$ (under the assumption that $J \subseteq \mathrm{rad}\, S$). Next, (2) follows from (1), since the ideal $(x) \subseteq R = S[[x]]$ is in $\mathrm{rad}(R)$, and $R/(x) \cong S$. Finally, let E be as in (3). By a theorem of Facchini [Fa] (see also [La_2, (20.15)]), $E/\mathrm{rad}(E)$ is either a division ring or a direct product of two division rings. Therefore, by (1) again (plus Corollary 2.4), $\mathrm{ssr}(E) = 1$. $\square$

Our next theorem gives an analogue of Vaserstein's result on the passage of stable range one to corner rings [Va_2, (2.8)]. A slight adaptation of Vaserstein's proof enables us to get a similar result in the case of square stable range 1.

Theorem 2.12. *If* $\mathrm{ssr}(S) = 1$, *then for any idempotent* $e \in S$, *we have* $\mathrm{ssr}(eSe) = 1$.

Proof. Let $S' = eSe$, and let $f = 1 - e$. Suppose $a, b \in S'$ are such that $aS' + bS' = S'$; say $as + br = e$, where $r, s \in S'$. Then $(a + f)S + bS$ contains both $(a + f)s + br = e$ and $(a + f)f = f$, so it contains 1. Since $\mathrm{ssr}(S) = 1$, there exists $x \in S$ such that $u := (a + f)^2 + bx = a^2 + f + bx \in \mathrm{U}(S)$. Also, $v := 1 - bxf \in \mathrm{U}(S)$ (with inverse $1 + bxf$). Letting $w := vu = a^2 + f + bx - bxf = a^2 + f + bxe \in \mathrm{U}(S)$, we have

$$S' = e(wS)e = e\big(a^2 + f + bxe\big)Se = \big(a^2 + bxe\big)eSe = \big(a^2 + b(exe)\big)S'. \tag{2.13}$$

According to Corollary 2.9, this is enough to show that $\mathrm{ssr}(S') = 1$. (Actually, the use of Corollary 2.9 is not essential here, since a calculation similar to the one above also shows that $S' = S'(a^2 + b(exe))$, so $a^2 + b(exe) \in \mathrm{U}(S')$.) □

In [Ch_1], a ring S is said to have (right) *idempotent stable range* 1 (written $\mathrm{isr}(S) = 1$) if $aS + bS = S \Rightarrow a + be \in \mathrm{U}(S)$ for some idempotent $e \in S$. According to Chen (see [Ch_1, Corollary 6]), this condition is *left–right symmetric.* As recalled in footnote 2, a ring S is *abelian* if all idempotents in S are central. For example, any reduced ring (and hence any domain) is abelian; see [La_4, p. 187]. From Example 2.6, we know that, *if S is abelian,* $\mathrm{sr}(S) = 1$ *need not imply that* $\mathrm{ssr}(S) = 1$. In the following result, we'll show, however, that this implication will hold if the condition $\mathrm{sr}(S) = 1$ is strengthened to $\mathrm{isr}(S) = 1$.

Theorem 2.14. *Let S be an abelian ring with* $\mathrm{isr}(S) = 1$. *Then S is a clean ring*; *that is, every element of S is the sum of a unit and an idempotent. Moreover, S is left and right quasi-duo, and S has left and right square stable range* 1.

Proof. To make the proof self-contained, we'll interpret $\mathrm{isr}(S) = 1$ as meaning "left idempotent stable range 1". For any $a \in S$, the equation $Sa + S(-1) = S$ implies that $u := a + e(-1) \in \mathrm{U}(S)$ for some idempotent $e \in S$. Thus, $a = u + e$, proving that S is clean. (The abelian assumption on S is not needed here.) It remains now to prove that S is right quasi-duo, and that $\mathrm{ssr}(S) = 1$ (since the rest will follow from Chen's result on the left–right symmetry of $\mathrm{isr}(S) = 1$). For any $x, y \in S$, we have $S(yx - 1) + Sx = S$, so the assumption that S has left idempotent stable range 1 implies that $yx - 1 + ex \in \mathrm{U}(S)$ for some idempotent $e \in S$. Since S is abelian, $ex = xe$. Thus, we have $(yx - 1)S + xS = S$. As this holds for all $x, y \in S$, [LD, Theorem 3.2] implies that S is right quasi-duo. On the other hand, S having left idempotent stable range 1 obviously implies that $\mathrm{sr}(S) = 1$. Thus, Theorem 2.3 gives $\mathrm{ssr}(S) = 1$, as desired. □

We conclude this section with the following strengthening of a result of Estes and Ohm [EO] on the stable range of rings of algebraic integers.

Theorem 2.15. *Let S be the full ring of algebraic integers in a finite number field K. Then* $\mathrm{ssr}(S) \neq 1$ (*so in particular* $\mathrm{sr}(S) \neq 1$).

Proof. The proof here is an adaptation of the argument originally given in [EO, (7.6)]. Let S_0 be an integrally closed domain with quotient field K_0, and let S be the integral closure of S_0 in a finite separable extension K of K_0. Following [EO], one can show that there exists an integer n (which can be taken to be $[K' : K_0]$ where K' is the normal hull of K over K_0), such that, for any $a, b \in S_0$ with

$aS_0 + bS_0 = S_0$ and $a^2 + bs \in \mathrm{U}(S)$ for some $s \in S$, there exists $s_0 \in S_0$ such that $a^{2n} + bs_0 \in \mathrm{U}(S_0)$.[4] Given this fact, let $S_0 = \mathbb{Z}$, $K_0 = \mathbb{Q}$, and let S and K be as in the statement of Theorem 2.14. After coming up with the integer n as above, let $a = 2$, and let b be any odd integer not dividing $2^{2n} \pm 1$. Then $aS_0 + bS_0 = S_0$ implies that $aS + bS = S$, but the choice of n and a, b above guarantees that there is no $s \in S$ such that $a^2 + bs \in \mathrm{U}(S)$. This shows that $\mathrm{ssr}(S) \neq 1$. □

Remark 2.16. In the result above, it is essential to assume that K is a *finite* number field. If K was the field of all algebraic numbers, the ring of all algebraic integers in K is known to have stable range 1 (and hence also square stable range 1): see [Va$_2$, (1.2)].

3. Artin–Schreier rings

In the class of (possibly noncommutative) rings with square stable range 1, there are some natural subclasses of rings that are defined by a sequence of curious (but highly "symmetrical") strengthenings of Definition 2.0. These subclasses of rings of square stable range 1 will be introduced and discussed in this section, and various illustrative examples will be given to put them in perspective. We thank Professor R.G. Swan for his kind input into the formulation of the results in this section. The acronym "AS" below stands for "Artin–Schreier", and is chosen to honor the seminal work [AS] of these authors in 1927 on the role of sums of squares in abstract algebra.

Definition 3.0. For a given integer $n \geqslant 0$, we say that a ring S has the AS_n property (or simply S is AS_n)[5] if, for any $a_0, \ldots, a_n \in S$, $\sum_{i=0}^{n} a_i S = S \Rightarrow \sum_{i=0}^{n} a_i^2 \in \mathrm{U}(S)$. If S is AS_n for all n, we simply say that it is AS (or an Artin–Schreier ring).

Clearly, AS_0 rings are just the Dedekind-finite rings. Also, if $n \geqslant 1$, any AS_n ring is AS_{n-1}, *and it has always square stable range* 1 since, in the condition (A) of Definition 2.0, we can simply choose x to be b. However, if $(S, \mathfrak{m})$ is a local ring with $2 \in \mathfrak{m}$, then $\mathrm{sr}(S) = \mathrm{ssr}(S) = 1$, but S is not AS_n for any $n \geqslant 1$.

Note that the nature of the condition $\sum_{i=0}^{n} a_i^2 \in \mathrm{U}(S)$ in Definition 3.0 often makes the AS_n notion easier to work with than the condition $\mathrm{ssr}(S) = 1$. The routine proofs for the three properties of AS_n rings in Proposition 3.1 below can be safely omitted.

Proposition 3.1.

(1) *Let $\{S_i\}$ be a family of AS_n subrings of a ring R. Then the subring $S := \bigcap_i S_i \subseteq R$ is also AS_n.*
(2) *If R is an AS_n ring, and $T \subseteq R$ is a full subring (that is, a subring such that $T \cap \mathrm{U}(R) \subseteq \mathrm{U}(T)$), then T is also AS_n.*
(3) *The center of an AS_n ring is always AS_n. The same facts also hold for AS rings.*

Of course, these facts cannot be expected to have analogues for rings of (square) stable range 1. For instance, $\mathbb{Z}$ is the intersection of its localizations at all maximal ideals, which all have (square) stable range 1, but $\mathrm{sr}(\mathbb{Z}) \neq 1 \neq \mathrm{ssr}(\mathbb{Z})$.

In the case of *commutative* rings, it turns out that Definition 3.0 can be substantially "simplified". We'll make this point explicit by proving the following result.

Theorem 3.2. *Let $n \geqslant 1$ be a fixed integer. For any commutative ring S, the condition that "(1) S is AS_n" is equivalent to each of the following*:

[4] The proof given for Proposition 7.6 in [EO] was based on the unstated assumption that $b \neq 0$. But if $b = 0$, then $a \in \mathrm{U}(S_0)$, in which case the desired conclusion is true for *any* choice of $s_0 \in S_0$.

[5] Again, a more proper term to use here would have been "right AS_n". But in any case, we shall be able to prove later that "right AS_n" is the same as "left AS_n": see Theorem 4.2.

(2) $1 + x_1^2 + \cdots + x_n^2 \in \mathrm{U}(S)$ *for all* $x_i \in S$.
(3) *For each maximal ideal* $\mathfrak{m} \subseteq S$*, the field* $S/\mathfrak{m}$ *has level* $> n$.

In particular, S is AS *iff each* $S/\mathfrak{m}$ *above is a formally real field.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are both clear. To prove (3) $\Rightarrow$ (1), assume that there is a unimodular sequence $(a_0, \ldots, a_n)$ such that $a_0^2 + \cdots + a_n^2 \notin \mathrm{U}(S)$. Then, by Zorn's Lemma, this element lies in some maximal ideal $\mathfrak{m}$. Since some $a_i \notin \mathfrak{m}$, working modulo $\mathfrak{m}$ shows that -1 is a sum of n squares in $S/\mathfrak{m}$; that is, $S/\mathfrak{m}$ has level $\leqslant n$. $\square$

The following two remarks about AS_1 are pertinent toward giving some perspective on the general AS_n conditions.

Remark 3.3. Let S be a (not necessarily commutative) ring given with an involution $*$. The condition "$1 + x^*x \in \mathrm{U}(S)$ for every $x \in S$" is known as the G–N (Gelfand–Naimark) property in functional analysis. (In some books, e.g. [Be, (1.33)], this is also referred to as the "symmetric" property.) Most importantly, any C*-algebra with identity has this property. In Theorem 3.2 above where S is a commutative ring, we can take the involution $*$ to be the identity map. In this case, AS_1 is exactly the G–N property for $(S, *)$. The more general property "$1 + x_1^*x_1 + \cdots + x_n^*x_n \in \mathrm{U}(S)$ for all $x_i \in S$" has also shown up in functional analysis and linear algebra.

Remark 3.4. For $n = 1$, the implication (2) $\Rightarrow$ (1) for commutative rings in Theorem 3.2 can also be seen directly (without using Zorn's Lemma) as follows. Assuming (2) (for $n = 1$), consider any equation $aS + bS = S$. Writing $ar + bs = 1$ for some r, s and using the 2-square identity, we have

$$\left(a^2 + b^2\right)\left(r^2 + s^2\right) = (ar + bs)^2 + (as - br)^2 = 1 + (as - br)^2 \in \mathrm{U}(S).$$

Thus, $a^2 + b^2 \in \mathrm{U}(S)$, so S is AS_1. (The same argument works for AS_3 and AS_7.) However, in the *noncommutative* case, (2) $\Rightarrow$ (1) in Theorem 3.2 no longer holds, even for $n = 1$. For instance, there exist quaternion division algebras S (over suitable fields) such that $1 + x_1^2 \in \mathrm{U}(S) = S \setminus \{0\}$ for all x_1, but there exist nonzero elements $a, b \in S$ such that $a^2 + b^2 = 0$; see, e.g. [OV, Corollary 3.5(b)].

We record the following two easy consequences of Theorem 3.2. The first one follows from the fact that the level of a field is always a power of 2 or ∞ (see, e.g. [La$_6$, XI.2.2]), and the second one follows by a routine localization argument.

Corollary 3.5. *Let S be a commutative ring. If* $0 \leqslant i < r = 2^k$*, then S is* AS_r *iff it is* AS_{r+i}*. In particular, for commutative rings, it suffices to work with the notion of* AS_{2^k} *rings*: *these are the commutative rings S for which the level of* $S/\mathfrak{m}$ *is* $\geqslant 2^{k+1}$ *for each maximal ideal* $\mathfrak{m} \subseteq S$.

Corollary 3.6. *Let R be a semireal commutative ring* (i.e. *R has infinite level*)*, and let S be the ring obtained from R by inverting* $1 + x_1^2 + \cdots + x_n^2$ *for all* x_i*'s and all n. Then S is an Artin–Schreier ring*; *in particular,* $\mathrm{ssr}(S) = 1$.

Returning to noncommutative rings for the moment, note that a good number of the facts on rings S with $\mathrm{ssr}(S) = 1$ carry over to AS_1 rings, with only slightly modified proofs. We leave it to the reader to check that the following results all fall within this category: Example 2.5, Example 2.6, Theorem 2.8, Corollary 2.9, Proposition 2.10(2), Theorem 2.12, and the first two parts of Theorem 2.11. As for Proposition 2.10(1) (passage to factor rings), the situation is more subtle. To make it work for AS_1 rings, some extra assumption is needed on S, as follows.

Proposition 2.10′. *Let* $J \subseteq S$ *be an ideal, and assume that either* S/J *is commutative or* $\mathrm{sr}(S) = 1$*. If S is* AS_1*, then so is* $\bar{S} := S/J$.

Proof. If $\bar{S}$ is commutative, the desired conclusion follows easily from Theorem 3.2. In the general case, assume instead that $\mathrm{sr}(S) = 1$, and start with any equation $\bar{a}\bar{S} + \bar{b}\bar{S} = \bar{S}$. For some $r, s \in S$, we have $ar + bs + j = 1$ where $j \in J$. Since $\mathrm{sr}(S) = 1$, there exists $t \in S$ such that $a + (bs + j)t \in \mathrm{U}(S)$. As S is AS_1, $(a + jt) + bst \in \mathrm{U}(S)$ implies that $(a + jt)^2 + b^2 \in \mathrm{U}(S)$. Passing to $\bar{S} = S/J$ then gives $\bar{a}^2 + \bar{b}^2 \in \mathrm{U}(\bar{S})$. This shows that $\bar{S}$ is AS_1. □

For the rest of this section, *we shall restrict our attention to commutative rings.* Using Theorem 3.2, we can produce many examples of AS_n Prüfer domains by exploiting a result of A. Dress [Dr] (see also [La$_1$, (11.4)]). In particular, we get many examples of Artin–Schreier Prüfer domains (all of which have square stable range 1).

Theorem 3.7. *For a fixed integer $n \geqslant 1$, let F be any field of level $> n$, and let*

$$\Sigma_n = \left\{ \left(1 + x_1^2 + \cdots + x_n^2\right)^{-1} :\ x_i \in F \right\} \subseteq F. \tag{3.8}$$

(1) *Any subring $S \subseteq F$ containing Σ_n is an AS_n Prüfer domain with quotient field F.*
(2) *A valuation ring V of F is AS_n iff $V \supseteq \Sigma_n$.*
(3) *Let $\{(V_i, \mathfrak{m}_i)\}$ be any family of valuation rings of F such that $V_i/\mathfrak{m}_i$ has level $> n$. Then $H := \bigcap_i V_i$ is an AS_n Prüfer domain with quotient field F.*

Proof. (1) Since Σ_n contains $(1 + x^2)^{-1}$ for all $x \in F$, the fact that S is a Prüfer domain with quotient field F follows from [Dr, p. 390]. Thus, it only remains to show that S is AS_n. For any $x_1, \ldots, x_n \in S$, we have both $s := 1 + x_1^2 + \cdots + x_n^2 \in S$ and $s^{-1} \in \Sigma_n \subseteq S$. This implies that $s \in \mathrm{U}(S)$, so S is AS_n according to Theorem 3.2.

(2) In view of (1), it suffices to prove the "only if" part, so assume that V is AS_n. For any $x_1, \ldots, x_n \in F$, we want to show that the inverse of $s := 1 + x_1^2 + \cdots + x_n^2$ is in V. After a reindexing, we may assume that $x_i/x_1 \in V$ for all i. If $x_1 \in V$, then all $x_i \in V$, and hence $s \in \mathrm{U}(V)$ (since V is AS_n). In this case, certainly $s^{-1} \in V$. Now assume $x_1 \notin V$. Then $1/x_1 \in V$ (since V is a valuation ring of F), and so

$$v := s/x_1^2 = (1/x_1)^2 + 1 + (x_2/x_1)^2 + \cdots + (x_n/x_1)^2 \in \mathrm{U}(V). \tag{3.9}$$

Thus, $s^{-1} = (1/x_1)^2 v^{-1} \in V \cdot V \subseteq V$, as desired.

(3) By Theorem 3.2, each V_i is AS_n. Thus, by (2) above, we must have $\Sigma_n \subseteq V_i$. This implies that $\Sigma_n \subseteq H$, so by (1), H is an AS_n Prüfer domain with quotient field F. (Of course, the fact that H is AS_n also follows from Proposition 3.1(1).) □

Example 3.10. An important special case of the construction in Theorem 3.7(3) is as follows. Let F be any formally real field, and let $\{(V_i, \mathfrak{m}_i)\}$ be the family of *all* valuation rings of F such that the residue field $V_i/\mathfrak{m}_i$ is formally real. The intersection $\mathcal{H}(F) := \bigcap_i V_i$ in this case is known as the *real holomorphy ring* of F (see [La$_1$, Section 9]). By Theorem 3.7(3) above (applied for all n), $\mathcal{H}(F)$ is an Artin–Schreier Prüfer domain with quotient field F. In particular, $\mathrm{ssr}(\mathcal{H}(F)) = 1$.

Next, we'll show that combining Theorem 3.7 with a result of Swan leads to interesting examples of commutative domains S that are AS (so that $\mathrm{ssr}(S) = 1$), but have $\mathrm{sr}(S) \neq 1$. In fact, S may be chosen to be a Prüfer domain with arbitrary finite Krull dimension. To do this, we first recall a construction of Swan from [Sw, Section 1]. Let F be a field that has no element with square -1. For any subring $B \subseteq F$, let $B^{\#}$ be the subring of F generated by B and the set $\Sigma_1 = \{(1 + x^2)^{-1}:\ x \in F\}$. In [Sw], elements of the form $x/(1 + x^2)$ are also adjoined to B to form $B^{\#}$. However, in our situation this is unnecessary since $1/2 \in \Sigma_1$, and Dress's identity [Dr, (10)]

$$x/\left(1 + x^2\right) = \left[\left(1 + y^2\right)^{-1} - \left(1 + y^{-2}\right)^{-1}\right]/2 \quad \left(\text{with } y := (x - 1)/(x + 1)\right) \tag{3.11}$$

implies that $x/(1+x^2) \in B^{\#}$ for all $x \in F$. By Theorem 3.7, $B^{\#}$ is an AS_1 Prüfer domain with quotient field F; in particular, $\mathrm{ssr}(B^{\#}) = 1$. To produce examples of $B^{\#}$ with finite Krull dimension and with stable range not 1, we can now simply invoke the following result of Swan [Sw, Theorem 5] (see also [He]), where the general stable range of a ring is defined as in [Ba] or [Va_1].

Theorem 3.12. *Let B_n ($n \geqslant 1$) be the real coordinate ring of the n-sphere; that is, $B_n = \mathbb{R}[x_0, \ldots, x_n]$, with the relation $x_0^2 + \cdots + x_n^2 = 1$. Let F_n be the quotient field of B_n. Then the ring $B_n^{\#} \subseteq F_n$ is an AS_1 Prüfer domain of Krull dimension n, and the unimodular sequence $(x_0, \ldots, x_n)$ is not "stabilizable" to a shorter unimodular sequence over $B_n^{\#}$. In particular, $\mathrm{ssr}(B_n^{\#}) = 1$, but $\mathrm{sr}(B_n^{\#}) = n + 1 \geqslant 2$.*

In private communication to the authors, Professor Swan has pointed out that the topological methods used in [Sw] to prove Theorem 3.12 can be applied in similar fashion to show that the equation $\mathrm{sr}(S) = n + 1$ also holds if S is the n-dimensional ring denoted by $A_n^{\#}$ or $A_n^{\%}$ in [Sw]. In particular, from the discussions in Section 6 of [Sw], it follows that the real holomorphy ring $\mathcal{H}_n$ of the rational function field in n variables over $\mathbb{R}$ is a Prüfer domain with stable range $n+1$. Here, $\mathcal{H}_n$ is an Artin–Schreier ring by Example 3.10; in particular, its *square* stable range is 1. Noetherian examples can be obtained by applying Vaserstein's results in [Va_1] to the ring S constructed in Corollary 3.6, taking the ring R there to be the real polynomial ring $\mathbb{R}[t_1, \ldots, t_n]$.

Motivated in part by the ideas used earlier in this section, we shall now discuss another major source of commutative rings S with $\mathrm{ssr}(S) = 1$, but possibly $\mathrm{sr}(S) \neq 1$. These examples are drawn from the study of (commutative) algebras of functions. Following [GJ], for any topological space X, let $C(X)$ denote the ring of continuous real-valued functions on X, and let $C^*(X)$ denote the subring of $C(X)$ consisting of the bounded functions.

Theorem 3.13. *For any topological space X, let S be a full subring of $C(X)$ or of $C^*(X)$ (in the sense of Proposition 3.1(2)). Then S is an AS ring. In particular, $\mathrm{ssr}(S) = 1$.*

Proof. By Proposition 3.1(2), it is enough to handle the cases where $S = C(X)$ or $S = C^*(X)$. First let $S = C(X)$, and consider any equation $\sum_i f_i S = S$. Clearly, the functions f_i's have no common zero on X. Thus, $\sum_i f_i^2$ is nowhere zero on X, and is therefore in $\mathrm{U}(S)$ (see [GJ, (1.12)]). This shows that S is an AS ring. Next, let $S = C^*(X)$. This *may not* be a full subring of $C(X)$, so we cannot invoke the case we have settled above. However, to show that S is Artin–Schreier, we can appeal to Theorem 3.2. For any $f_1, \ldots, f_n \in S = C^*(X)$, the continuous function $(1 + f_1^2 + \cdots + f_n^2)^{-1}$ is clearly bounded by 1. Therefore, $(1 + f_1^2 + \cdots + f_n^2)^{-1} \in S$, which shows that $1 + f_1^2 + \cdots + f_n^2 \in \mathrm{U}(S)$. From Theorem 3.2, we conclude that S is an AS ring. $\square$

In general, commutative rings of the type $S = C(X)$ are rarely of stable range one. Indeed, if X is a completely regular Hausdorff space, then according to Vaserstein [Va_1], $\mathrm{sr}(C(X)) = 1$ iff $\dim(X) = 0$. (Here, the topological invariant "$\dim(X)$" is as defined in [Va_1]; see also [Az], and [FY].) Thus, any completely regular Hausdorff space X with $\dim(X) > 0$ leads to a commutative ring $S = C(X)$ with $\mathrm{ssr}(S) = 1 \neq \mathrm{sr}(S)$.

As is pointed out by Professor M. Rieffel, the fact that in Theorem 3.13 we can take any full subring $S \subseteq R = C(X)$ allows us to name many more examples of Artin–Schreier rings. For instance, if $X = \mathbb{R}$, take S to be the subring of infinitely differentiable 2π-periodic functions. (We can also think of S as a subring of the ring $C(T)$, where T denotes the unit circle.) Since S is obviously a full subring of $R = C(X)$, Theorem 3.13 implies that S is an Artin–Schreier ring; in particular, $\mathrm{ssr}(S) = 1$. Similar examples can be constructed readily upon replacing "infinitely differentiable" by various other conditions (e.g. in terms of the Fourier coefficients of a function). In conclusion, we might also observe that, for possible future applications to analysis, it should be worthwhile to generalize the material of this section by working with *rings with involution* $(S, *)$, and developing the theory using the expression $\sum_i x_i^* x_i$ throughout in place of $\sum_i x_i^2$: see Remark 3.3.

4. Right quasi-duo rings and exchange rings

We begin this section by proving Theorem 4.1 below, which provides some pretty strong necessary conditions on rings S of square stable range one, including the fact that S *must be right quasi-duo.* The basic idea of this theorem comes from the construction in Example 2.5. However, a proper formulation of this idea is best given in the setting of the Density Theorem in the classical theory of rings.

Theorem 4.1. *For any ring S with $\operatorname{ssr}(S) = 1$, we have the following.*

(1) *S is right quasi-duo.*
(2) *Any right-primitive factor ring of S is a division ring.*
(3) *Any semiprimitive factor ring of S (e.g. $S/\operatorname{rad} S$) is a subdirect product of division rings (and is thus a reduced ring).*
(4) *If $R = \mathbb{M}_n(R_0) \neq \{0\}$, then $\operatorname{ssr}(R) = 1$ iff $n = 1$ and $\operatorname{ssr}(R_0) = 1$.*

Proof. We start by recalling (say, from [KKJ, Lemma 7]) that (1) *and* (2) *are equivalent for all rings S* (without assuming $\operatorname{ssr}(S) = 1$). Indeed, (1) $\Rightarrow$ (2) appeared in [Yu_2, p. 23] and in [LD, (4.1)]. The converse can be proved by "reversing" the argument in the latter. Indeed, suppose (2) holds. If $\mathfrak{m} \subseteq S$ is any maximal right ideal, then the simple right S-module $S/\mathfrak{m}$ has annihilator J that is a right-primitive ideal. By (2), S/J is a division ring, so we must have $\mathfrak{m} = J$, which is an ideal. This shows that S is right quasi-duo. In the following, it thus suffices to prove (2), (3), and (4).

(2) Since the hypothesis that $\operatorname{ssr}(S) = 1$ is preserved by factor rings (by Proposition 2.10(1)), we may assume that S itself is a right-primitive ring. By the basic structure theorem on such rings (see, e.g. [La_2, (11.19)]), S acts faithfully on the right as a *dense* ring of linear endomorphisms on a left vector space V over some division ring k. If we can show that $\dim_k V = 1$, then $S \cong k$, as desired. *Assume for the moment that* $\dim_k V > 1$. Let v_1, v_2 be k-linearly independent vectors in V. By density, there exists $f \in S$ such that $v_1 f = v_2$ and $v_2 f = 0$. Likewise, there also exists $g \in S$ such that $v_2 g = v_1$. Since $\operatorname{ssr}(S) = 1$, Proposition 2.1 implies that there exists some $h \in S$ such that $f^2 + (1 - fg)h \in \mathrm{U}(S)$. Then

$$v_1\big(f^2 + (1 - fg)h\big) = v_2 f + \big(v_1 - (v_2 g)\big)h = 0 + (v_1 - v_1)h = 0,$$

contradicting the fact that $f^2 + (1 - fg)h$ is an *invertible* operator on V.

(3) A semiprimitive factor ring of S has the form S/J where J is a semiprimitive ideal of S. By [La_2, (11.5)], J is the intersection of all of the right-primitive ideals $\{J_i\}$ containing J. Thus S/J is a subdirect product of the right-primitive rings $\{S/J_i\}$. By Proposition 2.10(1), $\operatorname{ssr}(S/J_i) = 1$ for every i. This proves (3) since each S/J_i is a division ring by (2).

The reducedness of $S/\operatorname{rad} S$ as a necessary condition for $\operatorname{ssr}(S) = 1$ will be used rather heavily in the sequel. For this reason, it is of interest to record here a direct proof of this fact, *independently of any use of the density theorem.* Letting $J := \operatorname{rad}(S)$, it suffices to show that, if $a \in S$ is such that $a^2 \in J$, then $a \in J$. Consider any element $s \in S$. By Proposition 2.1, there exists $x \in S$ such that $a^2 + (1 - as)x \in \mathrm{U}(S)$. With $a^2 \in J$, this implies that $(1 - as)x \in \mathrm{U}(S)$. Thus, $1 - as$ is right-invertible. Since this holds for all $s \in S$, a standard result on the Jacobson radical [La_2, (4.1)] guarantees that $a \in J$.

(4) It suffices to prove the "only if" part. So assume that $R = \mathbb{M}_n(R_0)$ and $\operatorname{ssr}(R) = 1$. Since $R/\operatorname{rad} R \cong \mathbb{M}_n(R_0/\operatorname{rad} R_0)$, (3) implies that the matrix ring $\mathbb{M}_n(R_0/\operatorname{rad} R_0)$ is reduced. This is possible only if $n = 1$, in which case $\operatorname{ssr}(R_0) = \operatorname{ssr}(R) = 1$. $\square$

A remarkable application of Theorem 4.1 is the following result on left–right symmetry.

Theorem 4.2. *For any $n \geqslant 0$, a ring S is left AS_n iff it is right AS_n. (In particular, being AS is a left–right symmetric property for rings.)*

Proof. If $n = 0$, both conditions mean that S is Dedekind-finite. For $n \geqslant 1$, it suffices to prove the "if" part, so assume that S is right AS_n. Then $\mathrm{ssr}(S) = 1$, so by Theorem 4.1(1) above, S is right quasi-duo. Consider any equation $\sum_{i=0}^{n} Sa_i = S$. Then we also have $\sum_{i=0}^{n} a_i S = S$ by [LD, Theorem 3.2], and so the fact that S is right AS_n implies that $\sum_{i=0}^{n} a_i^2 \in \mathrm{U}(S)$. This work shows that S is left AS_n! □

Next, we shall apply Theorem 4.1 to the well-studied class of *exchange rings.* Theorem 4.3 below shows why this class of rings is particularly interesting for our investigation. The focus of the rest of the section will be on this class of rings, and on some of its subclasses; e.g. regular rings, π-regular rings, and strongly π-regular rings. The basic results on exchange rings and on the lifting of idempotents modulo ideals in exchange rings (from [Wa] and [Ni_1]) will be used freely throughout.

Theorem 4.3. *Let S be an abelian exchange ring. Then* (1) $\mathrm{ssr}(S) = 1$, *and* (2) *S is an AS_1 ring iff, in every division factor ring of S, -1 is not a product of two squares. In this case, $S/\mathrm{rad}\, S$ is a subdirect product of division rings in which -1 is not a product of two squares.*

Proof. (1) Assume for the moment that $\mathrm{ssr}(S) \neq 1$. Then there exist $a, b \in S$ such that $aS + bS = S$, but $a^2 + bx \notin \mathrm{U}(S)$ for every $x \in S$. Let $\mathcal{C}$ be the family of ideals $I \subseteq S$ such that $\bar{a}^2 + \bar{b}\bar{x} \notin \mathrm{U}(S/I)$ for every $\bar{x} \in S/I$. Clearly, $(0) \in \mathcal{C}$. After checking easily that $\mathcal{C}$ is an inductive family (with respect to inclusion), we can apply Zorn's Lemma to show that $\mathcal{C}$ has a maximal member, say I_0. Then S/I_0 is indecomposable as a ring. (For, if $S/I_0 = I_1/I_0 \times I_2/I_0$ is a nontrivial ring decomposition, then $I_1, I_2 \notin \mathcal{C}$, and we can quickly get a contradiction to $I_0 \in \mathcal{C}$ by a direct product argument.) Now idempotents can be lifted modulo I_0 (since S is an exchange ring), so S/I_0 remains an abelian exchange ring. As it has only trivial central idempotents, it must have only trivial idempotents. This means that the exchange ring S/I_0 is a local ring (see [Ni_1]); in particular, by Corollary 2.4, $\mathrm{ssr}(S/I_0) = 1$. This contradicts the fact that $I_0 \in \mathcal{C}$.

(2) It suffices to prove the first statement, since it implies the second statement via Theorem 4.1(3). For the "only if" part of the first statement, assume that S is AS_1, and consider any division factor ring D of S. Since S is an abelian exchange ring, Yu's result in [Yu_1] gives $\mathrm{sr}(S) = 1$.[6] Therefore, by Proposition 2.10′ (in Section 3), D is also AS_1. This means that -1 is not a product of two squares in D. For the "if" part, assume S is not AS_1. Repeating the Zorn's Lemma argument in the proof of (1) above, we get an ideal $I_0 \subseteq S$ such that S/I_0 is a local ring that is not AS_1. From this, we see easily as before that -1 is a product of two squares in the residue division ring D of S/I_0. This gives what we want, since D is also a division factor ring of S. □

Combining Theorem 4.1(3) and Theorem 4.3(1) with some results from [Yu_2] and [LD] leads to the following characterization theorem for $\mathrm{ssr}(S) = 1$ in case $S/\mathrm{rad}\, S$ is an exchange ring. (In particular, the theorem applies when S itself is an exchange ring.)

Theorem 4.4. *Let S be any ring such that S/J is an exchange ring, where $J := \mathrm{rad}(S)$. The following four statements are equivalent*:

(1) $\mathrm{ssr}(S) = 1$. (2) *S/J is reduced.* (3) *S/J is abelian.* (4) *S is right quasi-duo.*

In particular, since (2) *and* (3) *are left–right symmetric, it follows that* (1) *and* (4) *are also left–right symmetric.*

Proof. The equivalence of (2), (3) and (4) is proved in [Yu_2, (4.1)]; see also [BS], [St, (4.10)], and [LD, (4.6)].[7] To relate these statements to (1), note that Theorem 4.1(1) gives (1) ⇒ (4) without any

[6] In fact, according to Theorem 4.4 below, Yu's result that $\mathrm{sr}(S) = 1$ will actually follow from $\mathrm{ssr}(S) = 1$, which is already proved in (1). But at this point, it is probably easier to use Yu's original result.

[7] In Corollary 3.12 of [Yu_3], more equivalent conditions are given for (2) and (3) above. Some of these conditions involve, for instance, properties of the factor rings of S/J.

conditions on S. (Of course, Theorem 4.1(3) also gives (1) $\Rightarrow$ (2) without any conditions on S.) To see that (3) $\Rightarrow$ (1), assume that S/J is abelian. Since S/J is an exchange ring, Theorem 4.3(1) implies that $\mathrm{ssr}(S/J) = 1$. Therefore, by Theorem 2.11(1), $\mathrm{ssr}(S) = 1$, proving (1). (Alternatively, we can use again Yu's result [Yu$_1$] that the abelian exchange ring S/J has stable range 1. Since S/J is also right quasi-duo (by (4)), Theorem 2.3 gives $\mathrm{ssr}(S) = 1$.) □

For exchange rings, the three stable range conditions introduced so far ("sr", "ssr", and Chen's "isr" in Theorem 2.14) are related to one another as in the following corollary.

Corollary 4.5. *Let S be an exchange ring. Then*

$$(*) \qquad S \text{ is abelian} \quad \Rightarrow \quad \mathrm{ssr}(S) = 1 \quad \Rightarrow \quad \mathrm{isr}(S) = 1 \quad \Rightarrow \quad \mathrm{sr}(S) = 1.$$

If S is a commutative exchange ring, all four conditions hold.

Proof. Let $\bar{S} = S/J$, where $J = \mathrm{rad}(S)$. The first implication is just Theorem 4.3(1). For the second implication, assume that $\mathrm{ssr}(S) = 1$. Since $\bar{S}$ remains an exchange ring, it is abelian by Theorem 4.4. Thus, by Chen's Theorem 12 in [Ch$_1$], $\mathrm{isr}(\bar{S}) = 1$, and Theorem 9 in the same paper implies that $\mathrm{isr}(S) = 1$ (keeping in mind that idempotents can be lifted modulo J). This proves the second implication, and the third one is a tautology. The last statement in Corollary 4.5 is clear since commutative rings are obviously abelian. □

Examples 4.6. The first implication in $(*)$ above cannot be reversed (in the noncommutative case). For instance, let S be the ring of $n \times n$ upper triangular matrices over a division ring, where $n > 1$. This is an exchange ring, with $\mathrm{ssr}(S) = 1$ by Theorem 2.11(1). However, S is not abelian, since the idempotent $\mathrm{diag}(1, 0, \ldots, 0) \in S$ is not central. The second implication in $(*)$ above also cannot be reversed. Indeed, if $S = \mathbb{M}_n(D)$ where $n > 1$ and D is any division ring, then S is an exchange ring with $\mathrm{ssr}(S) \neq 1$ (by Theorem 4.1(4)). However, the main result in [WC] implies that $\mathrm{isr}(S) = 1$. As for the third implication in $(*)$, we have no examples to show that it cannot be reversed. Indeed, in [WC, (3.7)], it is shown that this implication can be reversed *for regular rings.*

Corollary 4.7. (1) *A* (*von Neumann*) *regular ring S has* $\mathrm{ssr}(S) = 1$ *iff S is reduced, iff S is abelian, iff S is right quasi-duo.* (2) *A semilocal ring S has* $\mathrm{ssr}(S) = 1$ *iff* $S/\mathrm{rad}\,S$ *is a finite direct product of division rings.*

Proof. (1) We have here $\mathrm{rad}(S) = 0$, and S is an exchange ring. Thus, (1) follows directly from Theorem 4.3. For (2), in view of Theorem 2.11(1), we may replace S by $S/\mathrm{rad}\,S$ to assume that S is semisimple. By the Wedderburn–Artin Theorem, S is a finite direct product of matrix rings over division rings. In this case, (2) follows from (1) and the obvious fact that, if D is a division ring, the regular ring $\mathbb{M}_n(D)$ is abelian iff $n = 1$. □

Concerning (1) above, we shall be using freely the well-known fact that abelian regular rings are exactly the *strongly regular rings* in Corollary 2.4 (i.e. rings S in which $b \in b^2S$ for every $b \in S$); see [La$_4$, Exercise 12.6A].[8] On the other hand, regular rings of stable range 1 are exactly the unit regular rings [La$_4$, Exercise 20.10D]. Therefore, if S is any unit-regular ring that is not strongly regular, then $\mathrm{sr}(S) = 1$, but $\mathrm{ssr}(S) \neq 1$. This adds to our stock of examples of $\mathrm{sr}(S) = 1 \nRightarrow \mathrm{ssr}(S) = 1$. Nevertheless, it is of interest to note (again) that, for *commutative rings* (or even right quasi-duo rings), $\mathrm{sr}(S) = 1 \Rightarrow \mathrm{ssr}(S) = 1$, while for *regular rings* (or even exchange rings), $\mathrm{ssr}(S) = 1 \Rightarrow \mathrm{sr}(S) = 1$.

Recall that a ring S is *π-regular* (resp. *strongly π-regular*) if, for every $a \in S$, a^n is regular for some $n \geqslant 1$ (resp. the chain $aS \supseteq a^2S \supseteq \cdots$ stabilizes). We conclude this section by proving the following

[8] Indeed, the fact that a regular ring S is strongly regular iff $\mathrm{ssr}(S) = 1$ can be proved directly in a few lines without using the more general Theorem 4.4. (The "only if" part is already covered by Corollary 2.4.)

characterization result for π-regular rings of square stable range 1, which may be viewed as a "π-version" of Corollary 4.7(1).

Theorem 4.8. *For any ring S with $J = \operatorname{rad}(S)$, the following are equivalent*:

(1) *S is π-regular and $\operatorname{ssr}(S) = 1$.*
(2) *S is strongly π-regular and S/J is reduced.*
(3) *S is strongly π-regular and S/J is abelian.*
(4) *S is strongly π-regular and right quasi-duo.*
(5) *S/J is strongly regular, and J is nil.*

If these statements hold, then $\operatorname{sr}(S) = 1$.

Proof. Recall that strongly π-regular rings are π-regular (see [La$_4$, Exercise 23.6(1)]), and π-regular rings are exchange rings. Thus, by Corollary 4.5, (1) implies that $\operatorname{sr}(S) = 1$. On the other hand, it is clear that a ring S is right quasi-duo iff $S/\operatorname{rad} S$ is right quasi-duo. Thus, the equivalence of (2), (3), and (4) follows as in the proof of Theorem 4.4. It remains only to prove that (3) $\Rightarrow$ (1) $\Rightarrow$ (5) $\Rightarrow$ (2).

(3) $\Rightarrow$ (1). Assume (3). Then S is an exchange ring, so Theorem 4.4 shows that $\operatorname{ssr}(S) = 1$.

(1) $\Rightarrow$ (5). Assume (1); in particular, S is an exchange ring. For any $a \in J$, there exists $n \geqslant 1$ such that $a^n = a^n x a^n$. Thus, $(1 - a^n x)a^n = 0$. Since $1 - a^n x \in \mathrm{U}(S)$, we have $a^n = 0$. This shows that J is nil. From $\operatorname{ssr}(S) = 1$, we also know (from Theorem 4.1(3)) that S/J is reduced, and hence abelian. Since S/J remains π-regular, a result of Badawi [B, Theorem 3] on abelian π-regular rings implies that S/J is regular, and therefore strongly regular.

(5) $\Rightarrow$ (2). Assume (5). Then S is an exchange ring (since S/J is an exchange ring and idempotents can be lifted modulo the nil ideal $J = \operatorname{rad} S$: see [Wa]). As the strongly regular ring S/J is reduced, Theorem 4.4 implies that S is right quasi-duo. Also, since J is nil, it coincides with the prime radical of S, and the reducedness of S/J implies that J is exactly the set of all nilpotent elements of S.[9] Given these facts on S, we can infer from [KKJ, Proposition 8] that S is strongly π-regular, proving (2). □

Remark 4.9. As far as the stable range is concerned, Ara [Ar] has proved that *any* strongly π-regular ring S has $\operatorname{sr}(S) = 1$. Theorem 4.8 above covers this result only in the special case where S is 1-sided quasi-duo. But of course, the main focus of Theorem 4.8 is on the case of *square* stable range 1. A prototypal example for the rings satisfying the conditions in Theorem 4.8 is the ring of $n \times n$ upper triangular matrices over a division ring, or a commutative ring of Krull dimension zero (see [La$_4$, Exercise 4.15]).

5. Strongly regular elements and square stable range one

In this section, we'll study the problem of square stable range one from an "element-wise" point of view. Unless otherwise stated, S denotes any ring below. At the end of the section, however, we'll return to the theme of exchange rings, and provide (in addition to Theorem 4.4) another criterion for such rings S to have $\operatorname{ssr}(S) = 1$ in terms of the behavior of the (von Neumann) regular elements in S.

We begin by recalling the "element-wise" perspective in studying the stable range. In [KL], the first two authors defined an element a in a ring S to have (right) stable range 1 (written $\operatorname{sr}(a) = 1$) if $aS + bS = S$ (for any $b \in S$) implies that $a + bx \in \mathrm{U}(S)$ for some $x \in S$. In this spirit, we can similarly define $a \in S$ to have (right) square stable range 1 (written $\operatorname{ssr}(a) = 1$) if $aS + bS = S$ (for any $b \in S$) implies that $a^2 + by \in \mathrm{U}(S)$ for some $y \in S$. Thus, $\operatorname{ssr}(S) = 1$ *amounts to* $\operatorname{ssr}(a) = 1$ *for all* $a \in S$. The advantage of working with $\operatorname{ssr}(a) = 1$ is that it allows us to look at rings where only certain elements have this property.

[9] A ring is called "2-primal" if all its nilpotent elements lie in the prime radical. This step showed that S is 2-primal, which is a hypothesis needed for the ensuing application of [KKJ, Proposition 8].

To properly study the element-wise condition $\mathrm{ssr}(a) = 1$, let us first set up the notations for several basic sets of elements in a ring S which arise from notions related to (von Neumann) regularity. Recall that an element $a \in S$ is called *regular* if $a \in aSa$, *unit-regular* if $a \in a\,\mathrm{U}(S)a$, and *strongly regular* if $a \in a^2S \cap Sa^2$. The sets of regular, unit-regular, and strongly regular elements in S are denoted, respectively, by $\mathrm{reg}(S)$, $\mathrm{ureg}(S)$, and $\mathrm{sreg}(S)$. It is not hard to show that $\mathrm{sreg}(S) \subseteq \mathrm{ureg}(S) \subseteq \mathrm{reg}(S)$; see, e.g. [Ni$_2$, p. 3283], where one of the many membership criteria for $\mathrm{sreg}(S)$ gives the crucial first inclusion. We begin with a lemma on Dedekind-finiteness which is probably well known. For the sake of completeness, we'll include its proof.

Lemma 5.1. *Let $a \in \mathrm{reg}(S)$; say $a = axa$ $(x \in S)$, and let $e = ax$, which is an idempotent. The following are equivalent*:

(1) *The right module aS is Dedekind-finite.*
(2) *The left module Sa is Dedekind-finite.*
(3) *The ring eSe is Dedekind-finite.*

Proof. Note that $aS = eS$. Thus, $\mathrm{End}_S(aS) = \mathrm{End}_S(eS) \cong eSe$. This shows that (1) $\Leftrightarrow$ (3). Next, recall that $aS = eS \Rightarrow Sa \cong Se$ (see [La$_4$, Exercise 1.17]). Thus, we also have $\mathrm{End}_S(Sa) \cong \mathrm{End}_S(Se) \cong eSe$. From this, we have (2) $\Leftrightarrow$ (3). $\square$

In [KL, (3.2), (3.5)], it is proved that $a \in \mathrm{ureg}(S)$ *implies that* $\mathrm{sr}(a) = 1$, *and that the converse holds if* $a \in \mathrm{reg}(S)$. In the following, we'll prove a (somewhat) analogous result for $\mathrm{ssr}(a) = 1$ in relation to the set of strongly regular elements $\mathrm{sreg}(S)$.

Theorem 5.2. *If $a \in \mathrm{sreg}(S)$, then* $\mathrm{ssr}(a) = 1$. *The converse holds if $a \in \mathrm{reg}(S)$ and the right module aS is Dedekind-finite.*

Proof. Assume that $a \in \mathrm{sreg}(S) \subseteq \mathrm{ureg}(S)$, and consider any equation $aS + bS = S$. By [Ni$_2$, p. 3283], we can write $a = aua$ for some $u \in \mathrm{U}(S)$ commuting with a. Thus, $a^2 = a^2u^2a^2$, so $a^2 \in \mathrm{ureg}(S)$; in particular, by [KL, Theorem 3.2], $\mathrm{sr}(a^2) = 1$. Since $aS = a^2S$, $aS + bS = S \Rightarrow a^2S + bS = S$. Therefore, $a^2 + bx \in \mathrm{U}(S)$ for some $x \in S$. This shows that $\mathrm{ssr}(a) = 1$.

Conversely, assume that $\mathrm{ssr}(a) = 1$, and that $a \in \mathrm{reg}(S)$, with aS Dedekind-finite. Write $a = axa$ (for some x), and let $f = ax$, so that $fS = aS$. Since $aS + (1-f)S = S$, $\mathrm{ssr}(a) = 1$ implies that $u := a^2 + (1-f)y \in \mathrm{U}(S)$ for some y. Left-multiplying this by f gives $fu = fa^2 = a^2$. This shows that a^2 is a product of an idempotent and a unit, so $a^2 \in \mathrm{ureg}(S) \subseteq \mathrm{reg}(S)$ by [La$_4$, Exercise 4.14B)]. It also shows that $a^2S = fuS = fS = aS$, which implies that $Sa^2 \cong Sa$ (again by [La$_4$, Exercise 1.17]). Since $a^2 \in \mathrm{reg}(S)$, Sa^2 is a direct summand of S, and hence of Sa. But by (1) $\Rightarrow$ (2) of Lemma 5.1, Sa is Dedekind-finite. This forces $Sa^2 \subseteq Sa$ to be an *equality*; that is, $Sa = Sa^2$. This equation, together with $aS = a^2S$, shows that $a \in \mathrm{sreg}(S)$, as desired. $\square$

In [KL], a ring S is said to have the *internal cancellation property* (or, S is an IC ring) if, for any decompositions $S = A \oplus B = C \oplus D$ where A, B, C, D are right ideals, $A \cong C \Rightarrow B \cong D$. In [KL, (1.1) and (4.2)], it was shown that

$$S \text{ is IC} \quad \Leftrightarrow \quad \mathrm{reg}(S) = \mathrm{ureg}(S) \quad \Leftrightarrow \quad \mathrm{sr}(a) = 1 \quad \text{for all } a \in \mathrm{reg}(S). \tag{5.3}$$

(In particular, "IC ring" is a left–right symmetric notion.) In the following theorem, we'll obtain similar results, with the equation $\mathrm{reg}(S) = \mathrm{ureg}(S)$ strengthened to $\mathrm{reg}(S) = \mathrm{sreg}(S)$.

Theorem 5.4. *For any ring S, the following are equivalent*:

(1) $\mathrm{reg}(S) = \mathrm{sreg}(S)$.
(2) $ax + e = 1$ *with* $e = e^2$ *implies that* $a^2 + ey \in \mathrm{U}(S)$ *for some* $y \in S$.

(3) *$ax + e = 1$ with $e = e^2$ and $a \in \mathrm{reg}(S)$ implies that $a^2 + ey \in \mathrm{U}(S)$ for some $y \in S$.*
(4) *For all $b \in S$, $b \in \mathrm{reg}(S) \Rightarrow b \in b^2 S$.*
(5) $\mathrm{ssr}(a) = 1$ *for all $a \in \mathrm{reg}(S)$.*

If any one of these conditions holds for S, we say that S is a strongly IC *ring. For instance, any ring S with* $\mathrm{ssr}(S) = 1$ *is strongly* IC.

Proof. We first prove the equivalence of (1)–(4). After this, we'll complete the proof by showing (1) $\Rightarrow$ (5) $\Rightarrow$ (3). To begin with, (2) $\Rightarrow$ (3) is a tautology, and (3) $\Rightarrow$ (4) can be proved by the argument in the second paragraph of the proof of Theorem 5.2.

(4) $\Rightarrow$ (1). Assuming (4), we first show that S is Dedekind-finite. If $xy = 1$, then $y = yxy \in \mathrm{reg}(S)$ implies that $y = y^2 z$ for some $z \in S$ (by (4)). Left-multiplying by x then gives $1 = xy^2 z = yz$, so $y \in \mathrm{U}(S)$. To prove (1), consider any $b \in \mathrm{reg}(S)$. By (4), $b^2 S = bS$, which is generated by an idempotent. Thus, $b^2 \in \mathrm{reg}(S)$, and $Sb \cong Sb^2$ (again by [La$_4$, Exercise 1.17]). As S is Dedekind-finite, the direct summand Sb^2 in Sb must equal Sb. Therefore, $b \in Sb^2$, which, together with (4), shows that $b \in \mathrm{sreg}(S)$.

(1) $\Rightarrow$ (2). This is perhaps the most subtle implication in the proof of the theorem. Since S is not assumed to be regular, the key step in proving (2) is to "create" a regular element from the equation $ax + e = 1$ given there. For this, we'll use an idea from the proof of Theorem 3 in [CY] (although our ring S is not assumed to be an exchange ring here). Write $f := ax = f^2$, and let $b := axa = fa$. Then

$$bxb = (fa)x(fa) = f^3 a = fa = b, \tag{5.5}$$

so $b \in \mathrm{reg}(S)$. By (1), $b = b^2 s$ for some $s \in S$. Using this, we have

$$b^2(sx) + e = bx + e = (fa)x + e = f^2 + e = f + e = 1. \tag{5.6}$$

Now $b^2 = (fa)b = (1-e)ab = ab - eab \in a^2 S + eS$ (since $b = axa$). Combining this with (5.6), we get $1 \in a^2 S + eS$; thus, $a^2 S + eS = S$. But (1) implies that $\mathrm{reg}(S) = \mathrm{ureg}(S)$, so by (5.3), S is IC. According to [KL, Theorem 4.3], the condition (3) in that theorem is then valid. Applying this to $a^2 S + eS = S$, we see that $a^2 + ey \in \mathrm{U}(S)$ for some $y \in S$, proving (2).

To complete the proof, note that (1) $\Rightarrow$ (5) follows directly from Theorem 5.2. To prove (5) $\Rightarrow$ (3), start with $ax + e = 1$, where $e = e^2$, and $a \in \mathrm{reg}(S)$. Since $\mathrm{ssr}(a) = 1$ by (5), we have by definition $a^2 + ey \in \mathrm{U}(S)$ for some $y \in S$, proving (3). $\square$

Remark 5.7. Of course, the expression $\mathrm{ssr}(a)$ in (5) above is still understood to be the "right square stable range". But the condition (1) is obviously left–right symmetric. From this, it follows that the conditions (4) and (5) are left–right symmetric as well.

To conclude this section, we now return to exchange rings. In [CY, Theorem 3], Camillo and Yu proved that an exchange ring S has $\mathrm{sr}(S) = 1$ iff S is an IC ring (that is, $\mathrm{reg}(S) = \mathrm{ureg}(S)$). In the following, we obtain a complete analogue of this result for the square stable range — by proving that S has $\mathrm{ssr}(S) = 1$ iff S is a strongly IC ring (that is, $\mathrm{reg}(S) = \mathrm{sreg}(S)$). Of course, this result shows once more that $\mathrm{ssr}(S) = 1$ is a left–right symmetric notion for the class of exchange rings.

Theorem 5.8. *If S is an exchange ring, then* $\mathrm{ssr}(S) = 1$ *iff S is strongly* IC. *In this case, S is a clean ring.*

Proof. The "only if" part is true without the exchange ring assumption, as we have noted in the last statement of Theorem 5.4. For the "if" part, suppose S is an exchange ring that is strongly IC. To check that $\mathrm{ssr}(S) = 1$, start with any equation $aS + bS = S$. Since S is an exchange ring, there exists an idempotent $e \in S$ such that $e = by$ and $1 - e = ax$ for some $x, y \in S$ (see [Ni$_1$]). Then $ax + e = 1$, so the condition (2) in Theorem 5.4 implies that there exists $z \in S$ such that $a^2 + ez = a^2 + b(yz) \in \mathrm{U}(S)$.

This checks that $\mathrm{ssr}(S) = 1$, as desired. Finally, from the second implication in Corollary 4.5(∗), it follows that $\mathrm{isr}(S) = 1$. This implies that S is a clean ring, as we have observed in the proof of Theorem 2.14. (Note that the exchange ring assumption on S is essential for the "if" part above. For instance, a domain S is always strongly IC, since $\mathrm{reg}(S) = \{0\} \cup \mathrm{U}(S) = \mathrm{sreg}(S)$. But if $S = \mathbb{Z}$ (say), we have $\mathrm{ssr}(S) \neq 1$.) □

6. Infinitude of unit group under (square) stable range one

In this section, we study the unit groups of (possibly noncommutative) rings with stable range 1 or square stable range 1. The first main result of the section (Theorem 6.2) utilizes Ara's stable range 1 theorem in [Ar] and Dischinger's theorem in [Di] to give a new characterization of strongly π-regular rings in terms of their unit groups. A more or less parallel result in the case of square stable range 1 is given in Theorem 6.4. These culminate in a final result (Theorem 6.8) of the section, which shows that an infinite domain S with $\mathrm{sr}(S) = 1$ or $\mathrm{ssr}(S) = 1$ must have a *non-artinian* unit group. This result calls for a rather long proof since the case where S is a division ring requires a separate argument (which is completed only in Section 7).

The methods we use in this section are generalizations of the commutative methods used by Estes and Ohm in [EO]. After [EO] (but more generally in the noncommutative setting), we first introduce the following notation. For any right ideal $J \subseteq S$, let $K(J) := \mathrm{U}(S) \cap (1 + J)$. It is a routine exercise to check that $K(J)$ is a *subgroup* of $\mathrm{U}(S)$. If $J' \subseteq J$ is another right ideal, we have clearly $K(J') \subseteq K(J)$. If J happens to be an *ideal*, then $K(J)$ is precisely the kernel of the natural homomorphism $\mathrm{U}(S) \to \mathrm{U}(S/J)$. In particular, in this case, $K(J)$ would be a normal subgroup of $\mathrm{U}(S)$. The beginning point of our considerations is the following observation on the groups $K(J)$ in a ring of stable range one.

Lemma 6.1. *Assume that* $\mathrm{sr}(S) = 1$, *and let* $c, d \in S$ *be such that* $cS \supseteq dS$ *and* $(1 + c)S + dS = S$. *Then* $cS = dS$ *iff* $K(cS) = K(dS)$.

Proof. We need only prove the "if" part, so assume $K(cS) = K(dS)$. Since $\mathrm{sr}(S) = 1$, we have $u := (1 + c) - ds \in \mathrm{U}(S)$ for some $s \in S$. Then $1 - u = ds - c \in cS$ implies that $u \in K(cS) = K(dS)$, so $1 - u \in dS$. Thus, $c = ds - (1 - u) \in dS$, and hence $cS = dS$. □

Theorem 6.2.

(A) *A ring S is strongly π-regular iff* $\mathrm{sr}(S) = 1$ *and all group-chains of the form* $K(aS) \supseteq K(a^2S) \supseteq \cdots$ (*with* $a \in S$) *stabilize.*
(B) *A ring S with an artinian unit group is strongly π-regular iff* $\mathrm{sr}(S) = 1$.

Proof. (B) obviously follows from (A). To prove (A), first assume S is strongly π-regular Then any right principal ideal chain $aS \supseteq a^2S \supseteq \cdots$ already stabilize, and Ara's main theorem in [Ar] gives $\mathrm{sr}(S) = 1$. Conversely, assume that $\mathrm{sr}(S) = 1$ and that all group-chains in (A) of the theorem stabilize. Let $a \in S$. For any $i, j > 0$, note that $(1 + a^i)S + a^jS = S$. (This equation holds with S replaced by the commutative subring of S generated by $\mathbb{Z} \cdot 1$ and a, so it also holds for S itself.) Applying Lemma 6.1 to $c = a^i$ and $d = a^{i+1}$, we see that the group-chain condition implies that $a^iS = a^{i+1}S$ for sufficiently large i, so by Dischinger's theorem [Di], S is a strongly π-regular ring. □

Using Theorem 6.2, we'll show that, if $\mathrm{sr}(S) = 1$, then the subgroups $K(aS) \subseteq \mathrm{U}(S)$ are infinite groups of units for "many" choices of $a \in S$.

Corollary 6.3. *Let S be any ring with* $\mathrm{sr}(S) = 1$, *and let $a \in S$ be not right-invertible and not a left* 0*-divisor. Then* $K(aS) \supseteq K(a^2S) \supseteq \cdots$ *is a strictly descending chain of subgroups of* $\mathrm{U}(S)$. *In particular,* $K(aS)$ *is a non-artinian* (*and hence infinite*) *group.*

Proof. If the conclusion was false, we would have $K(a^iS) = K(a^{i+1}S)$ for some i, and hence $a^i \in a^{i+1}S$ by the proof of Theorem 6.2. Since a is not a left 0-divisor, this would imply that $1 \in aS$, contradicting the assumption that a is not right-invertible. □

Next, we would like to draw "similar" conclusions in the case where $\mathrm{ssr}(S) = 1$. In principle, we would try to recycle the proof method for Theorem 6.2 and Corollary 6.3, but we must take extra steps to adapt the arguments for proving Lemma 6.1 to the case $\mathrm{ssr}(S) = 1$. This calls for a considerably more subtle argument.[10]

Theorem 6.4. *Let $a \in S$ where S is any ring with $\mathrm{ssr}(S) = 1$. If either $2 \in \mathrm{U}(S)$ or a is not a left 0-divisor in S, then $aS \supseteq a^2S \supseteq \cdots$ stabilize iff $K(aS) \supseteq K(a^2S) \supseteq \cdots$ stabilize.*

Proof. ("If" part.) To prove this, we argue in the following two cases.

Case 1. $2 \notin \bigcap_{i \geqslant 1} a^iS$. Fix an integer $k \geqslant 1$ such that $2 \notin a^kS$. Pick some n such that $K(a^nS) = K(a^{n+i}S)$ for all $i \geqslant 0$. We may assume that $n \geqslant k$. Consider the equation $(1 + a^n)S + a^{2n}S = S$ (as in the proof of Theorem 6.2). Since $\mathrm{ssr}(S) = 1$, we have

$$\left(1 + a^n\right)^2 - u = 1 + 2a^n + a^{2n} - u \in a^{2n}S \quad \text{for some } u \in \mathrm{U}(S). \tag{6.5}$$

Then $1 - u \in a^nS$, so $u \in K(a^nS) = K(a^{n+k}S)$. This means that $1 - u \in a^{n+k}S$, so (in view of $n \geqslant k$) (6.5) implies that $2a^n \in a^{n+k}S$. If $2 \in \mathrm{U}(S)$, this gives the desired conclusion $a^n \in a^{n+k}S$. If a is not a left 0-divisor, we have $2 \in a^kS$, which is not the case.

Case 2. $2 \in \bigcap_{i \geqslant 1} a^iS$. This time, we work with the equation $(1 + a^n)S + a^{2n+1}S = S$ (where n is chosen as in Case 1). As before, we have

$$\left(1 + a^n\right)^2 - v = 1 + 2a^n + a^{2n} - v \in a^{2n+1}S \quad \text{for some } v \in \mathrm{U}(S). \tag{6.6}$$

Here, $2a^n = a^n \cdot 2 \in \bigcap_{i \geqslant 1} a^iS$. Thus, (6.6) implies that $1 - v \in a^{2n}S$, so $v \in K(a^{2n}S) = K(a^{2n+1}S)$. With this, (6.6) gives $a^{2n} \in a^{2n+1}S$, as desired. □

We can now prove the following "analogue" of Corollary 6.3 for the case $\mathrm{ssr}(S) = 1$. Note that there is a subtle difference between the statement below and that of Corollary 6.3.

Corollary 6.7. *Let S be any ring with $\mathrm{ssr}(S) = 1$, and let $a \in S$ be not right-invertible and not a left 0-divisor. Then the group-chain $K(aS) \supseteq K(a^2S) \supseteq \cdots$ does not stabilize. In particular, $K(aS)$ is a non-artinian (and hence infinite) group.*

Proof. We can repeat the proof of Corollary 6.3, using here Theorem 6.4 instead of Theorem 6.2. □

In the following final result of this section, we provide a "double" generalization of an earlier theorem of Estes and Ohm. Their original result in [EO, p. 351] is extended not only to the noncommutative case, but also to the case where $\mathrm{ssr}(S) = 1$.

Theorem 6.8. *Let S be an infinite domain with either $\mathrm{sr}(S) = 1$ or $\mathrm{ssr}(S) = 1$. Then $\mathrm{U}(S)$ is a non-artinian (and hence infinite) group.*

[10] We have already explained in Section 2 that the case $\mathrm{ssr}(S) = 1$ is in general logically independent of the case $\mathrm{sr}(S) = 1$. Thus, the results obtained in Theorems 6.2 and 6.4 are basically independent ones. It is true, however, that the conclusions of Theorem 6.2 are more penetrating than that of Theorem 6.4, since their proofs made use of two of the deepest results for strongly π-regular rings.

Proof. First assume that $\mathrm{ssr}(S) = 1$. If S is a division ring, then $\mathrm{U}(S) = S \setminus \{0\}$. To prove that this group is non-artinian requires some field-theoretic and number-theoretic techniques. Since these techniques have relatively little to do with the theme of stable range 1 and square stable range 1, it seems better if we defer the proof in this division ring case to the next section, in order to keep our present focus on the analysis of $\mathrm{sr}(S) = 1$ and $\mathrm{ssr}(S) = 1$. The division ring case will be taken up in full in Theorem 7.3.

For the rest of the proof, we may thus assume that S is not a division ring. Then there exists a nonzero element $a \in S$ that is not right-invertible (see [La$_4$, Exercise 1.2]). Since a is automatically not a left 0-divisor, Corollary 6.7 applies to show that $K(aS)$ is not artinian, proving what we want. The case where $\mathrm{sr}(S) = 1$ is similar (and in fact easier, using instead Corollary 6.3). □

7. A chain-condition theorem on division rings

In Section 6, we have invoked a certain result on chain conditions for the multiplicative groups of division rings. We return now to this theme, and present here the result that was used in the proof of Theorem 6.8. We begin with the following number-theoretic proposition, which was shown to (and proved for) us by Professor Michael Filaseta.

Proposition 7.1. *Let $n \geqslant 1$ be a fixed integer, and let E be an infinite set of positive integers. Then there are infinitely many primes dividing $n^e - 1$ as e varies over E.*

Proof. The case $n = 1$ is just Euclid's theorem on the infinitude of primes. Suppose now $n > 1$, and that the conclusion is false. Then there are finitely many primes $p_1, \ldots, p_r$ such that each integer $n^e - 1$ with $e \in E$ can be factored into $p_1^{e_1} \cdots p_r^{e_r}$ ($e_i \geqslant 0$ depending on e). We can rewrite this in the form

$$n^{i_0} x^3 - 1 = p_1^{i_1} \cdots p_r^{i_r} y^3, \quad \text{where } i_0, i_1, \ldots, i_r \in \{0, 1, 2\}, \text{ and } x, y \in \mathbb{Z}. \tag{7.2}$$

Indeed, by the division algorithm, $e = 3s + t$ for some $s \geqslant 0$ and $t \in \{0, 1, 2\}$. Since $n^e = n^t (n^s)^3$, we can take $i_0 = t$ and $x = n^s$. Doing the same thing with each $p_j^{e_j}$, we arrive at the diophantine equations (7.2). Since $e \in E$ (and hence s) can be arbitrarily large, we get infinitely many integral solutions (x, y) to (7.2) for at least one of the 3^{r+1} choices for $i_0, i_1, \ldots, i_r$. But (7.2) is a Thue equation of the form $Ax^3 - By^3 = 1$, which is known to have only *finitely many* integral solutions (see, e.g. [Co, Theorem 12.11.1]). This gives the desired contradiction. □

As was pointed out by Professor Jan Mináč, the proposition above can also be deduced from the existence of primitive prime divisors of $n^e - 1$ for $n > 1$ and $e > 6$: see, for instance, Theorem (P1.7) in Ribenboim's book [Ri] on Catalan's Conjecture.

We can now prove Theorem 7.3 below for a general division ring, which was actually the "beginning case" of our earlier result (Theorem 6.8) on unit groups of infinite domains with stable range 1 or square stable range 1. We include here a detailed proof for Theorem 7.3 since we have not been able to find a convenient reference for it in the division ring literature.

Theorem 7.3. *For any division ring D with multiplicative group $D^* = D \setminus \{0\}$, the following are equivalent*: (1) *D is finite.* (2) *D^* is artinian.* (3) *D^* is noetherian.*

Proof. We need only prove (2) ⇒ (1) and (3) ⇒ (1). *Assuming that D is infinite,* we would like to show that D^* is non-artinian and non-noetherian. In the following, we'll actually prove a much stronger result; namely, that

$$D^* \text{ contains an infinite direct sum of nontrivial cyclic groups.} \tag{7.4}$$

In the language of module theory, this statement basically amounts to the fact that D^* has infinite uniform dimension (see [La$_3$, Section 6]) as a "$\mathbb{Z}$-module". The only catch is that, if D is not a field, then D^* is not quite a "$\mathbb{Z}$-module" in the usual sense since its binary operation (multiplication) is not commutative. Nevertheless, it is conceptually beneficial to think of (7.4) as a fact on the "uniform dimension" of D^*.

To prove (7.4), let F be the center of D, and let $K \supseteq F$ be a maximal subfield of D. If $|K| < \infty$, then $\dim_F K < \infty$, and so $\dim_K(D_K) < \infty$ too (by [La$_2$, Theorem 15.8]). But then $|D| < \infty$, a contradiction. Thus, K must be an infinite field. We may then "replace" D by K, so in particular, we are now down to the *commutative* case. If $\mathrm{char}(K) = 0$, then we may further replace K by $\mathbb{Q}$, whose multiplicative group contains a free abelian group of infinite rank — with generators given by the primes, proving (7.4). We are now left with the case where $\mathrm{char}(K) = p$ (a prime). Let $\mathbb{F}_p$ be the prime field of K. If $K/\mathbb{F}_p$ is transcendental, then K contains a rational function field $\mathbb{F}_p(x)$, whose multiplicative group contains again a free abelian group of infinite rank — with generators given by the monic irreducible polynomials over $\mathbb{F}_p$.[11] Thus, we may now assume that K is an algebraic extension of $\mathbb{F}_p$. Since $[K : \mathbb{F}_p]$ is infinite, there exists an infinite chain of finite fields $\mathbb{F}_p \subsetneq K_1 \subsetneq K_2 \subsetneq \cdots$ within K. Let $e_i := [K_i : \mathbb{F}_p]$, and let $E := \{e_1, e_2, \ldots\}$, which is an infinite set of distinct positive integers. By Proposition 7.1, there are infinitely many primes, say $p_1, p_2, \ldots$, such that each p_j divides some $p^{e_i} - 1 = |K_i^*|$. By Cauchy's theorem, K_i^* (and hence also K^*) contains an element of (multiplicative) order p_j. Therefore, K^* contains a copy of the infinite direct sum $\bigoplus_{j=1}^{\infty} \mathbb{Z}/p_j\mathbb{Z}$, as desired. $\square$

Epilogue. We take this opportunity to point out that the main results of this paper can be further generalized, as follows. In modification of Definition 2.0, we may define a ring S to have (right) *power stable range* 1 (written $\mathrm{psr}(S) = 1$) if $aS + bS = S \Rightarrow a^n + bx \in \mathrm{U}(S)$ for some $x \in S$ and some integer $n \geqslant 2$ depending on $a, b \in S$. (Of course, $\mathrm{ssr}(S) = 1$ implies $\mathrm{psr}(S) = 1$.) Using this modified definition, one can check that all results in Sections 2, 4, and 5 can be carried over from the $\mathrm{ssr}(S) = 1$ case to the $\mathrm{psr}(S) = 1$ case. The proofs in the aforementioned sections will only need to be slightly adjusted, mostly by simply replacing a^2 by a^n (for some $n \geqslant 2$). In a few cases, a little more may need to be done. For instance, for the crucial Theorem 5.4, one would keep the five equivalent conditions (1)–(5) as stated, but add (2)′, (3)′ with a^2 replaced by a^n (for some $n \geqslant 2$), and also add "(5)′: $\mathrm{psr}(a) = 1$ for all $a \in \mathrm{reg}(S)$." The proof that these new conditions are equivalent to the original ones does not really require any new ideas. The upshot of all these changes is that they lead to a group of stronger and more encompassing results. For instance, the "new" version of Theorem 4.4 will say that, if $S/\mathrm{rad}\, S$ is an exchange ring, then $\mathrm{psr}(S) = 1$ is simply equivalent to $\mathrm{ssr}(S) = 1$; the "new" version of Theorem 5.8 will imply that, to check that an exchange ring S is strongly IC, it suffices to verify that $\mathrm{psr}(a) = 1$ for all $a \in \mathrm{reg}(S)$.

Among all the "new" results which come to light in the above manner, a most significant one is the following refinement of Remark 2.2 concerning power-substitution. In the notation of the power stable range introduced above, Goodearl's result [Go$_1$, (3.2)] simply states that *a commutative ring S has the power-substitution property iff* $\mathrm{psr}(S) = 1$. This is proved by first giving $\mathrm{psr}(S) = 1$ (for commutative S) the following natural matrix-theoretic interpretation (where $a, b, s \in S$):

$$(\dagger) \qquad as + b = 1 \quad \Rightarrow \quad aI_n + bX \in \mathrm{GL}_n(S) \quad \text{for some } X \in \mathbb{M}_n(S) \text{ with } n \geqslant 2,$$

after which [Go$_1$, Theorem 2.1] gives the equivalence with power-substitution. In particular, given $\mathrm{psr}(S) = 1$, power-cancellation holds for any right module A_k with $\mathrm{End}_k(A) \cong S$.

In general, if a ring S has $\mathrm{sr}(S) = 1$, then $\mathrm{psr}(S) = 1$ is simply equivalent to $\mathrm{ssr}(S) = 1$. (The "only if" part is proved as follows. If $\mathrm{psr}(S) = 1$, $aS + bS = S$ will give a unit $a^n + bx$ for some $x \in S$ and $n \geqslant 2$. But then $a^2 S + bS = S$, so $\mathrm{sr}(S) = 1$ guarantees a unit of the form $a^2 + by$ for some $y \in S$.) We note, however, that the class of rings S with $\mathrm{psr}(S) = 1$ is substantially larger than the class of rings with $\mathrm{ssr}(S) = 1$. For instance, if S is a right duo ring (in the sense that bS is an ideal for all $b \in S$), and if $\mathrm{U}(S/bS)$ is a torsion group for each $b \neq 0$, then we have obviously $\mathrm{psr}(S) = 1$, but not

[11] This case is similar to $\mathbb{Q}$, since $\mathbb{F}_p(x)$ is a function field analogue of the global field $\mathbb{Q}$.

necessarily $\mathrm{ssr}(S) = 1$. Thus, for instance, $\mathrm{psr}(\mathbb{Z}) = 1$, but $\mathrm{ssr}(\mathbb{Z}) \neq 1 \neq \mathrm{sr}(\mathbb{Z})$. This simple example also shows that Theorem 6.8 is not true in the case $\mathrm{psr}(S) = 1$, since $\mathbb{Z}$ is an infinite domain with this property, but it has only two units. Many other examples of rings with power-substitution are given in [Go_1].

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