Extensions of Jordan Bases for Invariant Subspaces of a Matrix

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ABSTRACT

A characterization is obtained for the matrices $A$ with the property that every (some) Jordan basis of every $A$-invariant subspace can be extended to a Jordan basis of $A$. These results are based on a criterion for a Jordan basis of an invariant subspace to be extendable to a Jordan basis of the whole space. The criterion involves two concepts: the constancy property and the depth property.

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1. INTRODUCTION

Let $A$ be an $n \times n$ complex matrix considered as a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$. A chain (for $A$) is a set of nonzero vectors

$$\{u, (A - \lambda I)u, \ldots, (A - \lambda I)^{k-1}u\} \quad (1.1)$$

such that $(A - \lambda I)^k u = 0$. The complex number $\lambda$ is necessarily an eigenvalue of $A$ and $(A - \lambda I)^{k-1}u$ is an eigenvector. A Jordan basis for an invariant subspace $W$ is a basis for $W$ which is the union of chains. A Jordan basis of $\mathbb{C}^n$ will be called a Jordan basis for $A$. That is, a Jordan basis for $A$ is a basis of the form

$$\{u_i, (A - \lambda_i I)u_i, \ldots, (A - \lambda_i I)^{k_i-1}u_i; i = 1, \ldots, t\} \quad (1.2)$$

where $u_i \in \mathbb{C}^n$ and $(A - \lambda I)^{k_i}u_i = 0$. The existence of a Jordan basis for any $n \times n$ matrix $A$ is well known and follows from the existence of the Jordan normal form of $A$.

Given a Jordan basis (1.2), certain $A$-invariant subspaces are seen immediately. Namely, for any choice of integers $m_i (i = 1, \ldots, t)$ such that $0 \leq m_i \leq k_i$, the subspace

$$M = \text{span}\{(A - \lambda I)^{m_i}u_i, (A - \lambda I)^{m_i+1}u_i, \ldots, (A - \lambda I)^{k_i-1}u_i; i = 1, \ldots, t\} \quad (1.3)$$

is $A$-invariant, i.e., $Ax \in M$ for every $x \in M$ [the equality $m_i = k_i$ for some $i$ is interpreted as the indication that $i$ is missing in the formula (1.3)]. The $A$-invariant subspaces that arise in this way, starting with any Jordan basis, are called marked in [2]. Equivalently, an $A$-invariant subspace $M$ is called marked if there is a Jordan basis for the restriction $A_M: M \to M$ which can be extended (by adjoining to it new vectors) to a Jordan basis for $A$ in $\mathbb{C}^n$.

Generally, not every $A$-invariant subspace is marked (an example is given in [2]). The existence of nonmarked invariant subspaces is sometimes overlooked in linear algebra texts. In this paper we characterize those matrices $A$ for which every invariant subspace is marked. We also characterize the matrices $A$ with a stronger property, namely, that every $A$-invariant subspace is strongly marked. Let us define this notion: an $A$-invariant subspace $M$ is strongly marked if every Jordan basis of $M$ can be extended (by adjoining
new vectors) to a Jordan basis for $A$ in $\mathbb{C}^n$. These notions call our attention to a more general question: when can a given Jordan basis for an $A$-invariant subspace be extended to a Jordan basis for the whole space $\mathbb{C}^n$? We solve this problem in Section 2 in terms of the height and depth of vectors and related properties. Another characterization (in different terms) of this extendability property is given in [1].

These results are used in subsequent sections to characterize marked and strongly marked subspaces. This characterization goes as follows. (The multiplicities of a matrix $A$ corresponding to its eigenvalue $\lambda_0$ are simply the sizes of the Jordan blocks with the eigenvalue $\lambda_0$ in the Jordan normal form of $A$.)

**Theorem 1.1.** Let $A$ be an $n \times n$ matrix. Then every $A$-invariant subspace is marked if and only if for every eigenvalue $\lambda_0$ of $A$ the difference between the biggest and the smallest multiplicity of $A$ corresponding to $\lambda_0$ does not exceed 1.

**Theorem 1.2.** Let $A$ be an $n \times n$ matrix. Then every $A$-invariant subspace is strongly marked if and only if for every eigenvalue $\lambda_0$ of $A$ all multiplicities of $A$ corresponding to $\lambda_0$ are equal.

To illustrate these results consider the following example. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

According to Theorems 1.1 and 1.2, every $A$-invariant subspace is marked, but there are $A$-invariant subspaces which are not strongly marked. For example, $K(A)$ is not strongly marked. Here and elsewhere in this paper $K(A)$ stands for the kernel (null space) of the matrix $A$. Indeed, a Jordan basis for $A_{K(A)}$ given by $(\alpha_1,0,\beta_2,0,\gamma_1)^T,(\alpha_2,0,\beta_2,0,\gamma_2)^T,(\alpha_3,0,\beta_3,0,\gamma_3)^T$ (here $\alpha_j,\beta_j,\gamma_j \in \mathbb{C}$) can be extended to a Jordan basis for $A$ if and only if there are two zeros among the numbers $\gamma_1,\gamma_2,\gamma_3$. An easy (but somewhat tedious) analysis shows that the following is a complete list of all $A$-invariant subspaces which are not strongly marked: $K(A)$; all 2-dimensional $A$-invariant subspaces spanned by eigenvectors, with the exception of Span$\{(1,0,0,0,0)^T, (0,0,1,0,0)^T\}$; all 4-dimensional $A$-invariant subspaces containing $K(A)$. 
As a corollary we recover the following result from [2] (Theorem 2.9.2). In fact the conclusion of Theorem 2.9.2 of [2] is weaker in the sense that only the marked property of every \( A \)-invariant subspace is asserted there.

**Corollary 1.2.** Let \( A \) be an \( n \times n \) matrix such that for every eigenvalue \( \lambda \) of \( A \) at least one of the following holds:

(a) the geometric multiplicity (i.e., the dimension of \( K(A - \lambda I) \)) is equal to the algebraic multiplicity;  
(b) \( \dim K(A - \lambda I) = 1 \).

Then every \( A \)-invariant subspace is strongly marked.

The proofs of Theorems 1.1 and 1.2 will be given in Sections 3 and 4, respectively.

We conclude the introduction by remarking that it is sufficient to prove Theorems 1.1 and 1.2 (and Theorem 2.1 stated below) for the case when \( A \) has a single eigenvalue \( \lambda_0 \) (without loss of generality it can be assumed that \( \lambda_0 = 0 \)). This follows readily from the well-known fact that every \( A \)-invariant subspace \( M \) can be written as

\[
M = [M \cap R_{\lambda_1}(A)] + \cdots + [M \cap R_{\lambda_r}(A)],
\]

where \( \lambda_1, \ldots, \lambda_r \) are all the distinct eigenvalues of \( A \) and

\[
R_{\lambda_j}(A) = K(A - \lambda_j I)^n
\]

is the root subspace of \( A \) corresponding to \( \lambda_j \). Thus, it will be assumed in Sections 2, 3, and 4 that \( A \) is nilpotent: \( A^n = 0 \).

2. **HEIGHT AND DEPTH**

Let \( A \) be an \( n \times n \) nilpotent complex matrix.

For a given \( x \in \mathbb{C}^n \) let the *height* of \( x \) [notation: \( \text{ht}(x) \)] be the minimal nonnegative integer \( k \) such that \( A^k x = 0 \) (as usual, we assume \( A^0 = I \); thus zero is the only vector of height zero). For \( x \neq 0 \), the *depth* of \( x \) [notation: \( \text{dpth}(x) \)] is by definition the maximal nonnegative integer \( k \) such that
x = A^k y for some y. Note the following easily verified properties: For complex numbers α_1, ..., α_s, and vectors x_1, ..., x_s, we have

\[ \text{ht} \left( \sum_{i=1}^{s} \alpha_i x_i \right) < \max \{ \text{ht}(x_i) : i = 1, \ldots, s \}, \]  \hspace{1cm} (i)

\[ \text{dpth} \left( \sum_{i=1}^{s} \alpha_i x_i \right) \geq \min \{ \text{dpth}(x_i) : i = 1, \ldots, s \}, \]  \hspace{1cm} (ii)

and the strict inequality

\[ \text{dpth}(x) \neq \text{dpth}(y) \Rightarrow \text{dpth}(x + y) = \min\{\text{dpth}(x), \text{dpth}(y)\}, \]  \hspace{1cm} (iii)

provided all vectors in (ii) and (iii) are nonzero. Also, for 0 \neq u \in \mathbb{C}^n, we have

\[ \text{ht}(Au) = \text{ht}(u) - 1, \]  \hspace{1cm} (iv)

\[ \text{dpth}(Au) > \text{dpth}(u), \]  \hspace{1cm} (v)

provided that Au \neq 0.

We address the question when a given Jordan basis B (1.2) for W can be extended to a Jordan basis for the whole space \( \mathbb{C}^n \), i.e., when there is a Jordan basis T in \( \mathbb{C}^n \) such that \( B \subseteq T \) (as sets of vectors). The answer is based on two notions that we call the constancy property and the depth property.

We say that a nonzero vector x has the constancy property (CP) if either \( Ax = 0 \) or \( Ax \neq 0 \) and

\[ \text{dpth}(Ax) = \text{dpth}(x) + 1. \]

A set S of nonzero vectors is said to have the CP if every vector in S has the CP. In particular, the notion of the constancy property can be applied to a chain \( S = \{x, Ax, \ldots, A^{k-1}x\} \); thus, this chain has CP if and only if

\[ \text{dpth}(A^{i-1}x) = \text{dpth}(x) + i - 1, \quad i = 1, \ldots, k. \]  \hspace{1cm} (2.1)
As by (iv) $ht(A^i x) = k - i$ ($0 \leq i \leq k - 1$), these equalities can be rewritten in the form

$$dpth(x) + ht(x) = dpth(A^i x) + ht(A^i x), \quad i = 0, \ldots, k - 1. \quad (2.2)$$

Also, if $dpth(A^{k-1} x) = k - 1$, then necessarily $dpth(x) = 0$ and (2.2) holds, and thus the chain $\{x, Ax, \ldots, A^{k-1} x\}$ has the CP.

In what follows we use the notation $\langle q \rangle$ for the set $\{1, \ldots, q\}$.

We say that a linearly independent set of vectors $\{x^i : i \in \langle q \rangle\}$ has the DP (the depth property) if $w = \sum_{i \in \langle q \rangle} \alpha_i x^i, \ w \neq 0$, implies that

$$dpth(w) = \min\{dpth(x^i) : i \in \langle q \rangle \text{ and } \alpha_i \neq 0\}. \quad (2.3)$$

The two properties CP and DP do not imply each other, as examples will presently show. First, note that every chain $\{x, Ax, \ldots, A^{k-1} x\}$ is linearly independent, by a standard argument. It follows from (iii) and (v) that every chain has the DP. An example of a chain without the CP (but with DP) is furnished by $\{u, Au\}$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $u = (0, 1, 0, 1)^T$. The following example shows a linearly independent set of chains without the DP. Let

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$u = (1, 0, 1)^T$, and $v = (0, 0, 1)^T$. Then each of the (singleton) chains $\{u\}$ and $\{v\}$ has the CP, the set $\{u, v\}$ is linearly independent, but $\{u, v\}$ does not have the DP. Indeed, $dpth(u) = dpth(v) = 0$, but for a nonzero vector $w = \alpha u + \beta v$ we have $dpth(w) = 0$ if $\alpha + \beta \neq 0$ and $dpth(w) = 1$ if $\alpha + \beta = 0$.

The main result of this paper is the following (which holds without the assumption that $A$ is nilpotent, though the proof is given only for nilpotent $A$; see the end of Section 1).

**Theorem 2.1.** Let $A$ be a complex $n \times n$ matrix. Let $B$ be a Jordan basis for an $A$-invariant subspace $W$. Then $B$ can be extended to a Jordan basis for
A in $\mathbb{C}^n$ if and only if $B$ has the CP and the DP.

Proof. “If”: Let

$$C = \{u_i, A u_i, \ldots, A^{k_i-1} u_i : i = 1, \ldots, t\}$$

be a Jordan basis for $\mathbb{C}^n$. It is enough to prove that $C$ has the CP and the DP, for then any subset of $C$ has the CP and the DP.

Let $w$ be a nonzero vector in $\mathbb{C}^n$. Then $w$ can be written uniquely as

$$w = \sum_{i=1}^{t} \sum_{j=0}^{k_i-1} \alpha_{ij} A^j u_i, \quad \alpha_{ij} \in \mathbb{C}, \quad j = 0, \ldots, k_i - 1, \quad i = 1, \ldots, t.$$  

Then, for $p \geq 0$, $A^p w$ has the unique representation

$$A^p w = \sum_{i=1}^{t} \sum_{j=0}^{k_i-1} \alpha_{i,j-p} A^j u_i, \quad j = 0, \ldots, k_i - 1, \quad i = 1, \ldots, t,$$

where $\alpha_{ij} = 0$ whenever $j < 0$, $i = 1, \ldots, t$. If $Aw \neq 0$, then it follows easily that

$$\text{dpth}(w) = \min\{ j : \text{at least one of } \alpha_{ij}, i = 1, \ldots, t, \text{ is nonzero} \}. \quad (2.4)$$

In particular,

$$\text{dpth}(A^j u_i) = j, \quad j = 0, \ldots, k_i - 1, \quad i = 1, \ldots, t. \quad (2.5)$$

Thus, by (2.1), $C$ has the CP, and, by applying (2.5) to (2.4) we see that $C$ has the DP.

Hence if $B$ is a subset of $C$, then $B$ has the CP and the DP.

“Only if”: We suppose that $W \neq \mathbb{C}^n$, for otherwise of course there is nothing to prove. We consider two cases. In each case we construct a subspace $W'$ which properly contains $W$ and a Jordan basis $B' \supseteq B$ for $W'$ such that $B'$ has the CP and the DP.

We say that a chain $(u, Au, \ldots, A^{k-1}u)$ is maximal if it is not contained (set theoretically) in a larger chain; in other words, a chain $(u, Au, \ldots, A^{k-1}u)$ is maximal if $\text{dpth}(u) = 0$.

Case I: Some chain of $B$ is not maximal. Suppose that $S = (u, \ldots, A^{k-1}u)$ is a chain of $B$, and that $\text{dpth}(u) = d > 0$. Let $y \in \mathbb{C}^n$ satisfy $A^d y = u$. 

Then \( \text{dpth}(y) = 0 \). Let \( S' = \{y, \ldots, A^{d+h-1}y\} \). We let \( B' \) consist of the chains of \( B \) with \( S \) replaced by \( S' \), and we let \( W' = \text{span}(B') \).

**Claim 1.1.** \( B' \) is linearly independent. Otherwise, there exists a nontrivial linear relation on \( B' \), and since this cannot be a nontrivial linear relation on \( B \), it must involve an element of form \( A^r y \), where \( r < d \). We choose the minimal such \( r \). Multiplying this linear relation by \( A^{d-r} \), we obtain a linear relation on the elements of \( B \), which is nontrivial, since it involves \( A^d y = u \). But this is impossible, since \( B \) is linearly independent.

**Claim 1.1.** The chain \( S' = \{y, \ldots, A^{d+h-1}y\} \) has the CP. Otherwise, by (2.1), \( \text{dpth}(A^{d+h-1}y) > d + h - 1 \), and there is a \( y' \in \mathbb{C}^n \) such that

\[
A^{d+h}y' = A^{d+h-1}y = A^{h-1}u.
\]

But then

\[
\text{dpth}(A^{h-1}u) - \text{dpth}(u) > d + h - d = h,
\]

which is impossible by (2.1), since \( S \) has the CP. Hence \( \text{dpth}(A^{d+h-1}y) = d + h - 1 \), and hence \( S' \) has the CP.

**Claim 1.3.** \( B' \) has the DP. Recall that every chain has the DP. Suppose \( B' \) does not. Since \( B \) and \( S' \) have the DP, it is easily shown using (iii) that there exists a \( w \in W \) and an

\[
x = \sum_{s \in \{r, \ldots, d-1\}} \gamma_s A^s y, \quad \gamma_s \in \mathbb{C}, \quad \gamma_r \neq 0.
\]  

(2.6)

where \( 0 \leq r < d \), such that \( w \neq 0 \), \( x \neq 0 \),

\[
\text{dpth}(w) = \text{dpth}(x) = r,
\]

(2.7)

and, for \( v = w + x \),

\[
\text{dpth}(v) > r.
\]

(2.8)

We then obtain

\[
\text{dpth}(A^{d-r}v) \geq \text{dpth}(v) + d - r > d.
\]

(2.9)

But this is impossible, for \( A^{d-r}v \) is a linear combination of nonzero elements
of $B$ one of which is $A^d y = u$, and $\text{dpth}(u) = d$. This proves the claim, and completes the proof of case I.

**Case II:** Every chain of $B$ is maximal.

**Claim II.1.** There exists $v \in \mathbb{C}^n$, $v \notin W$, with $ht(v) = 1$ (i.e., $v$ is an eigenvector of $A$). Let $u \in \mathbb{C}^n$, $u \notin W$. Let $ht(u) = h$. If $A^{h-1} u \notin W$, the claim is true. Otherwise there exists a least $r$, $0 < r < h - 1$, such that $A^r u \in W$. Thus $A^r u$ is a linear combination of $B$, and, since $B$ has the DP, it is a linear combination of elements of $B$ whose depth is at least $1$. Thus there exists a $w \in W$ such that $A^r w = A^r u$. Let $x = w - A^r u$. Then $x \in W$ and $Ax = 0$, which proves the claim.

We now choose a chain $S = \{u, \ldots, A^{h-1} u\}$ of maximal length such that $A^{h-1} u = v$ is not in $W$. Let $B' = B \cup S$. Then it is easy to prove that $W \cap \text{span}(S) = 0$, and it follows that $B'$ is a basis for $W' = W \oplus \text{span}(S)$.

**Claim II.2.** The chain $S$ has the CP. Since $S$ is a maximal chain beginning at $u$, clearly $\text{dpth}(u) = 0$. Suppose $S$ does not have the CP. Then, by (2.1), $\text{dpth}(A^{h-1} u) > h - 1$. Hence there is a $w \in \mathbb{C}^n$ such that $A^h w = A^{h-1} u$. Thus the chain $\{w, \ldots, A^{h-1} u\}$ has greater length than $S$, contrary to the assumption that $S$ is a maximal chain whose last element is not in $W$.

**Claim II.3.** $B'$ has the DP. Suppose $B'$ does not have the DP. Since $B$ and the chain $S$ have the DP, there must exist

$$v = w + x, \quad w \in W, \quad x \in \text{span}(S),$$

such that

$$\text{dpth}(v) > \min\{\text{dpth}(w), \text{dpth}(x)\}. \quad (2.10)$$

By (iii), we then have

$$\text{dpth}(w) = \text{dpth}(x) = d, \quad \text{say}, \quad (2.11)$$

when $0 \leq d < h$. By (2.11), we have

$$x = \sum_{r \in \{d, \ldots, h - 1\}} \gamma_r A^r u, \quad \gamma_r \in \mathbb{C}, \quad \gamma_d \neq 0.$$ 

By (2.10), there is a $z \in \mathbb{C}^n$ such that $A^{d+1} z = v$. Then

$$A^h z = A^{h-d-1} v = A^{h-d-1} w + A^{h-d-1} x = A^{h-d-1} w + \gamma_d A^{h-1} u.$$
Since $A^{h-d}w \in W$ and $\gamma_d A^{h-1}u \neq 0$, it follows that there is a chain of length $h+1$ which ends outside $W$, contrary to our assumption on $S$. This proves our claim, and completes the proof of case II.

Thus in either case, we have constructed an invariant subspace $W'$ with $\dim(W') > \dim(W)$ and a Jordan basis $B' \supset B$ with the DP such that $B'$ has the CP. By repeating this argument we obtain a Jordan basis for $\mathbb{C}^n$, which is an extension of $B$.

Another necessary condition for extendability of $B$ to Jordan basis if $\mathbb{C}^n$ can be given in terms of multiplicities, as follows. We write the list of all multiplicities (including repetitions, if necessary) in a nonincreasing order: $\lambda_1 \geq \cdots \geq \lambda_n$. A sequence of positive integers $\beta_1, \ldots, \beta_p$ will be called a sublist of multiplicities if $p \leq q$ and there is a one-to-one map $\xi : \{1, \ldots, p\} \to \{1, \ldots, q\}$ such that $\beta_i - \lambda_{\xi(i)}$ for $i = 1, \ldots, p$. By the index of a chain $S = \{x, \ldots, A^{h-1}x\}$, denoted $\text{ind}(S)$, we mean $\text{dpth}(A^{h-1}x) + 1$.

By (2.4), it is easy to see that if a Jordan basis $B$ of $W$ is extendable to a Jordan basis of $\mathbb{C}^n$, then the numbers $\text{ind}(S_i)$, where $S_1, \ldots, S_r$ are the chains in $B$, form a sublist of multiplicities. The following example shows that a Jordan basis $B$ with the CP and for which $\text{ind}(S_i)$, $i \in \langle r \rangle$, form a sublist of multiplicities need not be extendable to a Jordan basis of $\mathbb{C}^n$.

**Example 2.1.** Let

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Let $u = (1, 0, 1, 0)^T$, $v = (0, 0, 1, 0)^T$. Then $\{u, v\}$ forms a Jordan basis $B$ of the subspace $W = \text{span}\{(1,0,0,0)^T,(0,0,1,0)^T\}$. By Theorem 2.1 the basis $B$ cannot be extended to a Jordan basis in $\mathbb{C}^4$, since $\text{dpth}(u) = \text{dpth}(v) = 0$, while $\text{dpth}(u - v) = 1$. However, the basis $B$ has the CP and $\{1, 1\}$ is a sublist of multiplicities.

3. **PROOF OF THEOREM 1.1**

Let $A$ be an $n \times n$ nilpotent matrix. We start with the following:

**Proposition 3.1.** Suppose that every $A$-invariant subspace is marked. Then the lengths of any two maximal chains (in a Jordan basis of $A$) differ by less than two.
Proof. Arguing by contradiction, assume that
\[ u, Au, \ldots, A^{r-3}u \] (3.1)
is a maximal chain in a Jordan basis of \( A \), and
\[ v, Av, \ldots, A^{r-1}v \]
is a not necessarily maximal chain in the same Jordan basis of \( A \). Put 
\[ z = u + Av. \]
Note that \( A^{r-2}z = A^{r-2}v \). Further note that \( \text{dpth}(z) = 0 \) [indeed, if \( Ay = z \) for some \( y \), then \( u = A(y - v) \), which is a contradiction with the maximality in (3.1)]. We have \( \text{ht}(A^{r-2}z) = 1 \), \( \text{dpth}(A^{r-2}z) \geq r - 1 \), \( \text{ht}(z) = r - 1 \), and \( \text{dpth}(z) = 0 \); so the chain \( z, Az, \ldots, A^{r-2}z \) does not have the CP and hence by Theorem 2.1 cannot be extended to a Jordan basis for \( \mathbb{C}^n \). Observe that every Jordan basis for \( \text{span}\{Z, \ldots, A^{r-2}z\} \) has the form
\[ \{ w, Aw, \ldots, A^{r-2}w \}, \]
where
\[ w = \sum_{j=0}^{r-2} \alpha_j A^j z, \quad \alpha_j \in \mathbb{C}, \quad \alpha_0 \neq 0. \]
We see that \( \text{dpth}(z) = 0 \) and \( \text{dpth}(A^{r-2}w) = \text{dpth}(\alpha_0 A^{r-2}w) = \text{dpth}(\alpha_0 A^{r-2}z) \geq r - 1 \). Thus, the chain \( (A^j w)_{j=0}^{r-2} \) does not have the CP, and by Theorem 2.1 it cannot be extended to a Jordan basis for \( A \). Therefore, the \( A \)-invariant subspace \( \text{span}\{z, \ldots, A^{r-2}z\} \) is not marked. \( \blacksquare \)

Proposition 3.2. Let \( A \) be an \( n \times n \) nilpotent matrix with sizes of all Jordan blocks equal to \( q \) or \( q - 1 \). Then all chains have the CP.

Proof. Let \( w \) be a vector in \( \mathbb{C}^n \) such that \( Aw \neq 0 \). We shall show that
\[ \text{ht}(w) + \text{dpth}(w) = \text{ht}(Aw) + \text{dpth}(Aw). \] (3.2)
It follows from our assumptions that we may write
\[ w = u + v, \]
where \( u \) and \( v \) are linear combinations of vectors in Jordan chains of lengths,
respectively, $q$ and $q - 1$. Suppose that

$$ht(u) = h, \quad ht(v) = k.$$ 

Then $0 \leq h \leq q$, $0 \leq k \leq q - 1$. If $h = 0$ (i.e. $u = 0$) or $k = 0$ (i.e. $v = 0$), then we are basically in the situation [as far as (3.2) is concerned] when all the multiplicities of $A$ are equal. But in this case (3.2) follows easily [see also the equivalence (1) $\iff$ (2) in Theorem 1.2' of Section 4]. So suppose that $h, k \geq 1$. Note that we cannot have $h = k = 1$, for then $Aw = 0$, contrary to assumption. So either $h > 1$ or $k > 1$. It is easily checked that

$$ht(w) = \max\{h, k\},$$

$$dpth(w) = \min\{q - h, q - 1 - k\},$$

$$ht(Aw) = \max\{h - 1, k - 1\},$$

$$dpth(Aw) = \min\{q - h + 1, q - k\}.$$ 

If $h > k$, it follows that

$$ht(w) + dpth(w) = ht(Aw) = dpth(Aw) = q,$$

while if $h \leq k$

$$ht(w) + dpth(w) = ht(Aw) + dpth(Aw) = q - 1.$$ 

In either case, (3.2) holds and the proposition follows.

**Proposition 3.3.** Let $A$ be an $n \times n$ nilpotent matrix with sizes of all Jordan blocks equal to $q$ or $q - 1$. Let $M$ be an $A$-invariant subspace of $\mathbb{C}^n$, and let $B$ be a Jordan basis for $M$ with a maximal number of eigenvectors of depth $q - 1$. Then $B$ has the DP.

**Proof.** Let $B$ be the Jordan basis

$$\{g_i, Ag_i, \ldots, A^{k_i - 1}g_i, i = 1, \ldots, t\},$$
and let

\[ k = \max\{k_i : i = 1, \ldots, t\} . \]

Let \( B_h, h = 1, \ldots, k, \) be the subset of \( B \) consisting of vectors of height \( h \) or less, viz.

\[ B_h = \{ A^{k_i - j} g_i : j \in \langle h \rangle, i \in \langle t \rangle \} . \]

We shall prove by induction that \( B_h, h - 1, \ldots, k, \) has the DP.

We first consider \( B_1. \) In view of our assumptions on multiplicities, each vector in \( B_1 \) has depth \( q - 2 \) or \( q - 1 \) (since it is an eigenvector of \( A \)). Consider the linear combination of \( B_1: \)

\[ w = \sum_{i=1}^{p} \alpha_i A^{k_i - 1} g_i + \sum_{i=p+1}^{t} \alpha_i A^{k_i - 1} g_i, \quad (3.3) \]

where we may assume that

\[ \text{dpth}(A^{k_i - 1} g_i) = q - 2, \quad i = 1, \ldots, p, \]

\[ \text{dpth}(A^{k_i - 1} g_i) = q - 1, \quad i = p + 1, \ldots, t. \]

Now suppose that \( B_1 \) does not have the DP. Then we may find a vector \( w \) of form \((3.3)\) such that at least one of the coefficients \( \alpha_i, 1 \leq i \leq p, \) is nonzero and \( \text{dpth}(w) = q - 1. \) But then

\[ v = \sum_{i=1}^{p} \alpha_i A^{k_i - 1} g_i, \quad (3.4) \]

also satisfies \( \text{dpth}(v) = q - 1 \) by (iii). Let \( s, 1 \leq s \leq p, \) be an index for which \( \alpha_s \neq 0 \) in \((3.4)\) and such that \( k_s \) is minimal among \( k_i \) for which \( \alpha_i \neq 0 \) in \((3.4).\) Suppose, without loss of generality, that \( \alpha_i \neq 0, i = 1, \ldots, s \) and \( \alpha_i = 0, i = s + 1, \ldots, p. \) Let

\[ u = \sum_{i=1}^{s} \alpha_i A^{k_i - 1} g_i, \]
and in $B$ replace the Jordan chain

$$\{A^j g_j : j = 0, \ldots, k_y - 1\}$$

by the Jordan chain

$$\{A^j u : j = 0, \ldots, k_y - 1\}.$$  

The result is a Jordan basis for $M$ for which the number of eigenvectors of depth $q - 1$ is $t - p + 1$. But, since $B$ has $t - p$ such eigenvectors, this contradicts our assumption on $B$. Hence $B_1$ has the DP.

Now assume inductively that $1 < h < k$ and that $B_{h-1}$ has the DP. To prove that $B_h$ has the DP, we consider

$$0 \neq w = \sum_{i=1}^{t} \sum_{j=k_i'} \alpha_{ij} A^j g_i,$$  \hspace{1cm} (3.5)

where $k_i' = \max\{0, k_i - h\}$, $i = 1, \ldots, t$. We must prove that

$$\text{dpth}(w) = \min\{\text{dpth}(A^j g_i) : \alpha_{ij} \neq 0, j = k_i', \ldots, k_i - 1, i = 1, \ldots, t\}. \hspace{1cm} (3.6)$$

If $\alpha_{ij} = 0$ whenever $j = k_i - h$, then $w$ is a linear combination of elements of $B_{h-1}$, and (3.6) follows from our inductive assumption. So assume that $\alpha_{sj} \neq 0$ for some $j = k_y - h$ and $1 \leq s \leq t$. Note that it follows from our assumption on multiplicities that

$$\text{dpth}(A^{k_i'-1} g_i) \geq q - 2, \hspace{1cm} i = 1, \ldots, s$$

(since the above vectors are eigenvectors), and

$$\text{dpth}(A^{k_y-h} g_s) \leq q - 2,$$

since $h \geq 2$ (and hence this vector is not an eigenvector). Thus to prove (3.6) it is enough to prove

$$\text{dpth}(w) = \min\{\text{dpth}(A^j g_i) : \alpha_{ij} \neq 0, j = k_i', \ldots, k_i - 2, i = 1, \ldots, t\}. \hspace{1cm} (3.7')$$
To prove (3.7) we note that
\[ Aw = \sum_{i=1}^{t} \sum_{j=k_i}^{k_i-2} \alpha_{ij}A^{j+1}g_i \]
(and thus \( Aw \neq 0 \)). Since \( A^{j+1}g_i \in B_{h-1} \), \( j = k'_i, \ldots, k_i - 2 \), \( i = 1, \ldots, t \), our inductive assumption yields
\[ \operatorname{dpth}(Aw) = \min\{\operatorname{dpth}(A^{j+1}g_i) : \alpha_{ij} \neq 0, j = k'_i, \ldots, k_i - 2, i = 1, \ldots, t\}. \]
(3.8)

By Proposition 3.2, every chain has CP. Hence,
\[ \operatorname{dpth}(Aw) = \operatorname{dpth}(w) + 1, \]
\[ \operatorname{dpth}(A^{j+1}g_i) = \operatorname{dpth}(A^jg_i) + 1, \quad j = k'_i, \ldots, k_i - 2, \quad i, \ldots, t. \]

Hence (3.7) now follows from (3.8), and thus \( B_h \) has the DP. By induction, we obtain that \( B_k \) has the DP, and since \( B_k = B \), the result follows.

**Proof of Theorem 1.1.** We may assume that \( A \) is nilpotent. If the difference between the biggest and the smallest multiplicity of \( A \) is at least 2, then by Proposition 3.1 not every \( A \)-invariant subspace is marked. Conversely, assume that the multiplicities of \( A \) are equal to \( q \) and \( q - 1 \), for some \( q \geq 2 \). In view of Propositions 3.2 and 3.3, every \( A \)-invariant subspace \( M \) has a Jordan basis with the DP that also has the CP. The theorem now follows from Theorem 2.1.

We can augment Theorem 1.1 by the following statement.

**Theorem 3.4.** Assume \( A \) is a nilpotent. Then every \( A \)-invariant subspace is marked if and only if there is \( q \) such that the index of every vector in \( \mathbb{C}^n \) is either \( q \) or \( q - 1 \).

Theorem 1.1 was contained in an unpublished manuscript by the authors dated July 1988. A related result (in the framework of solutions of Riccati equations) was obtained independently in [3].
4. PROOF OF THEOREM 1.2

We will actually prove a more informative result.

**Theorem 1.2'.** The following are equivalent for a nilpotent matrix $A$:

1. All multiplicities are equal (to $q$).
2. For all $x \in \mathbb{C}^n \setminus \{0\}$, $ht(x) + dpth(x) = q$.
3. All invariant subspaces are strongly marked.

**Proof.** (1) $\Rightarrow$ (2): Let $C$ be a Jordan basis in $\mathbb{C}^n$. Then every element of $C$ of height $h$ has depth $q-h$, and (2) follows because $C$ has the DP (see the proof of the "if" part of Theorem 2.1).

(2) $\Rightarrow$ (3): Clearly, (2) implies that every chain has the CP. Let $W$ be an invariant subspace for $A$. Since all Jordan bases for $W$ contain the same number of eigenvectors, and by (2), all eigenvectors have the same depth $q-1$, it follows that every Jordan basis for $W$ satisfies the hypotheses of Proposition 3.3. Hence every Jordan basis for $W$ has the DP. We now obtain (3) by Theorem 2.1.

(3) $\Rightarrow$ (1): Suppose (1) is false, and $x$ and $y$ generate Jordan chains of lengths $q$ and $r$ respectively, where $r < q$. Let $u = A^{r-1}y - A^{q-1}x$ and $v = A^{q-1}x$. Then $dpth(u) = dpth(v) = q - 1$, but $dpth(u + v) = r - 1$. Hence the Jordan basis $\{u, v\}$ for the invariant subspace span $\{u, v\}$ does not have the DP. By Theorem 2.1, span$\{u, v\}$ is not strongly marked.

We now give a characterization of condition (1) in Theorem 1.2' in terms of the Weyr characteristic. Recall that the Weyr characteristic of a matrix $X$ corresponding to eigenvalue $\lambda$ is the vector $(w_1, w_2, \ldots, w_d)$, where

$$w_j = \dim K(X - \lambda I)^j - \dim K(X - \lambda I)^{j-1}, \quad j = 1, 2, \ldots, d,$$

and $d$ is the largest multiplicity of $X$ corresponding to $\lambda$.

**Proposition 4.2.** The following statements are equivalent for a nilpotent matrix $A$ (we denote by $d$ the largest multiplicity of $A$):

1. $K(A^{d-1}) \subset R(A)$;
2. the Weyr characteristic of $A$ is $(w_1, \ldots, w_d)$, where $w_1 = w_2 = \ldots = w_d$;
3. all multiplicities of $A$ equal $d$. 
Here $R(A)$ denotes the range of $A$. Proposition 4.2 can be easily proved by inspecting the Jordan form of $A$.

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