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Phase semantics for light linear logic

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Abstract

Light linear logic (Girard, Inform. Comput. 14 (1998) 175–204) is a refinement of the propositions-as-types paradigm to polynomial-time computation. A semantic setting for the underlying logical system is introduced here in terms of fibred phase spaces. Strong completeness is established, with a purely semantic proof of cut elimination as a consequence. A number of mathematical examples of fibred phase spaces are presented that illustrate subtleties of light linear logic. © 2002 Published by Elsevier Science B.V.

1. Introduction

Typed lambda calculi have long been recognized as analogous to formal logical calculi of intuitionistic logic. In technical terms this correspondence is known as the *Curry–Howard isomorphism* or the *propositions-as-types paradigm*. Logic provides not only basic input/output specifications (*i.e.*, types or formulas), but also a setting for well-typed programs (*i.e.*, terms or formal proofs), as well as a mode of execution of well-typed programs by means of term reduction or normalization [4]. The advent of linear logic [3] with its intrinsic ability to reflect computational resources has made

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it possible to refine the propositions-as-types paradigm to computational complexity specifications. A bounded version of linear logic (BLL) was introduced in [6], in which the reuse of resources is bounded in advance, and in which any functional term of appropriate type encodes a polynomial-time algorithm. Conversely, any polynomial-time function arises in this way. A detailed comparison of this approach to various other logical characterizations of polynomial-time functions may be found in [6, 5]. A major advantage of BLL is that the system itself is (locally) polynomial time. The run-time normalization complexity is implicit in the system and does not need to be enforced explicitly in the syntax.

From a strictly logical point of view, however, BLL still suffers from the presence of explicit resource parameters (whose technical role is to indicate input/output size ratios.) In this sense BLL is not a purely logical system. This difficulty is resolved in Girard's *light linear logic* (LLL) [5], which keeps all the advantages of BLL, but avoids mentioning the resources altogether. In LLL resources can be synthesized by purely logical means.

The basic idea in [5] is to set up the structural rules and the logical rules for modalities more carefully than in linear logic so that the computational power of normalization can be well controlled. In the course of setting up such well-controlled rules, central points are to dispense with the principles $!A \multimap A$ and $!A \multimap !!A$, but to retain the *exponential isomorphism* $!A \otimes !B \simeq !(A \& B)$. LLL shares some of these and other technical features with the systems studied in [1, 11]. A more subtle, but equally important point of LLL is to reject the principle $!A \otimes !B \multimap !(A \otimes B)$. In order to compensate for this, LLL adds a self-dual modality § that satisfies $!A \multimap \S A$ and $\S A \multimap ?A$ and $\S A \otimes \S B \multimap \S(A \otimes B)$. Although syntax of LLL is well-understood owing to Girard's careful analysis [1], semantics for LLL has remained an open question.

Surprisingly, an answer is suggested by another research direction, namely by the work of the first author and Ito on extensions of linear logic with certain features of temporal logic [7]. Models of temporal logic [16] distinguish among semantic objects at different "points in time" (much as Kripke models distinguish among semantic objects in different "worlds".) Temporal logic models also feature a semantic operator "*next*" such that "*next A*" at time t is A at time $t+1$. Our starting point is that not only does LLL modality § behave in many ways like the operator "*next*" (except for the self-dual nature of §), but that, for instance, the principle $!A \multimap A$ fails in such a stratified setting. Informally, consider a semantic setting for linear logic and repeat it, each time at a different level t. Let $(!A)_t$ be the given definition of ! applied to A_{t+1}. In this reading, for any t, $(!A)_t$ yields $(\S A)_t$ but there is no general reason why $(!A)_t$ should yield A_t. In fact, a closer analysis reveals that the semantic intuition of "levels" or "stages" t is related to the syntactic notion of *nesting depth of proof-boxes* in LLL. (The basic idea described above may be modified slightly by means of explicit transitions between levels so that semantic definitions at a given level t refer to transitions to t rather than to other levels, see Section 2.)

In this paper this analysis is applied in the context of phase semantics for linear logic [3, 9, 10, 12–15]. We explain how $!A \otimes !B \multimap !(A \otimes B)$ can fail "in nature." We also establish strong completeness for LLL (*i.e.*, valid formulas are provable without the cut rule), and thus we obtain a purely semantic proof of Cut Elimination (*i.e.*, provable formulas are also provable without the cut rule) following the technique in [12–14]. A preliminary version of this work appears in [8].

Similar analysis may also be carried out in other semantic settings such as coherence spaces, which will be discussed elsewhere. It would also be interesting to see if such semantic methods can also establish the stronger version of cut elimination that proof normalization reductions terminate. See [17] for some references on recent developments of Light Linear Logic and related Light Logics.

2. Fibred phase spaces

A *phase space* $(M, \perp)$ is a commutative monoid M with a distinguished subset $\perp \subseteq M$, called *bottom*. For any subset $\alpha \subseteq M$, define $\alpha^{\perp} =_{def} \{x \in M \mid x \cdot \alpha \subseteq \perp\} = \{x \in M \mid \forall y \in \alpha \;\; xy \in \perp\}$. A subset $\alpha \subseteq M$ is *closed* iff $\alpha^{\perp\perp} = \alpha$. Writing 1 for the neutral element of M, let $\mathbf{1}$ be the subset $\{1\}^{\perp\perp}$, that is, $\perp^{\perp}$. It is readily seen that $\mathbf{1}$ is a closed submonoid.

A *homomorphism of phase spaces*, or simply a *phase homomorphism*, is a monoid homomorphism $h : M \to M'$ such that $h(\perp) \subseteq \perp'$.

A phase space induces a natural preorder on the underlying monoid compatible with monoid multiplication:

$$x \preccurlyeq y \Leftrightarrow_{def} x \in \{y\}^{\perp\perp}.$$

Note that a phase homomorphism is not required to be monotone in the induced preorder.

More generally, a *phase structure* is a commutative monoid M with a *closure operator* on M, that is, a mapping Cl from subsets of M to subsets of M satisfying the following four properties for any $\alpha, \beta \subseteq M$:

(Cl1) $\alpha \subseteq Cl(\alpha)$,
(Cl2) $Cl(Cl(\alpha)) = Cl(\alpha)$,
(Cl3) $\alpha \subseteq \beta \Rightarrow Cl(\alpha) \subseteq Cl(\beta)$,
(Cl4) $Cl(\alpha) \cdot Cl(\beta) \subseteq Cl(\alpha \cdot \beta)$.

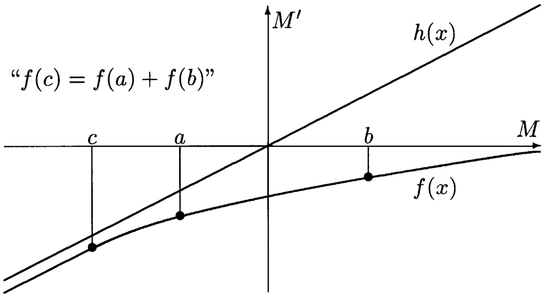
A subset $\alpha \subseteq M$ is said to be *closed* iff $Cl(\alpha) = \alpha$. One can again define a preorder compatible with monoid multiplication: $x \preccurlyeq y$ iff $Cl(\{x\}) \subseteq Cl(\{y\})$. A phase space is a special case where $Cl(\alpha) =_{def} \alpha^{\perp\perp}$.

For a given mapping $g : M \to M'$, let us consider its *lower approximations*, that is, mappings $f : M \to M'$ such that for every $a \in M$ there exists $b \in M$ such that $b \preccurlyeq a$ and $f(a) \preccurlyeq g(b)$. In this case we also say that f is *bounded by* g.

We are particularly interested in lower approximations that satisfy a certain continuity property. A mapping $f : M \to M'$ has *the intermediate value property* iff for every $a, b \in M$ such that $f(a) \in \mathbf{1}'$ and $f(b) \in \mathbf{1}'$, there exists $c \in M$ such that $c \preccurlyeq a$, $c \preccurlyeq b$, and

$f(a)f(b) = f(c)$. Note that the identity function has the intermediate value property with $c = ab$. However, in our applications, f will be bounded by $1'$, which will provide that $f(a) \preccurlyeq' 1'$ for all $a \in M$.

Example 2.1. Consider the reals with addition, where $\perp$ consists of the negative reals. In this phase space $a \preccurlyeq b$ iff $a \leq b$. Any linear function $h(x) = kx$, with a positive k, is certainly a phase homomorphism. Let f be a continuous function such that $f \preccurlyeq h$ and $f \preccurlyeq 0$.



f has the intermediate value property, both in the ordinary sense and as a mapping of phase spaces. The latter is a special case of the former on $(-\infty, \min\{a, b\}]$ because $\lim_{x \to -\infty} f(x) = -\infty$.

Example 2.2. Let the phase structure M_0 consist of the nonpositive integers, with $a \cdot b =_{def} \min\{a, b\}$. Let $Cl(\alpha) =_{def} \{z \in M_0 \mid \exists x \in \alpha . z \preccurlyeq x\}$. The properties (Cl1)–(Cl4) hold. Note that all elements of M_0 are idempotent, that is, $aa = a$ for all a. Also, $\mathbf{1}$ is the entire monoid.

Let M_1 be the integers with addition. The properties (Cl1)–(Cl4) again hold if $Cl(\alpha) =_{def} \{z \in M_1 \mid \exists x \in \alpha . z \preccurlyeq x\}$. In this case, $\mathbf{1} = (-\infty, 0]$, here meaning the *integers* $\preccurlyeq 0$.

Let $h_0 : M_1 \to M_0$ be the constant function 0 and let $f_0 : M_1 \to M_0$ be the function

$$f_0(a) = \begin{cases} 0, & \text{if } a > 0, \\ a - 1, & \text{if } a \preccurlyeq 0. \end{cases}$$

f_0 has the intermediate value property: if $a \preccurlyeq b$ take $c = a$, and if $b < a$ then take $c = b$. Then $f_0(a)f_0(b) = \min\{f_0(a), f_0(b)\} = f_0(c)$.

A *fibred phase space* is a family $\{(M_n, \perp_n), h_n, f_n\}_{n \geqslant 0}$, where for each integer $n \geqslant 0$, $(M_n, \perp_n)$ is a phase space, $h_n : M_{n+1} \to M_n$ is a phase homomorphism, and $f_n : M_{n+1} \to M_n$ is a mapping with the intermediate value property such that f_n is bounded by h_n. A *fibred phase structure* is defined similarly, but each h_n is only required to be a monoid homomorphism.

Given a fibred phase structure, consider a family $\alpha = \{\alpha_n\}_{n \geqslant 0}$, where each $\alpha_n \subseteq M_n$ is closed in M_n. One says that α is *closed*. For any closed $\alpha = \{\alpha_n\}_{n \geqslant 0}$ and $\beta = \{\beta_n\}_{n \geqslant 0}$

one defines $\mathbf{1}, \alpha \& \beta, \alpha \oplus \beta$, and $\alpha \otimes \beta$ in the natural way induced from the original definition in [3]:

$$
\begin{aligned}
&(\mathbf{1})_n = \mathbf{1}_n,\\
&(\top)_n = M_n,\\
&(\mathbf{0})_n = Cl_n(\emptyset),\\
&(\alpha \& \beta)_n = \alpha_n \cap \beta_n,\\
&(\alpha \oplus \beta)_n = Cl_n(\alpha_n \cup \beta_n),\\
&(\alpha \otimes \beta)_n = Cl_n(\alpha_n \cdot_n \beta_n),\\
&(\alpha \multimap \beta)_n = \{z \in M_n \mid z \cdot_n \alpha_n \subseteq \beta_n\}.
\end{aligned}
$$

$\S\alpha$ and $!\alpha$ are defined in the following way:

$$
\begin{aligned}
&(\S\alpha)_n = Cl_n(h_n(\alpha_{n+1})),\\
&(!\alpha)_n = Cl_n(f_n(\alpha_{n+1}) \cap \mathbf{1}_n \cap J_n),
\end{aligned}
$$

where $J_n \subseteq M_n$ is a submonoid of M_n such that every element of J_n is a *weak idempotent*, *i.e.*, $\forall a \in J_n, a \preccurlyeq_n a \cdot_n a$ (after Y. Lafont).

In a fibred phase space one further defines:

$$
\begin{aligned}
&(\bot)_n = \bot_n,\\
&(\alpha^\bot)_n = \alpha_n^{\bot_n},\\
&(\alpha \parr \beta)_n = (\alpha_n^{\bot_n} \cdot_n \beta_n^{\bot_n})^{\bot_n},\\
&\bar{\S}\alpha = (\S(\alpha^\bot))^\bot,\\
&(?\alpha)_n = (f(\alpha_{n+1}^{\bot_{n+1}}) \cap \mathbf{1}_n \cap J_n)^{\bot_n}
\end{aligned}
$$

Example 2.2 (*Continued*). Let $h_0, f_0 : M_1 \to M_0$ be as in Example 2.2 For $n \geqslant 1$, let $M_n = M_1$ and $h_n(x) = f_n(x) = x$ for all x. Let $J_0 = M_0$, $\alpha_1 = (-\infty, -1]$, and let α_n be any closed subset of M_n, $n \neq 1$. Then $(!\alpha)_0 = Cl(f_0(\alpha_1)) = (-\infty, -2]$. Thus $(!\alpha \otimes !\alpha)_0 = (-\infty, -2]$. We show that $(!(\alpha \otimes \alpha))_0 = (-\infty, -3]$, that is, $(!\alpha \otimes !\alpha)_0$ is not a subset of $(!(\alpha \otimes \alpha))_0$. Indeed, $\alpha_1 \otimes \alpha_1 = Cl(\alpha_1 + \alpha_1) = (-\infty, -2]$. Thus $(!(\alpha \otimes \alpha))_0 = Cl(f_0(\alpha_1 \otimes \alpha_1)) = Cl(f_0((-\infty, -2])) = (-\infty, -3]$.

Also note that $(!\mathbf{1})_0 = Cl(f_0((-\infty, 0])) = (-\infty, -1]$, which does not include the neutral element 0 of M_0.

3. Fibred phase semantics

In this section we define the fibred phase semantics for propositional LLL. We shall extend our fibred phase semantics to the second-order case (*i.e.*, to the full LLL [5]) in Section 5.

Let us recall basic elements of the syntax of propositional LLL from [5]. A propositional formula is defined in the same way as in linear logic, but one adds the new modalities § and $\bar{§}$, namely, if A is a formula then $§A$ and $\bar{§}A$ are formulas. As usual in linear logic, linear negation $A^\perp$ is used as an abbreviation in the sense of the de Morgan dual, except for atomic formulas $p^\perp$. $\bar{§}$ and § are duals of each other,[4] *i.e.*, $(§A)^\perp =_{def} \bar{§}(A^\perp)$, and $(\bar{§}A)^\perp =_{def} §(A^\perp)$.

In addition to formulas, the syntax of LLL involves several punctuation marks that facilitate the management of contexts. Intuitively, if A and B are formulas, an expression A, B is intended to represent $A \oplus B$, an expression $A; B$ is intended to represent $A ⅋ B$, and an expression $[A]$ is intended to represent $?A$. These expressions are themselves not formulas. Formally, a *block* is either a multiset $A_1, A_2, \ldots, A_\ell$ of formulas, where $\ell \geqslant 1$, or an expression $[A]$, where A is a formula. A *sequent* is an expression $\vdash \Gamma$, where Γ is a multiset $\mathbf{A}_1; \mathbf{A}_2; \ldots; \mathbf{A}_k$ of blocks, where $k \geqslant 0$. Note that sequents are allowed to be empty, but the blocks are not. We shall observe the following notation: Roman capitals for formulas, boldface Roman capitals for blocks, and Greek capitals for finite multisets of blocks mutually separated by semicolons. The inference rules of LLL are included in Appendix A.

Given a fibred phase space $\{(M_n, \perp_n), h_n, f_n\}_{n\geqslant 0}$, for each propositional formula A one associates a closed family $A^* = \{(A^*)_n\}_{n\geqslant 0}$ in the obvious way by using the semantic operations described in the previous section, starting with any *valuation*, *i.e.*, any assignment of closed families to propositional atoms. That is, given an assignment that to each propositional atom p associates a closed family p^*, one defines $1^* = \mathbf{1}$, $\perp^* = \perp$, $\top^* = \top$, $0^* = \mathbf{0}$, $(p^\perp)^* = p^{*\perp}$, $(A \otimes B)^* = A^* \otimes B^*$, $(A ⅋ B)^* = A^* ⅋ B^*$, $(A \& B)^* = A^* \& B^*$, $(A \oplus B)^* = A^* \oplus B^*$, $(!A)^* = !(A^*)$, $(?A)^* = ?(A^*)$, $(§A)^* = §(A^*)$, and $(\bar{§}A)^* = \bar{§}(A^*)$. A^* is called the *inner value* of A. It is readily shown that $(A^\perp)^* = A^{*\perp}$. A valuation *satisfies* a formula A iff for each n, $1_n \in (A^*)_n$. A formula is *valid* iff it is satisfied in any valuation in any fibred phase space. These notions are readily extended to sequents by using the intended representation of punctuation marks. That is, $[A]^* = ?A^*$, $(A_1, \ldots, A_\ell)^* = A_1^* \oplus \cdots \oplus A_\ell^*$, and $(\mathbf{A}_1; \ldots; \mathbf{A}_k)^* = \mathbf{A}_1^* ⅋ \cdots ⅋ \mathbf{A}_k^*$.

Lemma 3.1. *In any fibred phase structure*, $(!\alpha)_n \subseteq (§\alpha)_n$.

Proof. It suffices to show $f(\alpha_{n+1}) \cap \mathbf{1}_n \cap J_n \subseteq Cl_n(h_n(\alpha_{n+1}))$. But f_n is bounded by h_n and α_{n+1} is closed, hence $f_n(\alpha_{n+1}) \subseteq Cl_n(h_n(\alpha_{n+1}))$. □

Lemma 3.2. *Let α and β be closed families in a fibred phase structure. Then* $((!\alpha) \otimes (!\beta))_n \subseteq (!(\alpha \& \beta))_n$.

[4] Contrary to [5], we do not assume $§ = \bar{§}$ because this is not needed for the main features of LLL related to polynomial time. In particular, polynomial-time functions are naturally represented in an “intuitionistic” version of LLL [5], which, as a type system, is a refinement of system F [2, 4].

Proof. Since $(!\alpha)_n \otimes (!\beta)_n = Cl_n((!\alpha)_n \cdot (!\beta)_n)$, it suffices to show $(!\alpha)_n \cdot (!\beta)_n \subseteq (!(\alpha\&\beta))_n$. Then by (Cl4), it suffices to show

$$(f_n(\alpha_{n+1}) \cap 1_n \cap J_n) \cdot (f_n(\beta_{n+1}) \cap 1_n \cap J_n) \subseteq Cl_n(f_n(\alpha_{n+1} \cap \beta_{n+1}) \cap 1_n \cap J_n).$$

Take an arbitrary element d from the left hand side. d is of the form $f_n(a) \cdot f_n(b)$ for some $a \in \alpha_{n+1}$, $b \in \beta_{n+1}$. First, notice that $f_n(a) \cdot f_n(b) \in J_n$. This is because $f_n(a) \in J_n$, $f_n(b) \in J_n$ and J_n is a submonoid (of M_n). Also, $f_n(a) \cdot f_n(b) \in \mathbf{1}_n$; this is because $f_n(a) \in \mathbf{1}_n$, $f_n(b) \in \mathbf{1}_n$, and $\mathbf{1}_n$ is a submonoid of M_n. We need to show that $f_n(a) \cdot f_n(b) = f_n(c)$ for some $c \in (\alpha\&\beta)_{n+1}$. By the intermediate value property of f_n, $\exists c \in M_{n+1} c \preccurlyeq_{n+1} a, c \preccurlyeq_{n+1} b$ and $f_n(a) \cdot f_n(b) = f_n(c)$. But $c \preccurlyeq_{n+1} a \in \alpha_{n+1}$ implies $c \in \alpha_{n+1}$ and $c \preccurlyeq_{n+1} b \in \beta_{n+1}$ implies $c \in \beta_{n+1}$ since α_{n+1} and β_{n+1} are Cl_{n+1}-closed. □

Theorem 3.1 (Soundness). *If a formula is provable in propositional LLL, then it is valid.*

Proof. The argument is by induction on the length of LLL proof. Let us consider only the modality rules, since all other cases are standard [3]. Indices of monoid operations are omitted throughout the argument for the sake of readability.

Case 1. §-*rule*

(1.1) §-*rule of the form*:

$$\frac{\vdash B_1, \ldots, B_\ell; \ldots; C_1, \ldots, C_j; A_1; \ldots; A_i; D}{\vdash [B_1]; \ldots; [B_\ell]; \ldots; [C_1]; \ldots; [C_j]; \bar{\S}A_1; \ldots; \bar{\S}A_i; \S D}.$$

Let $\beta_1 = B_1^*, \ldots, \beta_\ell = B_\ell^*$, $\gamma_1 = C_1^*, \ldots, \gamma_j = C_j^*$, $\alpha_1 = A_1^*, \ldots, \alpha_i = A_i^*$, and let $\delta = D^*$. By the induction hypothesis, for any integer $n \geqslant 0$,

$$1_{n+1} \in (((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \ldots \ \cdot ((\gamma_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot (\alpha_1)_{n+1}^{\perp_{n+1}} \cdot \ \ldots \ \cdot (\alpha_i)_{n+1}^{\perp_{n+1}} \cdot \delta_{n+1}^{\perp_{n+1}})^{\perp_{n+1}},$$

that is,

$$((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \ldots \ \cdot ((\gamma_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot (\alpha_1)_{n+1}^{\perp_{n+1}} \cdot \ \ldots \ \cdot (\alpha_i)_{n+1}^{\perp_{n+1}} \cdot \delta_{n+1}^{\perp_{n+1}} \subseteq \perp_{n+1}.$$

Hence,

$$((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \ldots \ \cdot ((\gamma_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot (\alpha_1)_{n+1}^{\perp_{n+1}} \cdot \ \ldots \ \cdot (\alpha_i)_{n+1}^{\perp_{n+1}} \subseteq \delta_{n+1}.$$

Because each h_n is a monoid homomorphism,

$$h_n((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \ldots \ \cdot h_n((\gamma_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot h_n((\alpha_1)_{n+1}^{\perp_{n+1}}) \cdot \ \ldots \ \cdot h_n((\alpha_i)_{n+1}^{\perp_{n+1}}) \subseteq h_n(\delta_{n+1}).$$

Hence by the properties of Cl_n,

$$Cl_nh_n((\beta_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\beta_\ell)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot Cl_nh_n((\gamma_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot Cl_nh_n((\alpha_1)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot Cl_nh_n((\alpha_i)_{n+1}^{\perp_{n+1}})\subseteq Cl_nh_n(\delta_{n+1}).$$

Then by Lemmas 3.1 and 3.2,

$$Cl_n(f_n((\beta_1)_{n+1}^{\perp_{n+1}})\cap\mathbf{1}_n\cap J_n)\cdot\ \dots\ \cdot Cl_n(f_n((\beta_\ell)_{n+1}^{\perp_{n+1}})\cap\mathbf{1}_n\cap J_n)\cdot\ \dots$$
$$\cdot Cl_n(f_n((\gamma_1)_{n+1}^{\perp_{n+1}})\cap\mathbf{1}_n\cap J_n)\cdot\ \dots\ \cdot Cl_n(f_n((\gamma_j)_{n+1}^{\perp_{n+1}})\cap\mathbf{1}_n\cap J_n)$$
$$\cdot Cl_nh_n((\alpha_1)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot Cl_nh_n((\alpha_i)_{n+1}^{\perp_{n+1}})\subseteq Cl_nh_n(\delta_{n+1}).$$

Therefore,

$$[\beta_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\beta_\ell]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_j]^{\perp_n}\cdot\ (\bar{\S}\alpha_1)_n^{\perp_n}\cdot\ \dots\ \cdot(\bar{\S}\alpha_i)_n^{\perp_n}\subseteq(\S\delta)_n,$$

that is,

$$[\beta_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\beta_\ell]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_j]^{\perp_n}$$
$$\cdot(\bar{\S}\alpha_1)_n^{\perp_n}\cdot\ \dots\ \cdot(\bar{\S}\alpha_i)_n^{\perp_n}\cdot(\S\delta)_n^{\perp_n}\subseteq\perp_n.$$

In other words, the conclusion of the rule is satisfied.

(1.2) *§-rule of the form*:

$$\frac{\vdash B_1,\dots,B_\ell;\dots;C_1,\dots,C_j;A_1;\dots;A_i}{\vdash[B_1];\dots;[B_\ell];\dots;[C_1];\dots;[C_j];\bar{\S}A_1;\dots;\bar{\S}A_i}.$$

Let us continue the notation from (1.1). By the induction hypothesis, as in (1.1),

$$((\beta_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\beta_\ell)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot((\gamma_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot(\alpha_1)_{n+1}^{\perp_{n+1}}\cdot\ \dots\ \cdot(\alpha_i)_{n+1}^{\perp_{n+1}}\subseteq\perp_{n+1}.$$

Therefore,

$$h_n((\beta_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\beta_\ell)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot h_n((\gamma_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot h_n((\alpha_1)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot h_n((\alpha_i)_{n+1}^{\perp_{n+1}})\subseteq h_n(\perp_{n+1}).$$

Since $h_n(\perp_{n+1})\subseteq\perp_n$,

$$h_n((\beta_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\beta_\ell)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot h_n((\gamma_1)_{n+1}^{\perp_{n+1}}\cap\cdots\cap(\gamma_j)_{n+1}^{\perp_{n+1}})$$
$$\cdot h_n((\alpha_1)_{n+1}^{\perp_{n+1}})\cdot\ \dots\ \cdot h_n((\alpha_i)_{n+1}^{\perp_{n+1}})\subseteq\perp_n.$$

Then, as in (1.1),

$$[\beta_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\beta_\ell]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_1]_n^{\perp_n}\cdot\ \dots\ \cdot[\gamma_j]^{\perp_n}$$
$$\cdot(\bar{\S}\alpha_1)_n^{\perp_n}\cdot\ \dots\ \cdot(\bar{\S}\alpha_i)_n^{\perp_n}\subseteq Cl_n(\perp_n)=\perp_n$$

and the conclusion of the rule is satisfied.

Case 2. !-*rule*:

$$\frac{\vdash B_1,\dots,B_\ell; A}{\vdash [B_1];\dots;[B_\ell]; !A},$$

where $\ell \geqslant 1$. Let $\beta_1 = B_1^*,\dots,\beta_\ell = B_\ell^*$, and let $\alpha = A^*$. By the induction hypothesis, for any integer $n \geqslant 0$,

$$1_{n+1} \in (((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \alpha_{n+1}^{\perp_{n+1}})^{\perp_{n+1}},$$

that is,

$$((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cdot \ \alpha_{n+1}^{\perp_{n+1}} \subseteq \perp_{n+1},$$

and hence

$$(\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}} \subseteq \alpha_{n+1}.$$

Therefore,

$$f_n((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \subseteq f_n(\alpha_{n+1}),$$

and consequently,

$$Cl_n(f_n((\beta_1)_{n+1}^{\perp_{n+1}} \cap \cdots \cap (\beta_\ell)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \subseteq Cl_n(f_n(\alpha_{n+1}) \cap \mathbf{1}_n \cap J_n).$$

Then by Lemma 3.2,

$$[\beta_1]_n^{\perp_n} \cdot \ \cdots \ \cdot [\beta_\ell]_n^{\perp_n} \subseteq (!\alpha)_n,$$

that is,

$$[\beta_1]_n^{\perp_n} \cdot \ \cdots \ \cdot [\beta_\ell]_n^{\perp_n} \cdot (!\alpha)_n^{\perp_n} \subseteq \perp_n,$$

and therefore the conclusion of the !-rule is satisfied. Note that the argument does not apply if $\ell = 0$, which is not allowed in LLL. In the case when $\ell = 0$ the argument would apply if it were the case that $1_n \preccurlyeq_n f_n(1_{n+1})$ for each integer $n \geqslant 0$.

Case 3. ?-*rule*:

$$\frac{\vdash \Gamma; [A]}{\vdash \Gamma; ?A}.$$

This case is obvious since $(?A)^* = [A]^*$.

Case 4. M-*weakening*:

$$\frac{\vdash \Gamma}{\vdash \Gamma; [A]}.$$

By the induction hypothesis, for any integer $n \geqslant 0$, $(\Gamma^*)_n^{\perp_n} \subseteq \perp_n$. Because $\mathbf{1}_n = \perp_n^{\perp_n}$, it follows that $(\Gamma^*)_n^{\perp_n} \cdot \mathbf{1}_n \subseteq \perp_n$. But $\mathbf{1}_n$ is closed, hence

$$(\Gamma^*)_n^{\perp_n} \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \subseteq (\Gamma^*)_n^{\perp_n} \cdot \mathbf{1}_n \subseteq \perp_n,$$

and thus the conclusion of the M-weakening rule is satisfied.

Case 5. M-*contraction*:

$$\frac{\vdash \Gamma; [A]; [A]}{\vdash \Gamma; [A]}.$$

Recall that for any integer $n \geqslant 0$, every $a \in J_n$ is a weak idempotent, that is, $a \in Cl_n\{a \cdot a\}$. Thus

$$\begin{aligned} f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n &\subseteq Cl_n((f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \cdot (f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n)) \\ &\subseteq Cl_n(Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n)). \end{aligned}$$

Therefore,

$$\begin{aligned} Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \subseteq Cl_n(&Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \\ &\cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n)). \end{aligned}$$

On the other hand, by the induction hypothesis,

$$(\Gamma^*)_n^{\perp_n} \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \subseteq \perp_n.$$

But $(\Gamma^*)_n^{\perp_n}$ and $\perp_n$ are closed, thus by the properties of Cl_n,

$$\begin{aligned} &(\Gamma^*)_n^{\perp_n} \cdot Cl_n(Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n)) \\ &\quad \subseteq Cl_n(\perp_n) = \perp_n. \end{aligned}$$

Thus,

$$(\Gamma^*)_n^{\perp_n} \cdot Cl_n(f_n((A^*)_{n+1}^{\perp_{n+1}}) \cap \mathbf{1}_n \cap J_n) \subseteq \perp_n,$$

that is, the conclusion of the M-contraction rule is satisfied. □

The soundness of ordinary phase semantics for linear logic is a special case of Theorem 3.1 when for every n, $M_n = M_{n+1}$ and h_n and f_n are the identity functions. There is also an important generalization of Theorem 3.1 to fibred phase structures, where every valuation satisfies every formula provable in an “intuitionistic” version of propositional LLL, ILLL [5]. (A more detailed discussion of syntax and semantics of ILLL is included in Appendix A.) Viewed in this way, our Example 2.2 provides a natural mathematical setting in which the ILLL formulas $!A \otimes !A \multimap !(A \otimes A)$ and $!1$ are not satisfied. Note that in Example 2.2, redefining $f_0(0)$ to be 0 instead of -1 yields an example in which $!1$ is satisfied but $!A \otimes !A \multimap !(A \otimes A)$ is not.

4. Strong completeness

The completeness theorem may be proved in the following strong form.

Theorem 4.1 (Strong Completeness). *If a propositional formula is valid, then it is provable in propositional LLL without the cut rule.*

This implies the cut-elimination theorem.

Theorem 4.2 (Cut-Elimination). *If a formula is provable in propositional LLL, then it is provable in propositional LLL without the cut rule.*

Proof. If A is provable in LLL, then A is valid by the Soundness Theorem. Then by the Strong Completeness Theorem, A is cut-free provable in LLL. □

Remark. Cut-Elimination fails if one adds to LLL the !-rule with the empty context, *i.e.,* when $\ell = 0$ (see Appendix A) Indeed, let p be a propositional atom. The sequent $\vdash !(p^{\perp} \& 1); ?p; ?\perp$ is cut-free provable in LLL itself, the sequent $\vdash !1$ is cut-free provable as an instance of the new rule, and hence the sequent $\vdash !(p^{\perp} \& 1); ?p$ is provable by cut. But this sequent has no cut-free proofs.

We prove the Strong Completeness Theorem in the same manner as in Okada [12–14].

For that purpose, we consider the commutative monoid M of finite multisets of blocks, with multiset union as the monoid operation (which we continue to indicate by semicolon concatenation). The empty set $\emptyset$ is the neutral element of M.

Let us write

$$\vdash_{cf} \Delta$$

for

"$\vdash \Delta$ is provable in propositional LLL without the cut rule".

Given a sequent Δ, define the *outer value* $\|\Delta\|$ as

$$\|\Delta\| = \{\Gamma : \vdash_{cf} \Gamma; \Delta\}.$$

Recall that the original definition of the canonical phase model for linear logic in [3] uses, in the present notation, $\|\Delta\| = \{\Gamma : \vdash \Gamma; \Delta \text{ is provable}\}$.

Let $\perp \subseteq M$ be the subset $\|\emptyset\|$. Note that the outer value $\|\Delta\|$ is closed since $\Delta \in \|\Delta\|^{\perp}$.

Proposition 4.1. $\vdash_{cf} \Gamma; \perp$ *iff* $\vdash_{cf} \Gamma$. *That is,* $\|\emptyset\| = \|\perp\|$.

Proof. By a standard induction on cut-free LLL derivations. □

Let J be the submonoid $\{[A_1]; [A_2]; \ldots; [A_k] : A_i$ is a formula and $k \geqslant 0\}$. The *M-contraction* rule of LLL states precisely that every element of J is a weak idempotent.

In turn, *A-weakening* rule of LLL provides us with the following:

Proposition 4.2. *Let* $\mathbf{A}$ *be a comma expression. If* $\vdash_{cf} \mathbf{A}; \Delta$ *then* $\vdash_{cf} \mathbf{A}, B; \Delta$. *That is,* $\mathbf{A}, B \preccurlyeq \mathbf{A}$.

The fundamental Lemma 3.1 is reformulated in terms of our canonical model as follows:

Proposition 4.3. *If* $\vdash_{cf} \bar{\S}A;\varDelta$ *then* $\vdash_{cf} [A];\varDelta$*. In other words,* $[A] \preccurlyeq \bar{\S}A$*.*

Proof. The proof is by induction on the size of cut-free LLL derivations. All the cases are standard save the axiom case and the case where the formula $\bar{\S}A$ is a principal formula of §-*rule*:

$$\frac{\vdash A;\;\; B_1|\dots|B_k;A_1;\dots;A_{m-1};A_m}{\vdash \bar{\S}A;\;\; [B_1];\dots;[B_k];\bar{\S}A_1;\dots;\bar{\S}A_{m-1};\nabla A_m}.$$

(Where ∇ is either § or $\bar{\S}$, and where each | is either a semicolon or a comma, namely, $B_1|\dots|B_k$ means that formulas $B_1,\dots,B_n$ are separated by commas or semicolons.)
In the latter case, we apply the other version of just the same §-*rule*:

$$\frac{\vdash A;\;\; B_1|\dots|B_k;A_1;\dots;A_{m-1};A_m}{\vdash [A];\;\; [B_1];\dots;[B_k];\bar{\S}A_1;\dots;\bar{\S}A_{m-1};\nabla A_m}.$$

In the case of the axiom: $\vdash \bar{\S}A;\S A^{\perp}$, we may use the following one-step derivation:

$$\frac{\vdash A;A^{\perp}}{\vdash [A];\S A^{\perp}}\ \S\text{-rule.} \qquad \square$$

In order to establish a direct correlation between *comma* expressions and *plus* formulas, let us observe the following propositions, each readily shown by induction on the length of cut-free proofs.

Proposition 4.4. *If* $\vdash_{cf} A\oplus B;\varDelta$*, then* $\vdash_{cf} A\oplus(B\oplus C);\varDelta$*. That is,* $A\oplus(B\oplus C)\preccurlyeq A\oplus B$*.*

Proposition 4.5. $\vdash_{cf} A\oplus B;\varDelta$ *iff* $\vdash_{cf} B\oplus A;\varDelta$*.*

Proposition 4.6. $\vdash_{cf} (A\oplus B)\oplus C;\varDelta$ *iff* $\vdash_{cf} A\oplus(B\oplus C);\varDelta$*.*

Proposition 4.7. $\vdash_{cf} A;\varDelta$ *iff* $\vdash_{cf} A\oplus A;\varDelta$*.*

The direct correlation between *comma* expressions and *plus* formulas is established in the following two propositions by a standard induction on cut-free LLL derivations.

Proposition 4.8. *If* $\vdash_{cf} (A_1\oplus A_2\oplus\cdots\oplus A_\ell);\varDelta$*, then* $\vdash_{cf} A_1,A_2,\dots,A_\ell;\varDelta$*.*

Proposition 4.9. *If* $\vdash_{cf} A_1,A_2,\dots,A_\ell;\varDelta$*, then* $\vdash_{cf} (A_{\sigma(1)}\oplus A_{\sigma(2)}\oplus\cdots\oplus A_{\sigma(\ell)});\varDelta$ *for any permutation* σ *of* $1,2,\dots,\ell$*.*

Now, Proposition 4.3 is generalized as follows.

Proposition 4.10. *If* $\vdash_{cf} \bar{\S}(A_1 \oplus A_2 \oplus \cdots \oplus A_\ell); \Delta$ *then* $\vdash_{cf} [A_1]; [A_2]; \ldots; [A_\ell]; \Delta$, *which results in*

$$[A_1]; [A_2]; \ldots; [A_\ell] \preccurlyeq \bar{\S}(A_1 \oplus A_2 \oplus \cdots \oplus A_\ell).$$

Proof. Again we develop induction on the size of cut-free LLL derivations.

The only nonstandard case is that where the formula $\bar{\S}(A_1 \oplus A_2 \oplus \cdots \oplus A_\ell)$ is a principal formula of §-*rule*:

$$\frac{\vdash (A_1 \oplus A_2 \oplus \cdots \oplus A_\ell);\ B_1 | \ldots | B_k; C_1; \ldots; C_{m-1}; C_m}{\vdash \bar{\S}(A_1 \oplus A_2 \oplus \cdots \oplus A_\ell);\ [B_1]; \ldots; [B_k]; \bar{\S}C_1; \ldots; \bar{\S}C_{m-1}; \nabla C_m}.$$

(Where ∇ is either § or $\bar{\S}$, and where each | is either a semicolon or a comma.)
By the inductive hypothesis, Proposition 4.8 yields that

$$\vdash_{cf} A_1, A_2, \ldots, A_\ell;\ B_1 | \ldots | B_k; C_1; \ldots; C_{m-1}; C_m.$$

Now, by applying the corresponding version of §-*rule*, we obtain that

$$\vdash_{cf} [A_1]; [A_2]; \ldots; [A_\ell];\ [B_1]; \ldots; [B_k]; \bar{\S}C_1; \ldots; \bar{\S}C_{m-1}; \nabla C_m.$$

The case of the axiom:

$$\vdash \bar{\S}(A_1 \oplus A_2 \oplus \cdots \oplus A_\ell);\ \S(A_1^\perp \& A_2^\perp \& \cdots \& A_\ell^\perp)$$

is readily handled with the help of the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash A_1;\ A_1^\perp}{\vdots}}{\vdash A_1, A_2, \ldots, A_\ell;\ A_1^\perp} \quad \dfrac{\dfrac{\vdash A_2;\ A_2^\perp}{\vdots}}{\vdash A_1 A_2, \ldots, A_\ell;\ A_2^\perp} \cdots \dfrac{\dfrac{\vdash A_\ell;\ A_1^\perp}{\vdots}}{\vdash A_1, A_2, \ldots, A_\ell;\ A_\ell^\perp}}{\vdots}}{\dfrac{\vdash A_1, A_2, \ldots, A_\ell;\ A_1^\perp \& A_2^\perp \& \cdots \& A_\ell^\perp)}{\vdash [A_1]; [A_2]; \ldots; [A_\ell];\ \S(A_1^\perp \& A_2^\perp \& \cdots \& A_\ell^\perp)}} \quad \square$$

The following definition formalizes the intended meaning of punctuation marks. We assume a mapping π that orders formulas and blocks according to some canonical ordering. The connectives $\oplus$ and $\invamp$ are associated to the left. With these conventions, given a sequent Γ, the formula $\Gamma^{\invamp}$ is defined as

1. If $\Gamma = \emptyset$, then $\Gamma^{\invamp} = \perp$,
2. If $\Gamma = A_1, \ldots, A_\ell,\ \ell \geqslant 1$, then $\Gamma^{\invamp} = A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)}$,
3. If $\Gamma = [A]$, then $\Gamma^{\invamp} = ?A$,
4. If $\Gamma = \mathbf{A}_1; \ldots; \mathbf{A}_k$ where each $\mathbf{A}_i$ is a block and $k \geqslant 1$, then $\Gamma^{\invamp} = \mathbf{A}^{\invamp}_{\pi(1)} \invamp \cdots \invamp \mathbf{A}^{\invamp}_{\pi(k)}$.

Proposition 4.11. *If* $\vdash_{cf} \Gamma; \Delta$, *then* $\vdash_{cf} \Gamma^{\invamp}; \Delta$. *That is,* $\Gamma^{\invamp} \preccurlyeq \Gamma$.

Proof.

1. For the empty multiset Γ, it is readily seen that $\vdash_{cf} \Delta$ implies $\vdash_{cf} \perp; \Delta$.

2. Proposition 4.9 may be used to prove Proposition 4.11 for Γ being a comma expression: $\Gamma = A_1, \ldots, A_\ell$,
3. The *why-not* rule provides Proposition 4.11 for all Γ of the form $[A]$.
4. Finally, let Γ be a nondegenerated multiset of blocks:

$$\Gamma = \mathbf{A}_1; \mathbf{A}_2; \mathbf{A}_3; \ldots; \mathbf{A}_k,$$

where each $\mathbf{A}_i$ is a block and $k \geqslant 2$.
By repeatedly applying the above arguments to each of the blocks, we obtain the following cut-free derivable sequents:

$$\vdash_{cf} \mathbf{A}_1; \mathbf{A}_2; \mathbf{A}_3; \ldots; \mathbf{A}_k;\ \Delta,$$
$$\vdash_{cf} \mathbf{A}_1^{\parr}; \mathbf{A}_2; \mathbf{A}_3; \ldots; \mathbf{A}_k;\ \Delta,$$
$$\vdash_{cf} \mathbf{A}_1^{\parr}; \mathbf{A}_2^{\parr}; \mathbf{A}_3; \ldots; \mathbf{A}_k;\ \Delta,$$
$$\vdash_{cf} \mathbf{A}_1^{\parr}; \mathbf{A}_2^{\parr}; \mathbf{A}_3^{\parr}; \ldots; \mathbf{A}_k;\ \Delta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\vdash_{cf} \mathbf{A}_1^{\parr}; \mathbf{A}_2^{\parr}; \mathbf{A}_3^{\parr}; \ldots; \mathbf{A}_k^{\parr};\ \Delta.$$

Now, we may complete the proof with the help of the *par* rule:

$$\vdash_{cf} (\mathbf{A}_1^{\parr} \parr \mathbf{A}_2^{\parr} \parr \mathbf{A}_3^{\parr} \parr \cdots \parr \mathbf{A}_k^{\parr});\ \Delta. \qquad \square$$

Let us define a mapping $h : M \to M$ as

$$h(\emptyset) = \emptyset,$$
$$h(A_1, \ldots, A_\ell) = \bar{\S}(A_1, \ldots, A_\ell)^{\parr}, \quad \ell \geqslant 1,$$
$$h([A]) = \bar{\S}?A,$$
$$h(\mathbf{A}_1; \ldots; \mathbf{A}_k) = h(\mathbf{A}_1); \ldots; h(\mathbf{A}_k), \quad k \geqslant 1.$$

Proposition 4.12. *h is a phase homomorphism.*

Proof. Indeed, whatever multisets of blocks Γ and Δ:

$$\Gamma = \mathbf{A}_1; \mathbf{A}_2; \ldots; \mathbf{A}_k,$$
$$\Delta = \mathbf{B}_1; \mathbf{B}_2; \ldots; \mathbf{B}_n,$$

we take, by definition

$$h(\Gamma) = h(\mathbf{A}_1); h(\mathbf{A}_2); \ldots; h(\mathbf{A}_k),$$
$$h(\Delta) = h(\mathbf{B}_1); h(\mathbf{B}_2); \ldots; h(\mathbf{B}_n),$$

and

$$h(\Gamma; \Delta) = h(\mathbf{A}_1); h(\mathbf{A}_2); \ldots; h(\mathbf{A}_k); h(\mathbf{B}_1); h(\mathbf{B}_2); \ldots; h(\mathbf{B}_n).$$

It is readily seen that

$$h(\Gamma); h(\Delta) = h(\Gamma; \Delta).$$

It remains to show that $h(\|\emptyset\|) \subseteq \|\emptyset\|$, that is, if $\vdash_{cf} \Delta$, then $\vdash_{cf} h(\Delta)$. There are four cases.

Case 1: $\Delta = \emptyset$. Then $h(\Delta) = \emptyset$ and this case is a tautology.

Case 2: Δ is of the form $A_1, \ldots, A_\ell$, where $\ell \geqslant 1$. $h(A_1, \ldots, A_\ell) = \bar{\S}(A_1, \ldots, A_\ell)^{\wp} = \bar{\S}(A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)})$. But if $\vdash_{cf} A_1, \ldots, A_\ell$, then $\vdash_{cf} A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)}$ by Proposition 4.9, and therefore $\vdash_{cf} \bar{\S}(A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)})$ by the §-rule.

Case 3: Δ is of the form $[A]$. In this case $h(\Delta) = \bar{\S}?A$. But if $\vdash_{cf} [A]$, then $\vdash_{cf} ?A$ by the ?-rule, and hence $\vdash_{cf} \bar{\S}?A$ by the §-rule.

Case 4: Δ is of the form $\mathbf{A}_1; \ldots; \mathbf{A}_k$, $k \geqslant 1$. $h(\Delta) = h(\mathbf{A}_1); \ldots; h(\mathbf{A}_k) = \bar{\S}\mathbf{A}_1^{\wp}; \ldots; \bar{\S}\mathbf{A}_k^{\wp}$. But if $\vdash_{cf} \mathbf{A}_1; \ldots; \mathbf{A}_k$, then by several applications of Proposition 4.11 it follows that $\vdash_{cf} \mathbf{A}_1^{\wp}; \ldots; \mathbf{A}_k^{\wp}$, and therefore $\vdash_{cf} \bar{\S}\mathbf{A}_1^{\wp}; \ldots; \bar{\S}\mathbf{A}_k^{\wp}$ by the §-rule. □

Consider the function $f : M \to M$ defined as

$$f(\Gamma) = \begin{cases} [A_1]; \ldots; [A_\ell] \text{ if } \Gamma = A_1, \ldots, A_\ell & \text{for } \ell \geqslant 1, \\ [\Gamma^{\wp}] & \text{otherwise.} \end{cases}$$

In particular, if $\Gamma = \emptyset$, then $\Gamma^{\wp} = \bot$ by definition, and hence $f(\emptyset) = [\bot]$. The *M-weakening* rule of LLL implies that $f(\emptyset) \preccurlyeq \emptyset$. In fact, it is clearly the case that $f(\Gamma) \in \mathbf{1} \cap J$ for any sequent Γ. We also have the following lemma.

Lemma 4.1. *f has the intermediate value property. Furthermore, f is bounded by h.*

Proof. Let Γ and Δ be multisets of blocks. There are three cases to be considered.

Case 1: Suppose that Γ and Δ are nondegenerated comma expressions:

$$\Gamma = A_1, \ldots, A_\ell \quad \text{for } \ell \geqslant 2,$$
$$\Delta = B_1, \ldots, B_k \quad \text{for } k \geqslant 2.$$

Take the following comma expression $\mathbf{C}$:

$$\mathbf{C} := A_1, \ldots, A_\ell, B_1, \ldots, B_k.$$

By Proposition 4.2, both $\mathbf{C} \preccurlyeq \Gamma$ and $\mathbf{C} \preccurlyeq \Delta$.

According to the first line of the definition of f:

$$f(\Gamma); f(\Delta) = [A_1]; \ldots; [A_\ell]; [B_1]; \ldots; [B_k] = f(\mathbf{C}).$$

Case 2: Suppose that Γ is a nondegenerated comma expression, and Δ is not of that kind:

$$\Gamma = A_1, \ldots, A_\ell \quad \text{for } \ell \geqslant 2,$$
$$f(\Gamma) = [A_1]; \ldots; [A_\ell],$$
$$f(\Delta) = [\Delta^{\wp}].$$

In that case, take the desired comma expression $\mathbf{C}$ as

$$\mathbf{C} := A_1, \ldots, A_\ell, \Delta^{\wp}.$$

According to Propositions 4.2 and 4.11, both $\mathbf{C} \preccurlyeq \Gamma$ and $\mathbf{C} \preccurlyeq \Delta$.
A simple calculation shows that

$$f(\Gamma); f(\Delta) = [A_1]; \ldots; [A_\ell]; [\Delta^{\wp}] = f(\mathbf{C}).$$

Case 3: Finally, assume that both Γ and Δ are not nondegenerated comma expressions, and, hence,

$$f(\Gamma) = [\Gamma^{\wp}],$$
$$f(\Delta) = [\Delta^{\wp}].$$

Now we take the following comma expression $\mathbf{C}$:

$$\mathbf{C} := \Gamma^{\wp}, \Delta^{\wp}.$$

Again, by Propositions 4.2 and 4.11, both $\mathbf{C} \preccurlyeq \Gamma$ and $\mathbf{C} \preccurlyeq \Delta$.
And we complete the proof that f satisfies the intermediate value property by the equation:

$$f(\Gamma); f(\Delta) = [\Gamma^{\wp}]; [\Delta^{\wp}] = f(\mathbf{C}).$$

It remains to show that f is bounded by h. Namely, for any Γ we have to construct Γ' such that

$$\Gamma' \preccurlyeq \Gamma,$$
$$f(\Gamma) \preccurlyeq h(\Gamma').$$

If Γ is a nondegenerated comma expression:

$$\Gamma = A_1, \ldots, A_\ell \quad \text{for } \ell \geqslant 2,$$

we take $\Gamma' = \Gamma$, else we take $\Gamma' = \Gamma^{\wp}$.
According to Proposition 4.11, $\Gamma' \preccurlyeq \Gamma$.
If Γ is the above nondegenerated comma expression, then

$$f(\Gamma) = [A_1]; \ldots; [A_\ell],$$
$$h(\Gamma') = \bar{\S}(A_1, \ldots, A_\ell)^{\wp},$$

and, because of Proposition 4.10,

$$f(\Gamma) \preccurlyeq h(\Gamma').$$

Otherwise,

$$\Gamma' = \Gamma^{\wp},$$
$$f(\Gamma) = [\Gamma^{\wp}],$$
$$h(\Gamma') = \bar{\S}(\Gamma^{\wp}).$$

And Proposition 4.3 may be used to show that

$$f(\Gamma) \preccurlyeq h(\Gamma'). \qquad \square$$

Our canonical model is the fibred phase space $\{(M_n, \perp_n), h_n, f_n\}_{n\geqslant 0}$, where $M_n = M$, $\perp_n = \perp$, $h_n = h$, and $f_n = f$. We shall drop the indices for the rest of this section. Finally, we consider the valuation $p^* = \| p\|$ for any atomic formula p.

The following lemma is obtained in the manner similar to Okada [12–14].

Lemma 4.2 (Main Lemma). *For any propositional formula* A, $A^* \subseteq \|A\|$.

Strong Completeness follows from the Main Lemma.

Proof (*Strong Completeness*). Assume that A is valid. Hence $1 \in A^*$ for any model, in particular for this canonical model. Therefore, $\emptyset \in A^*$ in this model. On the other hand, $A^* \subseteq \|A\|$. Hence $\emptyset \in \|A\|$. By definition, this means "A is provable in LLL without the cut rule". $\square$

Let us also note another consequence of the Main Lemma.

Corollary 4.1. *For any propositional formula* A, $A \in A^{*\perp}$.

Proof. By the Main Lemma, $A^* \subseteq \|A\|$. It suffices to show $A^*; A \subseteq \perp$, namely, $\forall \Delta \in A^*(\Delta; A \in \perp)$. But $A^* \subseteq \|A\|$ means that for any $\Delta \in A^*$, $\vdash_{cf} \Delta; A$. Therefore $A^*; A \subseteq \perp$. $\square$

The Main Lemma has another formulation, which will be essential for the second-order case in the next section.

Lemma 4.3. *For any propositional formula* A, $A^\perp \in A^* \subseteq \|A\|$.

Proof. By the Main Lemma, $A^* \subseteq \|A\|$ for any A. By Corollary 4.1 for $A^\perp$, $A^\perp \in A^{*\perp\perp} = A^*$. $\square$

Before we prove the Main Lemma, let us observe

Proposition 4.13. *If* $\vdash_{cf} \Delta; B$ *then* $\vdash_{cf} h(\Delta); \S B$.

Proof.

Case 1: Δ is empty $\emptyset$. Then $h(\Delta) = \emptyset$. If $\vdash_{cf} B$, then $\vdash_{cf} \S B$ by the $\S$-rule.

Case 2: Δ is of the form $A_1, \ldots, A_\ell$, where $\ell \geqslant 1$. $h(A_1, \ldots, A_\ell) = \bar{\S}(A_1, \ldots, A_\ell)^{\parr} = \bar{\S}\ (A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)})$. But if $\vdash_{cf} A_1, \ldots, A_\ell; B$, then $\vdash_{cf} A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)}; B$ by Proposition 4.11, and thus $\vdash_{cf} \bar{\S}(A_{\pi(1)} \oplus \cdots \oplus A_{\pi(\ell)}); \S B$ by the $\S$-rule.

Case 3: Δ is of the form $[A]$. In this case $h(\Delta) = \bar{\S}?A$. But if $\vdash_{cf} [A]; B$, then $\vdash_{cf} ?A; B$ by the ?-rule, and hence $\vdash_{cf} \bar{\S}?A; \S B$ by the $\S$-rule.

Case 4: Δ is of the form $\mathbf{A}_1; \ldots; \mathbf{A}_k$, $k \geqslant 1$. $h(\Delta) = h(\mathbf{A}_1); \ldots; h(\mathbf{A}_k) = \bar{\S}\mathbf{A}_1^{\parr}; \ldots; \bar{\S}\mathbf{A}_k^{\parr}$. But if $\vdash_{cf} \mathbf{A}_1; \ldots; \mathbf{A}_k; B$, then by several applications of Proposition 4.11 it follows that $\vdash_{cf} \mathbf{A}_1^{\parr}; \ldots; \mathbf{A}_k^{\parr}; B$, and therefore $\vdash_{cf} \bar{\S}\mathbf{A}_1^{\parr}; \ldots; \bar{\S}\mathbf{A}_k^{\parr}; \S B$ by the $\S$-rule. $\square$

Proof (*Main Lemma*). By induction on the structure of the formula A.

Basis. There are five cases.

When A is of the form p where p is a propositional atom, the claim is obvious since $p^* = \|p\|$ by definition.

When A is of the form $\bot$, then $\bot^* = \|\emptyset\|$ by definition. But by Proposition 4.1, $\|\emptyset\| = \|\bot\|$.

When A is of the form 1, then by definition $1^* = \mathbf{1} = \|\emptyset\|^\perp$. Thus it suffices to show that $\|\emptyset\|^\perp \subseteq \|1\|$. Let $\Gamma \in \|\emptyset\|^\perp$, *i.e.*, for any Σ, if $\vdash_{cf} \Sigma$, then $\vdash_{cf} \Gamma; \Sigma$. In particular, if Σ is the formula 1, then $\vdash_{cf} \Gamma; 1$. In other words, $\Gamma \in \|1\|$, as required.

When A is of the form $\top$, $\top^* = M$ by definition. But $M = \|\top\|$ by the $\top$-rule.

When A is of the form 0, then by definition $0^* = \mathbf{0} = M^\perp$. It suffices to show that $M^\perp \subseteq \|0\|$. Let $\Gamma \in M^\perp$, *i.e.*, for any Σ, $\vdash_{cf} \Gamma; \Sigma$. In particular, if Σ is the formula 0, then $\vdash_{cf} \Gamma; 0$. In other words, $\Gamma \in \|0\|$, as required.

Induction step. There are nine cases, depending on the main connective.

Case 1: When A is of the form $p^\perp$ where p is a propositional atom, $p^\perp \in p^*$ since $p^* = \|p\|$. Assume $\Gamma \in p^{*\perp}$. Then $p^\perp; \Gamma \in \|\bot\|$, *i.e.*, $\vdash_{cf} p^\perp; \Gamma; \bot$, and thus $\vdash_{cf} p^\perp; \Gamma$. Hence $\Gamma \in \|p^\perp\|$.

Case 2. *A is of the form $B \,\&\, C$*: By the induction hypothesis, $B^* \subseteq \|B\|$ and $C^* \subseteq \|C\|$. Hence $B^* \,\&\, C^* = B^* \cap C^* \subseteq \|B\| \cap \|C\|$. On the other hand, since

$$\frac{\vdash_{cf} \Gamma; B \quad \vdash_{cf} \Gamma; C}{\vdash_{cf} \Gamma; B\&C},$$

$\|B\| \cap \|C\| \subseteq \|B \,\&\, C\|$. Therefore $B^* \,\&\, C^* \subseteq \|B \,\&\, C\|$.

Case 3. *A is of the form $B \oplus C$*: By the induction hypothesis, $B^* \subseteq \|B\|$ and $C^* \subseteq \|C\|$. Hence $B^* \cup C^* \subseteq \|B\| \cup \|C\|$. On the other hand, since

$$\frac{\vdash_{cf} \Gamma; B}{\vdash_{cf} \Gamma; B \oplus C} \quad \text{and} \quad \frac{\vdash_{cf} \Gamma; C}{\vdash_{cf} \Gamma; B \oplus C}.$$

Hence $\|B\| \subseteq \|B \oplus C\|$ and $\|C\| \subseteq \|B \oplus C\|$. Therefore $B^* \cup C^* \subseteq \|B \oplus C\|$. Since $\|B \oplus C\|$ is closed, $B^* \oplus C^* = Cl(B^* \cup C^*) \subseteq \|B \oplus C\|$.

Case 4. *A is of the form* $B \otimes C$: By the induction hypothesis, $B^* \subseteq \|B\|$ and $C^* \subseteq \|C\|$. Hence $B^*; C^* \subseteq \|B\|; \|C\|$. On the other hand,

$$\frac{\vdash_{cf} \Gamma; B \quad \vdash_{cf} \Delta; C}{\vdash_{cf} \Gamma; \Delta; B \otimes C}.$$

Therefore $B^*; C^* \subseteq \|B \otimes C\|$. $B^* \otimes C^* = Cl(B^*; C^*) \subseteq \|B \otimes C\|$ since $\|B \otimes C\|$ is closed.

Case 5. *A is of the form* $B ⅋ C$: Let $\Gamma \in B^* ⅋ C^* = (B^{*\perp}; C^{*\perp})^\perp$. Therefore, if $\Delta \in B^{*\perp}$, $\Sigma \in C^{*\perp}$ then $\vdash_{cf} \Gamma; \Delta; \Sigma$. On the other hand, by the induction hypothesis, $B^* \subseteq \|B\|$ and $C^* \subseteq \|C\|$. Then, $B \in B^{*\perp}$ and $C \in C^{*\perp}$ by Corollary 4.1. Take B for Δ and C for Σ above. Then $\vdash \Gamma; B; C$. Therefore, $\Gamma \in \|B ⅋ C\|$.

Case 6. *A is of the form* $!B$: Because $f(\Gamma) \in \mathbf{1} \cap J$ for any sequent Γ, it suffices to show that $Cl\{f(B^*)\} \subseteq \|!B\|$. But $\|!B\|$ is closed, hence it actually suffices to show that $f(B^*) \subseteq \|!B\|$. By the induction hypothesis, $B^* \subseteq \|B\|$. Let $\Gamma \in B^*$. Then $\vdash_{cf} \Gamma; B$. There are two cases. If $\Gamma = A_1, \ldots, A_\ell$ for $\ell \geqslant 1$, then by the !-rule, $\vdash_{cf} [A_1]; \ldots; [A_\ell]; !B$. That is, $f(\Gamma) \in \|!B\|$, as required. Otherwise, $\vdash_{cf} \Gamma^{⅋}; B$ by Proposition 4.11, and then $\vdash_{cf} [\Gamma^{⅋}]; !B$ by the !-rule. That is, again $f(\Gamma) \in \|!B\|$, as required.

Case 7. *A is of the form* $?B$: Assume $\Gamma \in (?B)^* = (f(B^{*\perp}))^\perp$, that is, $f(B^{*\perp}); \Gamma \subseteq \|\emptyset\|$. By the induction hypothesis, $B^* \subseteq \|B\|$. Hence by Corollary 4.1, $B \in B^{*\perp}$. Because $f(B) = [B]$, it is the case that $\vdash_{cf} [B]; \Gamma$, and thus $\vdash_{cf} ?B; \Gamma$ by the ?-rule. Therefore, $\Gamma \in \|?B\|$.

Case 8. *A is of the form* $\S B$. By the induction hypothesis, $B^* \subseteq \|B\|$. Hence for any $\Delta \in B^*$, it is the case that $\vdash_{cf} \Delta; B$. But then $\vdash_{cf} h(\Delta); \S B$ by Proposition 4.13. Hence $h(B^*) \subseteq \|\S B\|$. Therefore, $(\S B)^* = Cl(h(B^*)) \subseteq \|\S B\|$ since $\|\S B\|$ is closed.

Case 9. *A is of the form* $\bar{\S} B$: $(\bar{\S} B)^* = (h(B^{*\perp}))^\perp$, so it suffices to show that $(h(B^{*\perp}))^\perp \subseteq \|\bar{\S} B\|$. Let $\Gamma \in (h(B^{*\perp}))^\perp$, *i.e.*, for any $\Sigma \in h(B^{*\perp})$, $\vdash_{cf} \Gamma; \Sigma$. By induction hypothesis, $B^* \subseteq \|B\|$, and hence by Corollary 4.1, $B \in B^{*\perp}$. Thus, $\bar{\S} B = h(B) \in h(B^{*\perp})$ and we may pick Σ to be the formula $\bar{\S} B$. Thus $\vdash_{cf} \Gamma; \bar{\S} B$ as required. □

5. Second-order completeness

Girard [5] formulated LLL as a second-order propositional system. Let us adjust the underlying idea in Okada [12–14] to extend the fibred phase semantics to the second-order case so that the soundness, strong completeness and cut-elimination theorems apply to the full LLL. A further extension to higher-order (finite-order) LLL may also be possible using a modified version of higher-order phase models introduced in Okada [12–14].

Let us write $A\{X\}$ to indicate that X is a vector of propositional variables containing the free variables of A. Let $A\{B/X\}$ or $A\{B\}$ denote the formula obtained from $A\{X\}$ by substituting the vector of formulas B for X. Let $A^*\{\alpha/X\}$ or $A^*\{\alpha\}$ denote the result of the inner value construction starting with the vector of closed families α as the value of the variable list X. In this section we use *Form* to denote the set of second-order formulas.

Let $\{(M_n, \perp_n, f_n, h_n)\}_{n\geqslant 0}$ be a fibred phase space. Consider an assignment that to any formula A (possibly with free propositional variables), associates a set $\langle A\rangle$ of closed families. $\langle A\rangle_n$ denotes $\{(\alpha)_n : \alpha\in\langle A\rangle\}$. $\bigcap_{i\in\Lambda}\alpha_i$ denotes $\{\bigcap_{i\in\Lambda}(\alpha_i)_n\}_{n\geqslant 0}$, $\bigcup_{i\in\Lambda}\alpha_i$ denotes $\{\bigcup_{i\in\Lambda}(\alpha_i)_n\}_{n\geqslant 0}$, and $Cl(\bigcup_{i\in\Lambda}\alpha_i)$ denotes $\{Cl_n(\bigcup_{i\in\Lambda}(\alpha_i)_n)\}_{n\geqslant 0}$. In this notation the second-order propositional quantifiers may be interpreted as follows:

$$(\forall XA)^* =_{def} \bigcap_{\alpha\in\langle B\rangle, B\in Form} A^*\{\alpha/X\},$$

$$(\exists XA)^* =_{def} Cl\left(\bigcup_{\alpha\in\langle B\rangle, B\in Form} A^*\{\alpha/X\}\right).$$

More generally, second-order operators are defined as follows. Let $D=\{D_n\}_{n\geqslant 0}$ be a closed family. For any family of mappings $\xi=\{\xi_n\}_{n\geqslant 0}$, where $\xi_n : D_n\to D_n$, let

$$\begin{aligned}\forall X.\xi(X) &= \bigcap\{\xi(\alpha) : \alpha\in\langle B\rangle, B\in Form\}\\ &= \bigcap\{\xi_n(\alpha_n) : \alpha_n\in\langle B\rangle_n, B\in Form\}\}_{n\geqslant 0},\\ \exists X.\xi(X) &= Cl\left(\bigcup\{\xi(\alpha) : \alpha\in\langle B\rangle, B\in Form\}\right)\\ &= Cl_n\left(\bigcup\{\xi_n(\alpha_n) : \alpha_n\in\langle B\rangle_n, B\in Form\}\right)\Big\}_{n\geqslant 0}.\end{aligned}$$

Then

$$(\forall XA)^* = \forall XA^* = \left\{\bigcap_{\alpha_n\in\langle B\rangle_n, B\in Form} A_n^*\{\alpha_n/X\}\right\}_{n\geqslant 0},$$

$$(\exists XA)^* = \exists XA^* = \left\{Cl_n\left(\bigcup_{\alpha_n\in\langle B\rangle_n, B\in Form} A_n^*\{\alpha_n/X\}\right)\right\}_{n\geqslant 0}.$$

Note that the usual De Morgan equalities, $(\forall X A)^* = (\exists X A^\perp)^{*\perp}$ and $(\exists X A)^* = (\forall XA^\perp)^{*\perp}$, are shown easily.

A *second-order fibred phase model* is a fibred phase space $\{(M_n, \perp_n, f_n, h_n)\}_{n\geqslant 0}$ together with an assignment that associates a set $\langle A\rangle$ of closed families to any formula A, such that the following condition holds:

> For any formula $A\{X\}$, where $X = X_1,\dots,X_k$ is a vector of second-order propositional variables, for any vector of formulas $B = B_1,\dots,B_k$, for any vector of closed families $\alpha=\alpha_1,\dots,\alpha_k$, whenever $(\alpha_j)_n\in\langle B_j\rangle_n$ for all $n\geqslant 0$ and all $1\leqslant j\leqslant k$, then it is the case that $(A^*)_n\{(\alpha)_n/X\}\in\langle A\{B/X\}\rangle_n$ for all $n\geqslant 0$.

A formula is *closed* iff it has no free variables. A closed formula A is *valid* iff in any second-order phase model, $1_n\in(A^*)_n$ for all $n\geqslant 0$.

Theorem 5.1 (Soundness, second-order version). *Let $X = X_1,\dots,X_k$ be a vector of second–order propositional variables, containing all free second-order propositional variables of the formulas $A_1,\dots,A_\ell,\dots,B_1,\dots,B_m$. If the sequent*

$$\vdash A_1,\dots,A_\ell;\dots;B_1,\dots,B_m$$

is provable in LLL, then for any vector of formulas $C = C_1,\dots,C_k$, for any second-order fibred phase model, and for any vector of closed families $\alpha = \alpha_1,\dots,\alpha_k$, whenever $\alpha_j \in \langle C_j\rangle$ for all $1 \leqslant j \leqslant k$, then it is the case that

$$1_n \in ((A_1^*\{\alpha/X\} \oplus \cdots \oplus A_\ell^*\{\alpha/X\}) \parr \cdots \parr (B_1^*\{\alpha/X\} \oplus \cdots \oplus B_m^*\{\alpha/X\}))_n.$$

In particular, if a closed formula is provable in LLL, then it is valid.

Proof. The argument is carried out by the induction on the length of proof essentially in the same way as that of the propositional case in Section 3, except for the following second-order quantifier cases.
(1) $\forall$-rule

$$\frac{\vdash \Gamma\{X\}; A\{X,Y\}}{\vdash \Gamma\{X\}; \forall Y A\{X,Y\}},$$

where Y does not appear as a free variable in Γ.

By the induction hypothesis, for any $\alpha \in \langle D\rangle$, $D \in Form$, and for any $\beta \in \langle E\rangle$, $E \in Form$, $1_n \in (\Gamma^*\{\alpha\} \parr A^*\{\alpha,\beta\})_n$. Therefore, $(\Gamma^{*\perp}\{\alpha\})_n \cdot (A^{*\perp}\{\alpha,\beta\})_n \subseteq \perp_n$. Hence $(\Gamma^{*\perp}\{\alpha\})_n \subseteq (A^{*\perp\perp}\{\alpha,\beta\})_n = (A^*\{\alpha,\beta\})_n$. Since this holds for any $\beta \in \langle E\rangle$ and any $E \in Form$,

$$(\Gamma^{*\perp}\{\alpha\})_n \subseteq \bigcap_{\beta \in \langle E\rangle, E \in Form} (A^*\{\alpha,\beta\})_n = (\forall Y A^*\{\alpha,Y\})_n.$$

Hence $(\Gamma^{*\perp}\{\alpha\})_n \cdot ((\forall Y A^*\{\alpha,Y\})_n)^{\perp_n} \subseteq \perp_n$, which means $1_n \in (\Gamma^*\{\alpha\} \parr \forall Y A^*\{\alpha,Y\})_n$.
(2) $\exists$-rule

$$\frac{\vdash \Gamma\{X\}; A\{B\{X\},X\}}{\vdash \Gamma\{X\}; \exists Y A\{Y,X\}}.$$

By the induction hypothesis, for any $\alpha \in \langle C\rangle, C \in Form$, $1_n \in (\Gamma^*\{\alpha\} \parr A^*\{B^*\{\alpha\},\alpha\})_n$, namely $(\Gamma^{*\perp}\{\alpha\})_n \subseteq (A^*\{B^*\{\alpha\},\alpha\})_n$. By the condition on $\langle B\{C\}\rangle$, $B^*\{\alpha\} \in \langle B\{C\}\rangle$. Therefore,

$$(\Gamma^{*\perp}\{\alpha\})_n \subseteq \bigcup_{\beta \in \langle D\rangle, D \in Form} (A^*\{\beta,\alpha\})_n \subseteq (\exists Y A^*\{Y,\alpha\})_n.$$

Hence, for any n, $(\Gamma^*\{\alpha\})_n^{\perp_n} \cdot (\exists Y A^*\{Y,\alpha\})_n^{\perp_n} \subseteq \perp_n$, which means $1_n \in (\Gamma^*\{\alpha\} \parr \exists Y A^*\{Y,\alpha\})_n$. $\square$

The canonical phase space is defined as in the previous section, but $\vdash_{cf}$ now means "*provable in second-order LLL without the cut rule*". For any formula A, define $\langle A\rangle$ as

$$\langle A\rangle = \{\alpha \text{ closed}: A^{\perp} \in \alpha \subseteq \|A\|\}.$$

The set $\langle A\rangle$ corresponds to the set of candidates of reducibility of type A in [2, 4].

Lemma 5.1 (Main Lemma, second-order version). *For any formulas $A\{X\}$ and C, and for any $\alpha \in \langle C\rangle$, $A\{C/X\}^{\perp} \in A^*\{\alpha/X\} \subseteq \|A\{C/X\}\|$.*

Proof. The proof is carried out in the way similar to that of Lemma 4.3 with the help of Lemma 4.2 except for the following cases, which are treated essentially in the same way as Okada [12].

Case 1. *$A\{X\}$ is of the form X_i*: Then by the definition of $\langle C\rangle$, $A\{C/X\}^{\perp} \in A^*\{\alpha/X\} \subseteq \|A\{C\}\|$ is obvious for any $\alpha \in \langle C\rangle$ and any $C \in Form$.

Case 2. *$A\{X\}$ is of the form $\forall YB\{X,Y\}$*: We prove $\forall YB^*\{\alpha/X,Y\} \subseteq \|\forall YB\{C/X,Y\}\|$ for any $\alpha \in \langle C\rangle$ and any $C \in Form$. Assume that

$$\Gamma \in \forall YB^*\{\alpha, Y\} = \bigcap_{\beta \in \langle D\rangle, D \in Form} B^*\{\alpha, \beta\}.$$

By the induction hypothesis, $B^*\{\alpha,\beta\} \subseteq \|B\{C,D\}\|$. Hence, $\Gamma \in \|B\{C,D\}\|$. In particular, for a variable Y that does not occur in Γ, $\Gamma \in \|B\{C,Y\}\|$. On the other hand,

$$\frac{\vdash_{cf} \Gamma; B\{C,Y\}}{\vdash_{cf} \Gamma; \forall YB\{C,Y\}}$$

is a LLL-rule. Hence, $\Gamma \in \|\forall YB\{C,Y\}\|$.

Case 3. *$A\{X\}$ is of the form $\exists YB\{X,Y\}$*: We prove $\exists YB^*\{\alpha/X,Y\} \subseteq \|\exists YB\{C/X,Y\}\|$ for any $\alpha \in \langle C\rangle$ and any $C \in Form$. Take arbitrary $C \in Form$ and $\alpha \in \langle C\rangle$. Assume that $\Gamma \in \exists YB^*\{\alpha/X,Y\} = Cl(\bigcup_{\beta \in \langle D\rangle, D \in Form} B^*\{\alpha,\beta\})$. It suffices to show that $\exists YB\{C,Y\}$ belongs to $(\bigcup_{\beta \in \langle D\rangle, D \in Form} B^*\{\alpha,\beta\})^{\perp}$, because then $\Gamma; \exists YB\{C,Y\} \in \|\emptyset\|$, hence $\Gamma \in \|\exists YB\{C,Y\}\|$. By the induction hypothesis, $B^*\{\alpha,\beta\} \subseteq \|B\{C,D\}\|$ for any $D \in Form$ and any $\beta \in \langle D\rangle$. Therefore, for any $\Delta \in B^*\{\alpha,\beta\}$, $\vdash_{cf} \Delta; B\{C,D\}$, hence $\vdash_{cf} \Delta; \exists YB\{C,Y\}$ by the $\exists$-rule. Since this holds for any $D \in Form$ and any $\beta \in \langle D\rangle$, it is the case that $\exists XB\{C,Y\} \in (\bigcup_{\beta \in \langle D\rangle, D \in Form} B^*\{\alpha,\beta\})^{\perp}$, as required. □

In other words, the canonical phase space and the assignment $\langle A\rangle$ just defined from a second-order phase model. As before, we obtain strong completeness and hence cut elimination. Note that the condition for $\langle A\rangle$ on the stability under substitution is verified at the same time when the main lemma above is proved. Then with the same argument in the previous Section we have the following theorems.

Theorem 5.2 (Strong Completeness, second-order version). *If a closed formula is valid, then it is provable in LLL without the cut rule.*

Theorem 5.3 (Cut-Elimination, second-order version). *If a formula is provable in LLL, then it is also provable in LLL without the cut rule.*

The methods and results again extend to the intuitionistic version.

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Appendix A. LLL rules

Let us recall LLL inference rules from [5]. The § rules have been modified since we are not assuming that § is self-dual. The exchange rules are omitted because we are dealing with multisets.

Identity/Negation

$$\frac{}{\vdash A; A^{\perp}}\text{(identity)} \qquad \frac{\vdash \Gamma; A \quad \vdash A^{\perp}; \Delta}{\vdash \Gamma; \Delta}\text{(cut)}$$

Structure

$$\frac{\vdash \Gamma}{\vdash \Gamma; [A]}\text{(M-weakening)} \qquad \frac{\vdash \Gamma; \mathbf{A}}{\vdash \Gamma; \mathbf{A}, B}\text{(A-weakening)}$$

$$\frac{\vdash \Gamma; [A]; [A]}{\vdash \Gamma; [A]}\text{(M-contraction)} \qquad \frac{\vdash \Gamma; \mathbf{A}, B, B}{\vdash \Gamma; \mathbf{A}, B}\text{(A-contraction)}$$

Logic

$$\frac{}{\vdash 1}\text{(one)} \qquad \frac{\vdash \Gamma}{\vdash \Gamma; \perp}\text{(false)}$$

$$\frac{\vdash \Gamma; A \quad \vdash B; \Delta}{\vdash \Gamma; A \otimes B; \Delta}\text{(times)} \qquad \frac{\vdash \Gamma; A; B}{\vdash \Gamma; A \parr B}\text{(par)}$$

$$\frac{}{\vdash \Gamma; \top}\text{(true)} \qquad \text{(no rule for zero)}$$

$$\frac{\vdash \Gamma; A \quad \vdash \Gamma; B}{\vdash \Gamma; A \& B}\text{(with)} \qquad \frac{\vdash \Gamma; A}{\vdash \Gamma; A \oplus B}\text{(left plus)} \qquad \frac{\vdash \Gamma; B}{\vdash \Gamma; A \oplus B}\text{(right plus)}$$

$$\frac{\vdash B_1, \ldots, B_\ell; A}{\vdash [B_1]; \ldots; [B_\ell]; !A}\text{(of course)} \quad \text{(where } \ell \geqslant 1\text{)} \qquad \frac{\vdash \Gamma; [A]}{\vdash \Gamma; ?A}\text{(why not)}$$

$$\frac{\vdash B_1 | \ldots | B_k; A_1; \ldots; A_{m-1}; A_m}{\vdash [B_1]; \ldots; [B_k]; \bar{\S}A_1; \ldots; \bar{\S}A_{m-1}; \nabla A_m}\text{(neutral).}$$

(where $k, m \geqslant 0$, where ∇ is either § or $\bar{\S}$, and where each $|$ is either a semicolon or a comma, namely, $B_1|\ldots|B_k$ means that $B_1,\ldots,B_n$ are formulas separated by commas or semicolons.) Note that the conclusion contains at most one principal §.

$$\frac{\vdash \Gamma; A}{\vdash \Gamma; \forall XA}(\text{for all: } X \text{ is not free in } \Gamma) \qquad \frac{\vdash \Gamma; A\{B/X\}}{\vdash \Gamma; \exists XA}(\text{there is})$$

Intuitionistic propositional formulas are built from propositional atoms and the constant 1 by the connectives $\otimes, \multimap, \&$, and the modalities $!, \S$. Intuitionistic sequents are expressions of the form $\mathbf{A}_1; \mathbf{A}_2; \ldots; \mathbf{A}_k \vdash B$, where B and the formulas in the blocks $\mathbf{A}_i$ are intuitionistic. Because of the position of blocks to the left of the $\vdash$, the intended interpretation of the punctuation marks is dual to the one stated above, *i.e.*, A, B is intended to represent $A\&B$, $A; B$ is intended to represent $A \otimes B$, and $[A]$ is intended to represent $!A$. An intuitionistic sequent $\mathbf{A}_1; \mathbf{A}_2; \ldots; \mathbf{A}_k \vdash B$ may be interpreted in the language of LLL as the sequent $\vdash \mathbf{A}_1^{\perp}; \mathbf{A}_2^{\perp}; \ldots; \mathbf{A}_k^{\perp}; B$, where $\mathbf{A}_i^{\perp}$ denotes the block in which every formula in the block $\mathbf{A}_i$ is negated (where $(C \multimap D)^{\perp}$ is $C \otimes D^{\perp}$,) and the punctuation marks are left the same. The inference rules of ILLL are those that remain correct after this translation [5]. Note that the inner value for intuitionistic propositional formulas may be defined in any fibred phase structure in the straightforward way by interpreting each intuitionistic connective by its semantic counterpart. However, the definition of the inner value for intuitionistic sequents reflects the "left-handed" interpretation of the punctuation marks, *i.e.*, $[A]^* = !A^*$ and $(A_1,\ldots,A_\ell)^* = A_1^*\&\ldots\&A_\ell^*$. A valuation satisfies an intuitionistic sequent $\mathbf{A}_1; \ldots; \mathbf{A}_k \vdash B$ iff for each n, $(\mathbf{A}_1^* \otimes \cdots \otimes \mathbf{A}_k^*)_n \subseteq B_n^*$.

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