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ALGORITHMIC PROOFS OF TWO RELATIONS BETWEEN CONNECTIVITY AND THE 1-FACTORS OF A GRAPH*

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An algorithm is used to give simple proofs of these two known relations in the theory of matched graphs: A graph with a unique 1-factor contains a matched bridge; an n-connected graph with a 1-factor has at least n totally covered vertices, if $n \geq 2$. Conditions for exactly n, or more than n, totally covered vertices are also given.

1. Introduction

This note is based on an observation of Edmonds that properties of matched graphs can often be derived by analyzing an algorithm that finds a maximum matching [2]. Using a depth-first version of a cardinality matching algorithm, we derive two known relations between connectivity and 1-factors, strengthening one of them.

Specifically, we first show a graph with a unique 1-factor has a matched bridge. This result can be used to characterize graphs with a unique 1-factor [6]. It was previously proved by Kotzig, and then by Lovász [6]. (It also follows from a result in [1].)

Next we show an n-connected graph with a 1-factor has at least n totally covered vertices, if $n \geq 2$. This result is useful in analyzing the number of 1-factors of a graph [4, 7]. It was conjectured by Zaks [7], and proved by Lovász [6], as a consequence of a general theory of the structure of graphs with 1-factors.

Finally, we strengthen the second result. We give a sufficient condition for more than n totally covered vertices, and a necessary condition for exactly n totally covered vertices.

2. Definitions and notation

This section gives some relevant definitions and notation. For other standard terms from graph theory, see [5].

A *matching* M on an (undirected) graph is a set of edges, no two of which are

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incident to the same vertex. If edge $vw \in M$, it is a *matched edge*; vertices v and w are *matched* (to one another). A matching is specified by a function m, i.e., if $vw \in M$, then $v = m(w)$, $w = m(v)$. M is a *1-factor* if every vertex is matched. Throughout this paper, we consider only graphs that have a 1-factor. An *alternating path* (*cycle*) is a simple path (cycle) whose edges are alternately matched and unmatched. If C is an alternating cycle, the set of edges $M \oplus C = M \cup C - M \cap C$ is a 1-factor if M is. Fig. 1 shows a 1-factor, $\{12, 34, 56\}$. Adding edge 16 to the graph creates the alternating cycle $(1, 2, 4, 3, 5, 6, 1)$.

A graph is *n-connected* if it is connected and non-trivial whenever $<n$ vertices are deleted. A *bridge* is an edge whose removal increases the number of connected components.

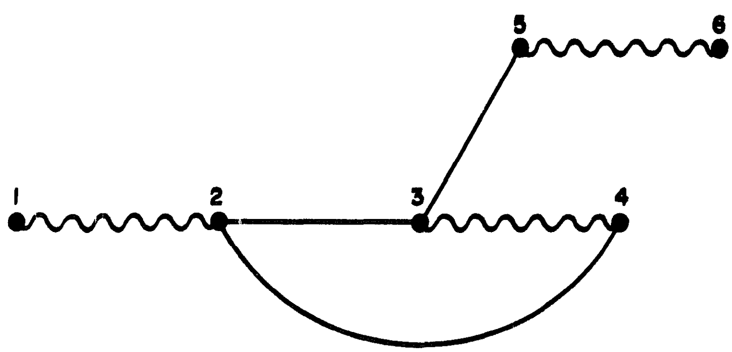

Fig. 1. A graph with a 1-factor.

3. Graphs with a unique 1-factor

This section describes how to search a graph with a 1-factor. The method is a simple modification of an algorithm that computes a maximum cardinality matching [3]. The search is used to analyze graphs with a unique 1-factor.

The idea is to build a long alternating path, such as $(1, 2, 4, 3, 5, 6)$ in Fig. 1. This is done by starting with a matched edge (edge 12), and repeatedly adding pairs of edges to the end of the path (first 24 and 43 are added, then 35 and 56). The process is complicated by odd length cycles, such as $B = (2, 3, 4, 2)$. For example, if the path $(1, 2, 3, 4)$ is constructed, it cannot be extended from 4. However, edge 24 (the last edge of cycle B) completes the alternating path $(1, 2, 4, 3)$, which can be extended. B is an example of a blossom, defined below.

We introduce some terminology; the botanical flavor comes from [2]. The subgraph built by a search is called the *stem*. It starts at the *root* vertex r; the first edge is matched. An *outer* vertex in the stem is joined to the root by an alternating path whose first and last edges are matched. If vertex v is outer, $P(v)$ denotes the associated alternating path; if x is a vertex in $P(v)$, then $P(v, x)$ denotes the portion of $P(v)$ from v to x. A vertex in the stem that is not outer is *inner.* In Fig. 1, vertex 1 is the root; 2, 3, 4 and 6 are outer; $P(6)=(6, 5, 3, 4, 2, 1)$.

The structure of the stem generalizes an alternating path, in the following sense. The outer vertices can be partitioned into sets, called *blossoms*, so if the graph is modified by contracting each blossom into a single vertex, the edges remaining in the stem form an alternating path. In this path, if blossom B is incident to the matched edge bc, where $b \in B$ is outer and $c \notin B$ is inner, then c is *matched* to B; b is the *base* of blossom B. In Fig. 1, the blossoms are $\{2, 3, 4\}$ and $\{6\}$; the alternating path is formed by edges 65, 53, 21; vertex 1 is matched to blossom $\{2, 3, 4\}$, whose base is 2.

The last blossom in the stem is called the *bud*. The stem is enlarged by adding edges incident to the bud (thereby changing the bud). In Fig. 1, the bud is 6.

Now we give the precise rules for a search. We use this notation: If $P \equiv (v_1, \ldots, v_n)$ and $Q \equiv (w_1, \ldots, w_m)$ are paths, and $v_n w_1$ is an edge, then path $P \cdot Q \equiv (v_1, \ldots, v_n, w_1, \ldots, w_n)$, and path rev $P \equiv (v_n, \ldots, v_1)$.

The search starts by making a matched edge $r, m(r)$ the stem. Vertex r is the root; $m(r)$ is the bud; $m(r)$ is outer, with alternating path $P(m(r)) \equiv (m(r), r)$.

The search continues by scanning (in a depth-first order) unmatched edges incident to the bud. Let v be a vertex in the bud, and let vw be an unmatched edge that has not been scanned. Edge vw is scanned as follows.

(1) If w is not in the stem, a *grow step* is done. Edges vw and $w, m(w)$ are added to the stem; $m(w)$ is the new bud; it is outer, with alternating path $P(m(w)) = m(w) \cdot w \cdot P(v)$.

(2) If w is an outer vertex not in the bud, a *blossom step* is done. Let B_i, $1 \leq i \leq m$, be the blossoms in the portion of the stem from v to w; thus $v \in B_1$, $w \in B_m$. Let c_i be the inner vertex matched to B_i. Then edge vw is added to the stem. The bud is a new blossom $B = B_1 \cup \cdots \cup B_m \cup \{c_1, \ldots, c_{m-1}\}$. B is matched to c_m. A new outer vertex c_i, $1 \leq i \leq m-1$, has alternating path $P(c_i) = \text{rev } P(v, c_i) \cdot P(w)$.

(3) In the remaining cases (vertex w is in the bud or w is inner), no changes are made to the stem.

The search continues scanning edges and building the stem, until all edges incident to the current bud are as in rule (3) above. At this point the search stops.

In Fig. 1, the stem is built by doing a grow, blossom, and grow step. If edge 64 is added to the graph and is scanned from the bud 6, a blossom step is done. Vertex 5 is made outer, with path $P(5)=(5, 6, 4, 3, 2, 1)$. The bud becomes $\{2, 3, 4, 5, 6\}$.

The following properties of the stem can be proved by a simple induction on the number of edges scanned. (For a more complete discussion, see [3, 2].) The stem consists of blossoms B_i and inner vertices c_i, for $1 \leq i \leq n$. For $1 \leq i \leq n$, B_i is joined to c_i by a matched edge $b_i c_i$, where b_i is the base of B_i. For $2 \leq i \leq n$, c_i is joined to B_{i-1} by an unmatched edge $c_i d_{i-1}$, where $d_{i-1} \in B_{i-1}$. If vertex $v \in B_i$, v is outer and $P(v)$ is an alternating path that starts and ends with matched edges. Further, $P(v) = P(v, b_i) \cdot c_i \cdot P(d_{i-1})$, i.e., $P(v)$ goes from v to the base of B_i, and then to the root. Note this implies $P(v)$ passes through blossoms B_i(and also through bases b_i and inner vertices c_j) for $i \geqslant j \geqslant 1$.

Now we analyze a graph with a unique 1-factor by searching it.

Theorem 1. *A graph with a unique* 1-*factor has a bridge that is matched.*

Proof. Consider the graph at the end of a search. Let B be the bud. We show the only edge joining B to a vertex not in B is the matched edge incident to B. This matched edge is thus the desired bridge (e.g., edge 56 in Fig. 1).

So let vw be an unmatched edge with $v \in B$. We must show $w \in B$. Since no grow or blossom step is possible, either $w \in B$ or w is inner (see rule (3)). But if w is inner, path $P(v, w) \cdot v$ is an alternating cycle, implying there are two 1-factors, contrary to assumption. Thus $w \in B$.

4. Higher connected graphs

This section shows graphs with 1-factors and connectivity $\geqslant 2$ have totally covered vertices. This is done by adding some rules to the search of the previous section.

We begin with some definitions. An edge is *matchable* if it is in some 1-factor. A vertex is *totally covered* if every edge incident to it is matchable. A set U of outer vertices whose paths contain unmatchable edges is defined as follows. Let vertex $v \neq m(r)$ be outer; let x be the first inner vertex in $P(v)$ after $m(v)$ $(x \neq m(v))$. (Note x exists since $v \neq m(r)$). Then

$$U = \{v \mid v \neq m(r) \text{ is outer and } P(v, x) \text{ contains an unmatchable edge}\};$$
$$W = \{v \mid v \text{ is outer and not in } U\}.$$

When a vertex v becomes outer, it enters set U or W. It may move from W to U when a subsequent blossom step changes the inner vertex x above. However, once v is in U, it remains so.

Vertices in U have the following useful property.

Lemma 1. *If $v \in U$ is in the bud and edge vw is unmatched, then w is not inner.*

Proof. We proceed by contradiction. Suppose w is inner. Then path $P(v, w) \cdot v$ is an alternating cycle, so every edge in it is matchable. In particular every edge in $P(v, x)$ is matchable, a contradiction.

Now we give the additional rules for choosing edges in a search. The idea is to extend the stem using unmatchable edges when possible. This makes set U large, and drives the search on.

Initially, the stem consists of the matched edge $r, m(r)$, where the root r is a totally covered vertex (if possible). An edge vw for scanning is chosen as follows:

(i) Suppose the bud is a non-totally covered vertex v. Scan an unmatchable edge vw. (This gives a grow or blossom step.)

(ii) Suppose the bud is a totally covered vertex v. If possible, scan an edge vw that gives a blossom step. Otherwise, scan an edge vw that gives a grow step such that the new outer vertex is not totally covered.

(iii) Suppose the bud is a blossom B, containing more than one vertex. Let b be the base. Scan an edge vw that gives a grow or blossom step, where $v \in B \cap U - b$.

The search ends when no edge vw can be chosen according to rules (i–iii) above.

Now we give some simple properties of a search.

Lemma 2. *If vertex* $z \in W - m(r)$, *then* z *is made outer in rule* (ii).

Proof. Let $z \neq m(r)$ be outer, and let x be the first inner vertex in $P(z)$ after $m(z)$, $x \neq m(z)$. If z is made outer in rule (i), $P(z, x)$ contains the unmatchable edge vw, so $z \in U$. If z is made outer in rule (iii), $P(z, x)$ contains the unmatchable edge of $P(v, b)$, so again $z \in U$. So if $z \in W - m(r)$, z can only be made outer in rule (ii).

Lemma 3. *If the bud is a totally covered vertex* v, $v \neq m(r)$, *then* $v \in U$.

Proof. Since $v \neq m(r)$ is the bud, it is made outer in a grow step. Since v is totally covered, it is not made outer in a grow step of rule (ii). So Lemma 2 shows $v \notin W$, i.e., $v \in U$.

Lemma 4. *In rule* (ii), *if edge* vw *gives a grow step, only vertex* $m(w)$ *enters* W; *if* vw *gives a blossom step, only* $m(v)$ *enters* W.

Proof. The Lemma is obvious for a grow step, so suppose edge vw gives a blossom step. This implies $v \neq m(r)$. So Lemma 3 shows $v \in U$. Let z be a new outer vertex. So path $P(z)$ contains rev $P(v, z)$. If $z \neq m(v)$, then $P(v, z)$ contains the unmatchable edge in $P(v)$; hence $z \in U$. Thus only $m(v)$ enters W.

Lemma 5. *Let* B *be a blossom with base* b *containing more than one vertex. Then* $B \cap U - b \neq \emptyset$.

Proof. Let vertex $z \in B - b$. Thus $z, m(z) \in B - b - m(r)$. (Vertex $m(r) \in B$ only if $m(r) = b$). Exactly one of the vertices $z, m(z)$ is made outer in a blossom step (see Section 3, or [3]); let it be z. It suffices to show that $z \in W$ implies $m(z) \in U$ (since the Lemma is obvious if $z \in U$). If $z \in W$, it is made outer in a blossom step of rule (ii), by Lemma 2. Thus $m(z)$ is totally covered, by Lemma 4. So Lemma 3 shows $m(z) \in U$.

Now we use the search to show when n-connected graphs with a 1-factor have totally covered vertices. If $n=1$, there need not be a totally covered vertex. (An example is a graph consisting of two odd length cycles plus an edge joining them.)

Theorem 2. *If $n \geq 2$, an n-connected graph G with a 1-factor has $\geq n$ totally covered vertices.*

Proof. We examine the bud at the end of a search and discover the desired totally covered vertices. There are three possibilities for the bud, corresponding to rules (i–iii) above. We discuss each possibility in turn.

(i) Suppose the bud is a non-totally covered vertex v. Vertex v is incident to an unmatchable edge vw. Vertex w is not inner (as in the proof of Lemma 1). So vw gives a grow or blossom step. In other words, the search does not end when the bud is a non-totally covered vertex.

(ii) Suppose the bud is a totally covered vertex v. If vw is an unmatched edge, then w is not inner. (If $v=m(r)$, this is obvious; if $v \neq m(r)$, it follows from Lemmas 3 and 1.) Thus vw gives a grow or blossom step. If the search has ended, no grow or blossom step can done according to the restrictions of rule (ii). We conclude vw gives a grow step where the new outer vertex is totally covered. The degree of v is $\geq n$. So there are $\geq n-1$ possible edges vw, i.e., $\geq n-1$ totally covered vertices that can be made outer in a grow step. Counting v gives $\geq n$ totally covered vertices. (In fact, since v is totally covered, vertex r is totally covered (by the paragraph preceding rule (i)). So there are actually $\geq n+1$ totally covered vertices).

(iii) Suppose the bud is a blossom B, with base b. We first show the graph $G-(B \cap W+b)$ is not connected. Let vertex $v \in B \cap U-b$, and consider an edge vw. Since the search has ended, vw does not give a grow or blossom step. Further, Lemma 1 shows w is not inner. Thus $w \in B$. So by Lemma 5, $B \cap U-b$ is a non-empty set of vertices in $G-(B \cap W+b)$, not connected to any other vertex (such as r). Thus $G-(B \cap W+b)$ is not connected.

Since G is n-connected, $|B \cap W+b| \geq n$. We show this implies $|W-m(r)| \geq n-1$. For $m(r) \in W$. If $m(r) \in B$, then $m(r)=b$, whence $B \cap W+b \subseteq W$. If $m(r) \notin B$, then $B \cap W \subseteq W-m(r)$. In both cases the desired inequality follows.

Each vertex in $W-m(r)$ corresponds uniquely to some totally covered vertex. (Specifically, a vertex $z \in W-m(r)$ is made outer in rule (ii) (Lemma 2). z corresponds to the totally covered vertex v that is the bud when z is made outer. Lemma 4 shows at most one z corresponds to a given totally covered vertex.) So there are $\geq n-1$ totally covered vertices. Since $n-1 \geq 1$, there are totally covered vertices, so r is totally covered. Since r is never the bud, it is not counted in the above correspondence. This gives $\geq n$ totally covered vertices.

In some cases Theorem 2 can be improved, as follows.

Corollary 1. *In an n-connected graph with a 1-factor, let a totally covered vertex r be adjacent to k totally covered vertices and $\geqslant 1$ non-totally covered vertex. Then there are $\geqslant n+k$ totally covered vertices.*

Proof. Choose a 1-factor with r matched to a non-totally covered vertex. Search the graph, with r as the root. It suffices to show at the end of the search, there are $\geqslant n-1$ totally covered vertices z such that z is outer and $P(z)$ contains an unmatchable edge. For then $z \neq r$ (as z is outer). Also, z is not adjacent to r. (If it is, $P(z) \cdot z$ is an alternating cycle with an unmatchable edge, a contradiction.) Thus z is not among the $k+1$ totally covered vertices hypothesized in the Corollary. Hence there are $\geqslant n+k$ totally covered vertices.

So to complete the proof, we must find the desired $n-1$ vertices z. The proof of Theorem 2 shows the search ends with the bud corresponding to rule (ii) or (iii). Now we consider each possibility.

(ii) Suppose the bud is a totally covered vertex v. The proof of Theorem 2 shows there are $\geqslant n-1$ totally covered vertices z that can be made outer in a grow step from v. Path $P(z)$ contains $P(v)$. Vertex $v \in U$, by Lemma 3 (Note $v \neq m(r)$: Vertex $m(r)$ is not totally covered, by the choice of 1-factor.) So $P(z)$ contains an unmatchable edge, as desired.

(iii) Suppose the bud is a blossom B. The proof of Theorem 2 shows there are $\geqslant n-1$ totally covered vertices z, where z was a bud at some point in the search. Lemma 3 shows $z \in U$ (note again, $z \neq m(r)$). Thus $P(z)$ contains an unmatchable edge.

For Corollary 1 to hold, we must assume r is adjacent to $\geqslant 1$ non-totally covered vertex. This is illustrated by the $(2n-1)$-connected graph K_{2n}.

As another illustration that Corollary 1 is tight, construct the following graph for n and k with $n \geqslant k$: The vertices are i, $1 \leqslant i \leqslant 2n$; the edges are $K_{n,n} \cup E$. Here $K_{n,n}$ is the set of edges ij, where $1 \leqslant i \leqslant n < j \leqslant 2n$; E is an arbitrary set of edges on vertices i, $1 \leqslant i \leqslant n-k$, containing $\geqslant 1$ edge incident to each vertex i. This graph is n-connected, with exactly $n+k$ totally covered vertices j, $n-k < j \leqslant 2n$.

Now we investigate when the bound of Theorem 2 is tight.

Corollary 2. *In an n-connected graph with a 1-factor, let there be exactly n totally covered vertices. Then these vertices form an independent set. Further, the graph has $\geqslant 2n$ vertices.*

Proof. Consider a totally covered vertex r. Since r is adjacent to $\geqslant n$ vertices, it is adjacent to a non-totally covered vertex. Thus Corollary 1 implies r is not adjacent to any totally covered vertex. So the totally covered vertices are independent. Further, there are $\geqslant n$ non-totally covered vertices, since deleting the non-totally covered vertices disconnects the graph. So the graph has $\geqslant 2n$ vertices.

This result and the tightness of Theorem 2 are illustrated by the example graphs for Corollary 1, with $k=0$.

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