Coexistence and stability of an unstirred chemostat model with Beddington–DeAngelis function

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1. Introduction

The chemostat is a laboratory apparatus used for the continuous culture of microorganisms. Mathematical models of the chemostat are surprisingly amenable to analysis. Basically, the chemostat consists of a nutrient input—with all nutrients needed for growth in abundance except one—pumped at a constant rate into a well-stirred culture vessel whose volume is kept constant by pumping the contents out at the same rate, and therefore its contents are spatially homogeneous. It is a model for a very simple lake where exploitative competition is easily studied. The mathematical analysis shows that two or more microbial populations cannot coexist indefinitely in competition for a single growth-limiting nutrient in a chemostat. The population which can grow at the lowest nutrient concentration effectively eliminates its rivals from the chemostat, see, for example [1]. This fact was subsequently verified by laboratory experiments. It was then natural to ask what additional factors can account for the apparent coexistence of competing species in nature. A candidate for an explanation is to remove the “well-stirred” hypothesis. A model often referred to as the “unstirred” chemostat, allows diffusion in one or more space variables, and thus involves a system of reaction–diffusion equations, such as [2–9]. Recently, the mathematical model with two resources in the unstirred chemostat had been well studied in [10–12]. The response function in these papers is mainly Holling type II functional response.

On the unstirred chemostat model with Holling type II functional response, early analyses can be found in [2], standard bifurcation theorems were used to show local coexistence in the one dimensional case, but without any stability results. Later, in [7], the corresponding results were generalized to the N-dimensional case by the monotone method and generalized maximum principle. And then the partial stability for the local coexistence solutions was established by the perturbation theorem for linear operators and the stability theorem for bifurcation solutions. Moreover, the global structure of the coexistence solutions was completely studied by the theorem of global bifurcation. In [5], the asymptotic behavior of
solutions was given as a function of the parameters by theory of uniform persistence in infinite-dimensional dynamical system and the theory of strongly order-preserving semidynamical system.

The most important advantage of the Holling type II functional response is that it is mathematically and mechanistically simple. However, in systems where predators compete directly for the available prey, the functional response should depend not only on the prey density but also on the predator density. The functional response introduced by Beddington [13] and DeAngelis et al. [14] is such a “predator-dependent” functional response. It is similar to the Holling type II functional response but has an extra term in the denominator which models mutual interference between predators. It can be derived mechanistically [13,15]. On the Beddington–DeAngelis functional response, a mathematical model of competition between two species for a growth-limiting nutrient in the unstirred chemostat was considered in [16]. There, the local coexistence solutions were studied and partial stability for the local coexistence solutions was established.

In this paper, we consider the unstirred chemostat model as follows:

\[
\begin{align*}
S_t &= S_x - m_1 u f(S, u) - m_2 v g(S, v), \\
u_t &= u_x + m_1 u f(S, u), \\
v_t &= v_x + m_2 v g(S, v), \\
S(x, 0) &= S_0(x) \\
u(x, 0) &= u_0(x), \\
v(x, 0) &= v_0(x),
\end{align*}
\]

where \(S(x, t)\) is the nutrient concentration at time \(t\), and \(u(x, t), v(x, t)\) are the concentrations of the two species in the culture vessel respectively. \(f(S, u) = \frac{S}{1 + k_1 S + \beta_1 u}, g(S, v) = \frac{S}{1 + k_2 S + \beta_2 v}\) are the Beddington–DeAngelis functions. \(m_1, k_1, \beta_1, i = 1, 2\) and \(\gamma\) are positive constants. \(m_i, i = 1, 2\) are the maximal growth rates of the two competitors (without an inhibitor), respectively. \(k_i, i = 1, 2\) are the Michaelis–Menten constants. \(\beta_i, i = 1, 2\) model mutual interference between predators \(u, v\) respectively. \(S_0(x), u_0(x), v_0(x) \in C([0, 1])\).

It can be easily seen that the solution of the problem (1) satisfies the relation \(S(x, t) + u(x, t) + v(x, t) = \frac{\gamma + 1}{\gamma} - x\) for all \(t \geq 0\) provided it satisfies this relation at \(t = 0\), which we assume to be the case for simplicity. Then the problem (1) reduces into the following apparently simpler problem:

\[
\begin{align*}
u_t &= u_x + m_1 u f(z - u - v, u), \\
v_t &= v_x + m_2 v g(z - u - v, v), \\
u(0, t) &= u(1, t) + \gamma u(1, t) = 0, \\
v(0, t) &= v(1, t) + \gamma v(1, t) = 0, \\
u(x, 0) &= u_0(x), \\
v(x, 0) &= v_0(x),
\end{align*}
\]

where \(z(x) = \frac{\gamma + 1}{\gamma} - x, x \in [0, 1]\).

We will discuss the existence of positive steady state solutions and the effects of the parameter \(\beta_1\) in the Beddington–DeAngelis functional response on coexistence states. Thus we will mainly concentrate on the following simplified elliptic system:

\[
\begin{align*}
u_u + m_1 u f(z - u - v, u) &= 0, \quad x \in (0, 1), \\
v_u + m_2 v g(z - u - v, v) &= 0, \quad x \in (0, 1), \\
u(0) &= u(1) = 0, \\
v(0) &= v(1) = 0,
\end{align*}
\]

where \(f(z - u - v, u) = \frac{z - u - v}{1 + k_1 (z - u - v) + \beta_1 u}, g(z - u - v, v) = \frac{z - u - v}{1 + k_2 (z - u - v) + \beta_2 v}\).

We only consider the case that \(S, u, v\) are nonnegative, so we redefine the response functions as follows:

\[
\hat{f}(S, u) = \begin{cases} f(S, u), & S \geq 0, u \geq 0 \\ 0, & \text{otherwise} \end{cases},
\]

\[
\hat{g}(S, v) = \begin{cases} g(S, v), & S \geq 0, v \geq 0 \\ 0, & \text{otherwise} \end{cases}.
\]

We will denote \(\hat{f}(S, u), \hat{g}(S, v)\) by \(f(S, u), g(S, v)\) respectively for the sake of simplicity.

In order to present the main results, we give some well-known conclusions and a few essential notations. Let \(\lambda_1, \sigma_1\) be the principal eigenvalues of the following problems, respectively,

\[
\begin{align*}
\phi_{xx} + \lambda_1 f(z, 0) \phi &= 0, \quad x \in (0, 1), \\
\psi_{xx} + \sigma g(z, 0) \psi &= 0, \quad x \in (0, 1),
\end{align*}
\]

where \(\phi_1(x) > 0, \psi_1(x) > 0\) are the corresponding eigenfunctions and they are uniquely determined by the normalization \(\|\phi_1(x)\| = 1, \|\psi_1(x)\| = 1\).
Setting \( v = 0 \) or \( u = 0 \) in (3), respectively, we get two scalar equations

\[
\begin{align*}
    u_{xx} + m_1 u f(z-u,u) &= 0, \quad x \in (0, 1), \quad u_x(0) = u_x(1) + \gamma u(1) = 0, \\
    v_{xx} + m_2 v g(z-v, v) &= 0, \quad x \in (0, 1), \quad v_x(0) = v_x(1) + \gamma v(1) = 0.
\end{align*}
\]

(6) (7)

The following results are proved in [16]. Similar results can also be found in [7].

**Lemma 1.1.** If \( m_1 \leq \lambda_1 \), then zero is the unique nonnegative solution of (6). If \( m_1 > \lambda_1 \), then (6) has a unique positive solution, denoted by \( \theta \), satisfying the following properties:

(i) \( 0 < \theta < z(x), x \in [0, 1] \);

(ii) \( \theta \) is continuously differentiable for \( m_1 \in (\lambda_1, +\infty) \), and is pointwisely increasing when \( m_1 \) increases;

(iii) \( \lim_{m_1 \to \lambda_1} \theta(m_1) = 0 \) uniformly for \( x \in [0, 1] \), and \( \lim_{m_1 \to +\infty} \theta(m_1) = z \) for almost every \( x \in [0, 1] \);

(iv) Let \( L_1 = \frac{d^2}{dx^2} + m_1(f(z - \theta, \theta) - \theta f'_1(z - \theta, \theta) + \theta f'_2(z - \theta, \theta)) \) be the linearized operator of (6) at \( \theta \). Then \( L_1 \) is a \( \text{Frechet} \) differentiable operator on \( C^2_\theta([0, 1]) = \{ u \in C^2([0, 1]) : u_x(0) = 0, u_x(1) + \gamma u(1) = 0 \} \), and all eigenvalues of \( L_1 \) are strictly negative.

**Remark 1.1.** For (7), we have the same conclusion as Lemma 1.1. Suppose \( m_2 > \sigma_1 \). We denote the unique positive solution by \( \Theta \). Let \( L_2 = \frac{d^2}{dx^2} + m_2(g(z - \Theta, \Theta) - \Theta g'_1(z - \Theta, \Theta) + \Theta g'_2(z - \Theta, \Theta)) \) be the linearized operator of (7) at \( \Theta \). Then all eigenvalues of \( L_2 \) are strictly negative.

Next, we introduce \( \hat{\lambda}_1, \hat{\sigma}_1 \) as the principal eigenvalues of the following two eigenvalue problems respectively

\[
\begin{align*}
    \phi_{xx} + \hat{\lambda} f(z - \Theta, 0) \phi &= 0, \quad x \in (0, 1), \quad \phi_x(0) = \phi_x(1) + \gamma \phi(1) = 0, \\
    \psi_{xx} + \hat{\sigma} g(z - \Theta, 0) \psi &= 0, \quad x \in (0, 1), \quad \psi_x(0) = \psi_x(1) + \gamma \psi(1) = 0,
\end{align*}
\]

where \( \hat{\phi}_1(x) > 0, \hat{\psi}_1(x) > 0 \) are the corresponding eigenfunctions and they are uniquely determined by the normalization \( \| \hat{\phi}_1(x) \| = 1, \| \hat{\psi}_1(x) \| = 1 \).

Now we are ready to state the main results of this paper.

**Theorem 1.1.** If \( m_1 > \lambda_1, m_2 > \sigma_1, \) and \( (m_1 - \hat{\lambda}_1)(m_2 - \hat{\sigma}_1) > 0 \), then (3) has a positive solution.

In addition, we use regular perturbation arguments to study the effects of the parameter \( \beta_1 \) in \( f(z - u - v, u) \) on the coexistence states of the system (3). We find that, for large \( \beta_1 \), any positive solution \((u, v)\) to (3) satisfies that \( \beta_1 u \) is close to a positive solution of the problem

\[
\begin{align*}
    w_{xx} + m_1 w \tilde{f}(z - \Theta, w) &= 0, \quad x \in (0, 1), \quad w_x(0) = w_x(1) + \gamma w(1) = 0,
\end{align*}
\]

(10)

where \( \tilde{f}(z - \Theta, w) = \frac{z - \Theta}{1 + k_1(z - \Theta) + w} \). That is, (10) governs almost all positive solutions of (3) when \( \beta_1 \) is very large. Thus by studying (10) carefully and employing the regular perturbation technique on the system (3), we obtain the following result.

**Theorem 1.2.**

(i) Suppose \( m_2 > \sigma_1 \) is fixed. For any \( \epsilon > 0 \), there exists \( M = M(\epsilon) > 0 \) large enough such that for any \( m_1 \in (\lambda_1, \hat{\lambda}_1 - \epsilon) \), \( \beta_1 \geq M \), (3) has no positive solution.

(ii) Suppose \( m_2 > \sigma_1 \) is fixed. For any \( \epsilon > 0 \), there exists \( M = M(\epsilon) > 0 \) large enough such that for each \( \beta_1 \geq M \),

(1) if \( \sigma_1 < m_2 < \hat{\sigma}_1 \), then (3) has no positive solution for \( m_1 \in (\hat{\lambda}_1, \epsilon) \);

(2) if \( m_2 > \hat{\sigma}_1 \), then (3) has no positive solution for \( m_1 = \hat{\lambda}_1 \) and exactly one positive solution for \( m_1 \in (\hat{\lambda}_1, \epsilon) \). Moreover, this unique solution is non-degenerate which means that the corresponding linearized operator is invertible, and linearly stable.

**Remark 1.2.** If \( \beta_1 \to \infty \), then \( \theta(\beta_1) \to 0 \) which implies that \( \hat{\sigma}_1 \to \sigma_1 \) as \( \hat{\sigma}_1 \) depends continuously on \( \theta \). Hence for \( \beta_1 \to \infty, m_1 \in (\lambda_1, \infty) \), \( \sigma_1 < m_2 < \hat{\sigma}_1 \), the nonexistence of (3) in Theorem 1.2 is easy to understand.

In fact, if \( \theta(\beta_1) \to 0 \) is not true as \( \beta_1 \to \infty \), then there exist \( \beta_{1i}, u_i = \theta(\beta_{1i}) \) satisfying \( \beta_{1i} u_i \to \infty, f(z - u_i, u_i) = \frac{x - u_i}{1 + k_1(z - u_i) + u_i} \to h_0 \) weakly in \( L^2 \) as \( \beta_{1i} \to \infty \) and

\[
    u_{xx} + m_1 u f(z - u_i, u_i) = 0, \quad x \in (0, 1), \quad u_x(0) = u_x(1) + \gamma u(1) = 0.
\]

Let \( \hat{u}_i = u_i / \| u_i \|_\infty \). Then

\[
    \hat{u}_{xx} + m_1 \hat{u} f(z - u_i, u_i) = 0, \quad x \in (0, 1), \quad \hat{u}_x(0) = \hat{u}_x(1) + \gamma \hat{u}(1) = 0.
\]

Since \( 0 < u_i < z \), by \( L^p \) estimates and the Sobolev embedding theorem, we assume \( \hat{u}_i \to \hat{u} \geq 0 \), \( \neq 0 \) in \( C^1 \) and

\[
    \hat{u}_{xx} + m_1 \hat{u} h_0 = 0, \quad x \in (0, 1), \quad \hat{u}_x(0) = \hat{u}_x(1) + \gamma \hat{u}(1) = 0.
\]
Since $0 \leq h_0 \leq \frac{1}{\xi_1}$, we obtain $\hat{u} > 0$ in $x \in [0, 1]$ from the strong maximum principle and Hopf lemma. Then $f(z - u_t, u_t) = \frac{1}{1 + k_1(z - u_t) + \beta_1|u_t|^{\infty}} \to 0$ in $L^2$. Hence $h_0 = 0$ and
\[ \hat{u}_{xx} = 0, \quad x \in (0, 1), \quad \hat{u}_x(0) = \hat{u}_x(1) + \gamma \hat{u}(1) = 0, \]
which implies $\hat{u} \equiv 0$. This is a contradiction.

The main tools in proving Theorems 1.1 and 1.2 include the linear stability theory, the fixed point index theory, the perturbation technique and the bifurcation theory. A key point of the proof for Theorem 1.2 is to make use of the limiting Eq. (10). Finally, the perturbation theory leads to the main result of this paper.

The rest of this paper is organized as follows. In Section 2, we first give some preliminary results and notations which will be used in the later section. Then, for the general case Theorem 1.1 is proved. In Section 3, for large $\beta_1$, we establish Theorem 1.2.

2. The proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. We first introduce some notations and preliminary results which will be used throughout this paper.

Let $X$ be a real Banach space and $W(\subset X)$ a closed convex set. $W$ is called a wedge provided that $\alpha W \subset W$ for all $\alpha \geq 0$. A wedge $W$ is said to be a cone if $W \cap [-W] = \{0\}$.

Let $y \in W$ and define a wedge $W_y = \text{cl}\{x \in X : y + \nu x \in W$ for some $\nu > 0\}$, where “cl” means the closure of the set. Let $S_y = \{x \in W_y : -x \in W_y\}$. Assume that $T$ is a compact and Fréchet differentiable operator on $X$ such that $y \in W$ is a fixed point of $T$ and $T(W) \subset W$. Then the Fréchet derivative $T'(y)$ of $T$ at $y$ leaves $W_y$ and $S_y$ invariant (see [17–19]). If there exists a closed linear subspace $X_y$ of $X$ such that $X = S_y \oplus X_y$, and $W_y$ is generating, then the index of $T$ at $y$ can be found by analyzing certain eigenvalue problems in $X_y$ and $S_y$ as follows.

**Lemma 2.1.** Let $Q : X \to X_y$ be the projection operator of $X_y$ along $S_y$. If the Fréchet derivative $T'(y)$ of $T$ at $y$ has no nonzero fixed point in $W_y$, then $\text{index}_W(T, y)$ exists. Furthermore,

(i) $\text{index}_W(T, y) = 0$, if $Q \circ T'(y)$ has an eigenvalue $\lambda > 1$;

(ii) $\text{index}_W(T, y) = \text{index}_S(T'(y), 0)$, if $Q \circ T'(y)$ has no eigenvalue equal to or greater than 1. Here $\text{index}_S(T'(y), 0)$ is the index of the linear operator $T'(y)$ at 0 in the space $S_y$.

We denote the Fréchet derivative of $T$ at the fixed point $y$ by $L$ and we say $L$ has property $\alpha$ at $y$ if there exist $t \in (0, 1)$ and $w \in W_y \setminus S_y$ such that $w - tw \in S_y$. Then the following statement is a general result of Dancer [18], Ruan and Wei [19] on the fixed point index with respect to the positive cone $W$.

**Lemma 2.2.**

(i) If $L - I$ is invertible on $X$, and $L$ has property $\alpha$ on $W_y$, then $\text{index}_W(T, y) = 0$;

(ii) If $L - I = L$ is invertible on $X$, and $L$ does not have property $\alpha$ on $W_y$, then $\text{index}_W(T, y) = (-1)^\sigma$, where $\sigma$ is the sum of the algebraic multiplicities of the eigenvalues of $L$ which are greater than 1;

(iii) If $L - I$ is not invertible on $X$, but $\text{Ker}(L - I) \cap W_y = \{0\}$, then $\text{index}_W(T, y) = 0$.

Finally, we introduce a well-known eigenvalue problem and the properties of its eigenvalues, which are crucial to prove our main results.

**Lemma 2.3.** Let $c(x), q(x) \in C([0, 1])$, and $c(x) \geq 0, q(x) \geq 0, 0$, and let $\lambda_i(c, q)$ be the $i$th eigenvalue and the corresponding eigenfunction of the problem
\[ -\Delta \varphi + c(x)\varphi = \lambda q(x)\varphi, \quad x \in (0, 1), \quad \varphi_x(0) = \varphi_x(1) + \gamma \varphi(1) = 0. \]

Then $0 < \lambda_1(c, q) < \lambda_2(c, q) \leq \cdots \to \infty$, $\varphi_1 > 0$ and
\[ \lambda_1(c, q) = \inf_{\varphi} \frac{\int_0^1 (\varphi'^2 + c(x)\varphi^2)dx + \gamma(1)\varphi^2(1)}{\int_0^1 q(x)\varphi^2dx} \]
is a simple eigenvalue. Moreover, $\lambda_i(c, q)$ is continuous on $c, q$, and the following comparison principles hold:

(i) $\lambda_1(c_1, q) \leq \lambda_1(c_2, q)$ if $c_1 \leq c_2$ on $[0, 1]$ and the strict inequality holds if $c_1 \neq c_2$;

(ii) $\lambda_i(c_1, q) \geq \lambda_i(c_2, q)$ if $q_1 \leq q_2$ on $[0, 1]$ and the strict inequality holds if $q_1 \neq q_2$.

**Lemma 2.4.** Let $c(x), q(x) \in C([0, 1])$, and $a, M$ be positive constants such that $M + aq(x) > c(x)$ for all $x \in [0, 1]$. Then we have the following statements:

(i) $a > \lambda_1(c, q) \Leftrightarrow \lambda_1(\Delta + aq(x) - c(x)) > 0 \Leftrightarrow r((-\Delta + M)^{-1}(aq(x) - c(x) + M)) > 1$;

(ii) $a < \lambda_1(c, q) \Leftrightarrow \lambda_1(\Delta + aq(x) - c(x)) < 0 \Leftrightarrow r((-\Delta + M)^{-1}(aq(x) - c(x) + M)) < 1$;

(iii) $a = \lambda_1(c, q) \Leftrightarrow \lambda_1(\Delta + aq(x) - c(x)) = 0 \Leftrightarrow r((-\Delta + M)^{-1}(aq(x) - c(x) + M)) = 1$. 
Here, $\lambda_1(\Delta + aq(x) - c(x))$ represents the principle eigenvalue of the operator $\Delta + aq(x) - c(x)$, and $r((-\Delta + M)^{-1}(aq(x) - c(x) + M))$ denotes the spectral radius of $((-\Delta + M)^{-1}(aq(x) - c(x) + M))$.

In addition, for (3) we also have the following a priori estimate.

**Lemma 2.5 ([16]).** Assume $(u, v)$ is a nonnegative solution of (3) with $u \not\equiv 0$ and $v \not\equiv 0$. Then

(i) $m_1 > \lambda_1, m_2 > \sigma_1$;

(ii) $0 < u \leq \theta, 0 < v \leq \Theta$;

(iii) $u + v < z$.

For the functional analytic framework of the degree theory, we introduce the following spaces:

- $C_0([0, 1]) = \{u \in C([0, 1]) : u(0) = 0, u_0(1) + \gamma u(1) = 0\}$
- $C_0^1([0, 1]) = \{u \in C^1([0, 1]) : u(0) = 0, u_0(1) + \gamma u(1) = 0\}$
- $X = C_0([0, 1]) \times C_0([0, 1]), \quad W = \{(u, v) \in X : u \geq 0, v \geq 0, x \in [0, 1]\}$
- $D = \{(u, v) \in W : u \leq \theta + 1, v \leq \Theta + 1, x \in [0, 1]\}$
- $D' = (\text{int} D) \cap W$.

$W$ is a cone of $X$.

**Lemma 2.6.**

(i) If $m_1 > \lambda_1, m_2 \not\equiv \sigma_1$ or $m_2 > \sigma_1, m_1 \not\equiv \lambda_1$, then $\text{index}_W(A, (0, 0)) = 0$; if $m_1 < \lambda_1$ and $m_2 < \sigma_1$, then $\text{index}_W(A, (0, 0)) = 1$.

(ii) $\text{index}_W(A, D') = 1$.

(iii) Assume $m_1 > \lambda_1$. Then $\text{index}_W(A, (\theta, 0)) = 0$ if $m_2 > \hat{\sigma}_1$; $\text{index}_W(A, (\theta, 0)) = 1$ if $m_2 < \hat{\sigma}_1$.

(iv) Assume $m_2 > \sigma_1$. Then $\text{index}_W(A, (0, \Theta)) = 0$ if $m_1 > \hat{\lambda}_1$; $\text{index}_W(A, (0, \Theta)) = 1$ if $m_1 < \hat{\lambda}_1$.

**Proof.** Define an operator $A_t : D' \rightarrow W$ as follows

$$A_t(u, v) = \left(-\frac{d^2}{dx^2} + M_1\right)^{-1}\begin{pmatrix}
 tm_1 uf(z - u - v, u) + M_1 u \\
 tm_2 vg(z - u - v, v) + M_1 v
\end{pmatrix},$$

where $\left(-\frac{d^2}{dx^2} + M_1\right)^{-1}$ is the inverse of the operator $-\frac{d^2}{dx^2} + M_1$ subject to the boundary conditions $u_0(0) = u_0(1) + \gamma u(1) = 0, M_1$ is a positive constant such that $tm_1 uf(z - u - v, u) + M_1 > 0, tm_2 vg(z - u - v, v) + M_1 > 0$ for all $(u, v) \in D'$ and $t \in [0, 1], A_t$ is a compact operator. Denote $A = A_1$, then $A : D' \rightarrow W$ is continuous and differentiable, and (3) has a nonnegative solution if and only if there is a fixed point of $A$ in $D'$.

(i) Let $y = (0, 0)$. Then $W_y = W, S_y = (0, 0), X_y = X, Q : X = X_y$. So $Q \equiv I$. Assume $(u, v)$ is a fixed point of $A'(0, 0)$ in $W_y$. Since

$$A'(0, 0) = \left(-\frac{d^2}{dx^2} + M_1\right)^{-1}\begin{pmatrix}
 m_1 f(z, 0) + M_1 \\
 m_2 g(z, 0) + M_1
\end{pmatrix},$$

we have

$$\begin{align*}
&-u_{xx} = m_1 f(z, 0) u, & x \in (0, 1), \\
&-v_{xx} = m_2 g(z, 0) v, & x \in (0, 1), \\
&u_0(0) = u_0(1) + \gamma u(1) = 0, \\
&v_0(0) = v_0(1) + \gamma v(1) = 0.
\end{align*}$$

It follows that $u \equiv 0, v \equiv 0$, since $m_1 \neq \lambda_1, m_2 \neq \sigma_1$. As a result, $I - A'(0, 0)$ is invertible on $W_y$.

Suppose $\lambda$ is the eigenvalue of $A'(0, 0)$, and $(\xi, \eta)$ is the corresponding eigenfunction. Then

$$\begin{pmatrix}
-\frac{d^2}{dx^2} + M_1 \end{pmatrix}^{-1}\begin{pmatrix}
m_1 f(z, 0) + M_1 \xi = \lambda \xi, & x \in (0, 1), \\
-m_2 g(z, 0) + M_1 \eta = \lambda \eta, & x \in (0, 1),
\end{pmatrix}$$

$$\begin{align*}
&\xi_0(0) = \xi_0(1) + \gamma \xi(1) = 0, \\
&\eta_0(0) = \eta_0(1) + \gamma \eta(1) = 0.
\end{align*}$$

If $\xi \equiv 0$, then $\lambda$ is an eigenvalue of the following problem

$$\begin{pmatrix}
-\frac{d^2}{dx^2} + M_1 \end{pmatrix}^{-1}\begin{pmatrix}
m_2 g(z, 0) + M_1 \eta = \lambda \eta, & x \in (0, 1), \\
\eta_0(0) = \eta_0(1) + \gamma \eta(1) = 0.
\end{pmatrix}$$

(11)
From Lemma 2.4, when $m_2 > \sigma_1 = \lambda_1(0, g(z(0), 0))$, we obtain $r \left( \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (m_1(g(z, 0) + M_1)) \right) > 1$, and when $m_2 < \sigma_1$, then $r \left( \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (m_1(g(z, 0) + M_1)) \right) < 1$. Hence for the eigenfunction with the form $(0, \eta)$, $A'(0, 0)$ has an eigenvalue greater than $1$ as $m_2 > \sigma_1$ and $A'(0, 0)$ has no eigenvalue equal to or greater than $1$ as $m_2 < \sigma_1$.

If $\xi \neq 0$, then $\lambda$ is an eigenvalue of the following problem

\[
\left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (m_1f(z, 0) + M_1)\xi = \lambda \xi, \quad x \in (0, 1), \quad \xi(0) = \xi(1) + \gamma \xi(1) = 0.
\]

Similarly, it follows from Lemma 2.4 that for the eigenfunction with the form $(\xi, \eta)$ satisfying $\xi \neq 0$, $A'(0, 0)$ has an eigenvalue greater than $1$ when $m_1 > \lambda_1$ and it has no eigenvalue equal to or greater than $1$ when $m_1 < \lambda_1$.

Hence, by Lemma 2.1 we see that if $m_1 > \lambda_1$, $m_2 \neq \sigma_1$ or $m_2 > \sigma_1$, $m_1 \neq \lambda_1$, then $\text{index}_W(A, (0, 0)) = 0$; if $m_1 < \lambda_1$ and $m_2 < \sigma_1$, then $\text{index}_W(A, (0, 0)) = 1$.

(ii) Let $\epsilon \in (0, 1)$ be small such that $\epsilon m_1 < \lambda_1$, $\epsilon m_2 < \sigma_1$, then $A_\epsilon$ has a unique nonnegative fixed point $(0, 0)$ in $D'$ and $\text{index}_W(A_\epsilon, (0, 0)) = 1$. Then $0 < \epsilon \ll 1$ from (i). Thus $\text{index}_W(A_\epsilon, D') = 1$. On the other hand, by virtue of an a priori estimate and the homotopic invariance property of the fixed point index, we obtain $\text{index}_W(A_\epsilon, D')$ is constant for all $t \in [0, 1]$. Hence, $\text{index}_W(A, D') = \text{index}_W(A_\epsilon, D') = 1$.

(iii) Let $y = (\theta, 0)$. Then $W_y = \{(u, v) \in X : v \geq 0\}$, $S_y = \{(u, 0) : u \in C_b([0, 1])\}$, $X_y = \{(0, v) : v \in C_b([0, 1])\}$, $Q : X \to X$. So $X = X_y \cup S_y, Q(u, v) = (0, v)$. Assume $(u, v)$ is a fixed point of $A'(\theta, 0)$ in $W_y$. Since

\[
A'(\theta, 0) = \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} \begin{pmatrix} M_1 + F & -m_1 \theta f_z(z, \theta, \theta) \\ 0 & M_1 + m_2 g(z, \theta, 0) \end{pmatrix},
\]

where $F = m_1[f(z, \theta, \theta) - \theta f_z(z, \theta, \theta) + \theta f_z(z, \theta, \theta)]$, we have

\[
\begin{align*}
-u_{xx} &= m_1[f(z, \theta, \theta) - \theta f_z(z, \theta, \theta) + \theta f_z(z, \theta, \theta)]u - m_1 \theta f_z(z, \theta, \theta) v, \quad x \in (0, 1), \\
u_{xx} &= m_2 g(z, \theta, 0) v, \quad x \in (0, 1), \\
u_x(0) &= \nu_x(1) + \gamma u(1) = 0, \\
u_x(0) &= \nu_x(1) + \gamma v(1) = 0.
\end{align*}
\]

It follows that $u \equiv 0, v \equiv 0$, since $m_2 \neq \sigma_1$. As a result, $I - A'(\theta, 0)$ is invertible on $W_y$. In the following we consider the eigenvalue of $Q \circ A'(\theta, 0)$. Since $Q(u, v) = (0, v)$, the eigenfunctions of $Q \circ A'(\theta, 0)$ have the form of $(0, v)$, where $v$ is a non-zero solution of the following problem,

\[
\left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (M_1 + m_2 g(z, \theta, 0)) v = \lambda v, \quad x \in (0, 1), \quad \nu_v(0) = \nu_v(1) + \gamma v(1) = 0.
\]

From Lemmas 2.4 and 2.3, we get that if $m_2 > \sigma_1 = \lambda_1(0, g(z(0), 0))$, then $r \left( \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (M_1 + m_2 g(z, \theta, 0)) \right) > 1$, i.e. $Q \circ A'(\theta, 0)$ has an eigenvalue greater than $1$ and $\text{index}_W(A, (\theta, 0)) = 0$; if $m_2 < \sigma_1$, then $r \left( \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (M_1 + m_2 g(z, \theta, 0)) \right) < 1$, $Q \circ A'(\theta, 0)$ has no eigenvalue equal to or greater than $1$ and $\text{index}_W(A, (\theta, 0)) = (-1)^\sigma$, where $\sigma$ is the sum of the algebraic multiplicities of the eigenvalues of $A'(\theta, 0)$ which are greater than $1$ in the space $S_y$. At last, we show that $\sigma = 0$. Let $\lambda$ be an eigenvalue of $A'(\theta, 0)$ in the space $S_y$, $(u, v)$ is the corresponding eigenfunction. Then $v = 0$, and $u \neq 0$ satisfies the following problem,

\[
\left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (M_1 + F) u = \lambda u, \quad x \in (0, 1), \quad u_x(0) = u_x(1) + \gamma u(1) = 0.
\]

Noted that $L_1 = \frac{d^2}{dx^2} + F$, and from Lemma 1.1(iv), all eigenvalues of $L_1$ are strictly negative, especially, $\lambda_1 \left( \frac{d^2}{dx^2} + F \right) < 0$. Then we obtain $r \left( \left( -\frac{d^2}{dx^2} + M_1 \right)^{-1} (M_1 + F) \right) < 1$ by Lemma 2.4, and $A'(\theta, 0)$ has no eigenvalue equal to or greater than $1$ in the space $S_y$. So $\sigma = 0$. Thus $\text{index}_W(A, (\theta, 0)) = 1$ if $m_2 < \sigma_1$.

(iv) Can be established by similar arguments as in (iii). □

Proof of Theorem 1.1. This result follows immediately from Lemma 2.6 and the degree theory. □
3. The proof of Theorem 1.2

In this section, we consider the effects of $\beta_1$ in $f(z - u - v, u)$ on the coexistence states.

**Theorem 3.1.** The problem (10) has a positive solution if and only if $m_1 > \hat{\lambda}_1$. Moreover, the positive solution is unique and asymptotically stable. We denote it by $w^*$.

**Proof.** Suppose $w$ is a positive solution of (10). Then, by Lemma 2.3, we have

$$m_1 = \lambda_1(0, \hat{f}(z - \theta, w)) > \lambda_1(0, \hat{f}(z - \theta, 0)) = \lambda_1(0, f(z - \theta, 0)) = \hat{\lambda}_1.$$ 

Therefore $m_1 > \hat{\lambda}_1$. Next, we show that if $m_1 > \hat{\lambda}_1$, then (10) has a unique positive solution. To this end, we first prove that there exists $C > 0$ such that $\|w\|_{C^1} \leq C$ for any positive solutions of (10) with $m_1 > \hat{\lambda}_1$. Suppose this is not true. We may assume that there exist $m_1 \to m_1 \geq \lambda_1$, as $t \to \infty$, $w_t$ is the positive solution of (10) with $m_1 = m_1$ and $\|w_t\|_{\infty} \to \infty$. Set $w_t = w_t/\|w_t\|_{\infty} \to h_1$ weakly in $L^2$ and

$$\hat{w}_{xx} + m_1 \hat{f}(z - \theta, w_t) = 0, \quad x \in (0, 1), \quad \|\hat{w}_t\|_{\infty} = 1, \quad \hat{w}_{tx}(0) = \hat{w}_{tx}(1) + \gamma \hat{w}(1) = 0.$$ 

By $L^p$ estimates and the Sobolev embedding theorem, we may assume $\hat{w} \to \hat{w} \geq 0, \neq 0$ in $C^1$, and $\hat{w}$ weakly satisfies

$$\hat{w}_{xx} + m_1 \hat{f}(z - \theta, w) = 0, \quad x \in (0, 1), \quad \hat{w}(0) = \hat{w}(1) + \gamma \hat{w}(1) = 0.$$ 

Since $0 \leq h_t \leq \frac{1}{\eta}$, the Harnack inequality (see Lemmas 2.1 and 2.2 in [20]) is applicable and we obtain $\hat{w} > 0$ in $x \in [0, 1]$. Then $\hat{f}(z - \theta, w) = \int \frac{z - \theta}{1 + k_1(z - \theta) + w_t/\|w_t\|_{\infty}} \to 0$ in $L^2$. Hence $h_1 = 0$ and

$$\hat{w}_{xx} = 0, \quad x \in (0, 1), \quad \hat{w}_x(0) = \hat{w}_x(1) + \gamma \hat{w}(1) = 0,$$

which implies $\hat{w} \equiv 0$. This is a contradiction to $\|\hat{w}\|_{\infty} = 1$. Thus there exists $C > 0$ such that $\|w\|_{\infty} < C$. For (10), by $L^p$ estimates and the Sobolev embedding theorem, the desired a priori estimate is established.

Finally, we show the stability and uniqueness. Suppose $m_1 > \hat{\lambda}_1$. Then $B_t(w) = \left(-\frac{\partial^2}{\partial x^2} + M_2\right)^{-1}(tm_1 \hat{f}(z - \theta, w) + M_2 w)$, where $M_2$ is a positive constant such that $tm_1 \hat{f}(z - \theta, w) + M_2 > 0$ for all $w \in D$ and $t \in [0, 1]$.

**For stability, suppose $w$ is a positive solution of (10).** Then

$$-w_{xx} - m_1 \hat{f}(z - \theta, w) = 0, \quad x \in (0, 1), \quad w_x(0) = w_x(1) + \gamma w(1) = 0.$$ 

We consider the eigenvalue problem

$$-h_x - m_1 [\hat{f}(z - \theta, w) + \hat{f}_2(z - \theta, w)] h = \eta h, \quad x \in (0, 1), \quad h_x(0) = h_x(1) + \gamma h(1) = 0.$$ 

**Denote the principle eigenvalue of (12) by $\eta_1$.** Then by Lemma 2.3,

$$\eta_1 = \lambda_1(-m_1 \hat{f}(z - \theta, w) + w_2(z - \theta, w), 1) > \lambda_1(-m_1 \hat{f}(z - \theta, w), 1) = 0,$$

since $w_2(z - \theta, w) < 0$.

Hence, if $m_1 > \hat{\lambda}_1$ then (12) has no eigenvalue equal to or less than 0. Therefore for any positive solution $w$ of (10), $w$ is non-degenerate, asymptotically stable and $\text{index}_p(B, w) = (-1)^0 = 1$ provided $m_1 > \hat{\lambda}_1$. It can be shown that (10) has at most finitely many positive solutions by the non-degeneracy of all positive solutions and the compactness of $B$. If we denote all the positive solutions of (10) by $\{w_i, 1 \leq i \leq l\}$, then

$$1 = \text{index}_p(B, D) = \text{index}_p(B, 0) + \sum_{i=1}^{l} \text{index}_p(B, w_i) = l,$$

which implies that (10) has a unique positive solution. □

Next, we will show all positive solutions of (3) are governed by problem (10) when $\beta_1$ is large.
Theorem 3.2. Let $m_2 > \sigma_1$ be fixed. For any $A > \lambda_1$, and small $\varepsilon > 0$, there exists a large $M = M(\varepsilon, A) > 0$ such that if $\beta_1 \geq M$ and $m_1 \in (\lambda_1, A)$, for any positive solution $(u, v)$ of\(^{(3)}\), we have $\|u\|_{C_1} + \|v - \Theta\|_{C_1} \leq \varepsilon$. Furthermore, by choosing $M(\varepsilon, A)$ suitably larger such that if $\beta_1 \geq M$ and $m_1 \in (\lambda_1, A)$, we have $\|\beta_1 u - w\|_{C_1} \leq \varepsilon$, where $w^*$ is the unique positive solution of\(^{(10)}\).

Proof. Suppose that the conclusion is not true for the first part. Then there exist $A_0 > \lambda_1$, $\beta_{\mathbb{H}} \rightarrow \infty$, $m_{11} \in (\lambda_1, A_0)$, and $(u_1, v_1)$ is a positive solution of\(^{(3)}\) with $m_1 = m_{11}$, $\beta_1 = \beta_{\mathbb{H}}$ such that $(u_1, v_1)$ is bounded away from $(0, \Theta)$. By Lemma 2.5, we have $v_1 < u_1$, $v_1 < z$, which implies that $\{ -u_{\mathbb{H}} \}$ and $\{ -v_{\mathbb{H}} \}$ are bounded in $L^\infty((0, 1))$ from the equations. Hence, by $L^p$ estimates and the Sobolev embedding theorem, we may suppose, choosing a subsequence when necessary, that $m_{11} \rightarrow m_1 \in [\lambda_1, A_0]$, $u_1 \rightarrow u \geq 0$, $v_1 \rightarrow v \geq 0$ in the $C^1$ norm for some $u, v \in C^1_0([0, 1])$. Then $f(z - u - v, u) = \frac{z}{1 + k_1(z - u - v) + \beta_{\mathbb{H}} u}$ weakly in $L^2$, and $u$ weakly satisfies

$$u_{xx} + m_{11}uh_2 = 0, \quad x \in (0, 1), \quad u_x(0) = u_x(1) + \gamma u(1) = 0.$$ 

Suppose $u \equiv 0$. Then $v$ satisfies

$$v_{xx} + m_{11}vg(z - v, v) = 0, \quad x \in (0, 1), \quad v_x(0) = v_x(1) + \gamma v(1) = 0.$$ 

It follows from Remark 1.2 that $v \equiv 0$ or $v = \Theta$ for $m_2 > \sigma_1$. If $v \equiv 0$, we let $\tilde{v}_1 = v/\|v\|_\infty$. Then

$$\tilde{v}_{1xx} + m_{11}\tilde{v}_1g(z - u - v, v) = 0, \quad x \in (0, 1), \quad \tilde{v}_x(0) = \tilde{v}_x(1) + \gamma \tilde{v}(1) = 0.$$ 

By $L^p$ estimates and the Sobolev embedding theorem, we may assume $\tilde{v}_i \rightarrow \tilde{v}$ in the $C^1$ norm. By passing to the limit in the equation of $\tilde{v}_i$, we get

$$\tilde{v}_{xx} + m_{11}\tilde{v}_1g(z, 0) = 0, \quad x \in (0, 1), \quad \tilde{v}_x(0) = \tilde{v}_x(1) + \gamma \tilde{v}(1) = 0.$$ 

Multiplying\(^{(13)}\) by $\psi_1$ and integrating on $(0, 1)$, then

$$\int_0^1 (m_2 - \sigma_1) \tilde{v}_1 g(z, 0) = 0.$$ 

From $m_2 > \sigma_1$, $\psi_1 > 0$, $g(z, 0) > 0$, it follows that $\tilde{v} \equiv 0$. This is impossible. Hence $v = \Theta$, then $(u_1, v_1) \rightarrow (0, \Theta)$ in $C^1$, which contradicts our assumption that $(u_1, v_1)$ is bounded away from $(0, \Theta)$.

Suppose that $u \geq 0, \not \equiv 0$. Then $u > 0$ by the Harnack inequality which implies $h_2 = 0$. Thus we have $u_x = 0, x \in (0, 1), u_x(0) = u_x(1) + \gamma u(1) = 0$, which means $u \equiv 0$, but this is a contradiction.

For the second part, it suffices to show that $\beta_1 u$ is close to some positive solution of\(^{(10)}\) in the $C^1$ norm when $\beta_1$ is large enough.

We begin with the proof that $\beta_1 \|u\|_\infty$ is uniformly bounded under the condition of Theorem 3.2. If this is not true, then there exist $A_0 > \lambda_1$, $\beta_{\mathbb{H}} \rightarrow \infty$, $m_{11} \in (\lambda_1, A_0)$, and $(u_1, v_1)$ is a positive solution of\(^{(3)}\) with $m_1 = m_{11}$, $\beta_1 = \beta_{\mathbb{H}}$ such that $\beta_{\mathbb{H}} \|u\|_\infty \rightarrow \infty$. Set $\tilde{u}_1 = u/\|u\|_\infty$. Then

$$\tilde{u}_{1xx} + m_{11}\tilde{u}_1f(z - u - v, v) = 0, \quad x \in (0, 1), \quad \tilde{u}_x(0) = \tilde{u}_x(1) + \gamma \tilde{u}(1) = 0.$$ 

By $L^p$ estimates and the Sobolev embedding theorem, we may assume $\tilde{u}_i \rightarrow \tilde{u} \geq 0, \not \equiv 0$ in the $C^1$ norm, $f(z - u - v, u) \rightarrow h_2$ weakly in $L^2$. By passing to the limit in\(^{(14)}\), we find that $\tilde{u}$ satisfies the following equation weakly:

$$\tilde{u}_{xx} + m_{11}\tilde{u}_2h_2 = 0, \quad x \in (0, 1), \quad \tilde{u}_x(0) = \tilde{u}_x(1) + \gamma \tilde{u}(1) = 0.$$ 

Therefore, $\tilde{u} > 0$ on $[0, 1]$ by the Harnack inequality. Thus

$$f(z - u - v, u) = \frac{z - u - v}{1 + k_1(z - u - v) + \beta_{\mathbb{H}} u} \rightarrow h_2 = 0,$$ 

as $i \rightarrow \infty$, which implies $\tilde{u} \equiv 0$. There is a contradiction.

Next, set $w_1 = \beta_{\mathbb{H}} u$. Then $w_i$ satisfies

$$w_{ixx} + m_{11}w_i f(z - u - v, w) = 0, \quad x \in (0, 1), \quad w_x(0) = w_x(1) + \gamma w(1) = 0.$$ 

Since $\|w_i\|_\infty$ is bounded, by $L^p$ estimates and the Sobolev embedding theorem, we may assume that $w_i \rightarrow w$ in $C^1$. Then we see that $w$ is a nonnegative solution of\(^{(10)}\) by letting $i \rightarrow \infty$ in\(^{(15)}\). There are two possibilities here:

(i) $m_1 = \hat{\lambda}_1$. In this case, $w_i = \beta_{\mathbb{H}} u_i \rightarrow w \equiv 0$. Since any positive solution of\(^{(10)}\) with $m_1 = m_{11}$ is close to zero when $m_{11} \rightarrow \hat{\lambda}_1$, $\beta_{\mathbb{H}} u_i$ is certainly close to positive solutions of\(^{(10)}\) with $m_1 = m_{11}$.

(ii) $m_1 > \hat{\lambda}_1$. In this case, it suffices to show that $w$ is a positive solution of\(^{(10)}\). If $w \equiv 0$, set $\tilde{w}_1 = w/\|w\|_\infty$. Then

$$\tilde{w}_{1xx} + m_{11}\tilde{w}_1 f(z - u - v, \tilde{w}) = 0, \quad x \in (0, 1), \quad \tilde{w}_x(0) = \tilde{w}_x(1) + \gamma \tilde{w}(1) = 0.$$ 

(16)
We may assume \( \tilde{w}_i \to \tilde{w} \) in \( C^1 \). By passing to the limit in (16), we obtain
\[
\tilde{w}_{xx} + m_1 \bar{w}f(z - \Theta, 0) = 0, \quad x \in (0, 1), \quad \tilde{w}_x(0) = \tilde{w}_x(1) + \gamma \tilde{w}(1) = 0.
\]
Since \( \tilde{w} \geq 0, \neq 0 \), we have \( \tilde{w} > 0 \) by the Harnack inequality. It follows that \( m_1 = \hat{\lambda}_1 \), but this is a contradiction. Thus \( w \geq 0, \neq 0 \), which implies \( w > 0 \) by the Harnack inequality. That is, \( \beta_i \mu_i \) converges to the unique positive solution \( w^* \) of (10). This completes the proof. \( \square \)

**Proof of Theorem 1.2.** (i) Suppose that the conclusion is not true. Then there exist \( \varepsilon_0 > 0, \beta_i \to \infty, m_i \to m_1 \in [\hat{\lambda}_1, \hat{\lambda}_1 - \varepsilon_0] \) such that \( (u_i, v_i) \) is a positive solution of (3) as \( (m_1, \beta_i) = (m_1, \beta_i) \). It follows from Theorem 3.2 that \( u_i \to 0, v_i \to \Theta \). We obtain that \( \beta_i \|u_i\|_{\infty} \) is uniformly bounded as in the proof of Theorem 3.2. Let \( u_i = \beta_i u_i \). Then \( u_i \) satisfies (15). By virtue of the standard regularity theory, we may assume \( w_i \to w \). Then \( w \) is a nonnegative solution of (10). Since \( m_1 \in [\hat{\lambda}_1, \hat{\lambda}_1 - \varepsilon_0] \), it follows from Theorem 3.1 that \( w \equiv 0 \). By the same way as the proof of the case \( m_1 > \hat{\lambda}_1 \) in Theorem 3.2, we get a contradiction.

(ii) We first show that any positive solution of (3) is non-degenerate and linearly stable in the condition of (ii). Suppose \((u, v)\) is a positive solution of (3), set \( \bar{w} = \beta \mu, \mu = 1/\beta \) and consider
\[
\begin{align*}
\bar{u}_x + m_1 \bar{w}f(z - \mu \bar{u} - v, \hat{u}) &= 0, \quad x \in (0, 1), \\
v_{xx} + m_2 \bar{w}g(z - \mu \bar{u} - v, \hat{u}) &= 0, \quad x \in (0, 1), \\
\bar{u}(0) &= \bar{u}(1) + \gamma \bar{u}(1) = 0, \\
v(0) &= v(1) + \gamma v(1) = 0.
\end{align*}
\tag{17}
\]

Clearly, \((u, v)\) solves (3) if and only if \((\beta \mu, u, v)\) solves (17) with \( \mu = 1/\beta \). It suffices to prove the non-degeneracy and stability of (17).

Suppose the conclusion is not true. Then we can find some \( \varepsilon_0 > \hat{\lambda}_1, m_i \to m_1 \in [\hat{\lambda}_1, \varepsilon_0], \beta_i \to \infty, \Re \eta_i \leq 0 \) and \( (h_i, k_i) \) smooth with \( \|h_i\|_{2} + \|k_i\|_{2} = 1 \) \((\|h\|_{2} \) denotes \( \|h\|_{2} \)) such that
\[
\begin{align*}
h_{ix} + m_1 \bar{w}f(z - \mu \hat{u} - v_i, \hat{u}_i) &= -\mu \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) + \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i \\
&\quad - m_1 \bar{w} f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i, \quad x \in (0, 1), \\
k_{ix} + m_2 \bar{w} g(z - \mu \hat{u} - v_i, \hat{u}_i) &= -m_2 \mu \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) + m_2 \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i \\
&\quad - m_2 \mu \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i, \quad x \in (0, 1), \\
\hat{u}_i(0) &= \hat{u}(1) + \gamma \hat{u}(1) = 0, \\
k_i(0) &= k_i(1) + \gamma k_i(1) = 0,
\end{align*}
\tag{18}
\]

where \((\hat{u}_i, v_i)\) is a positive solution to (17) with \((m_1, \mu) = (m_1, 1/\beta_i)\).

From (18), it follows that
\[
\begin{align*}
\int_0^1 |h_i|^2 &= -\gamma |h_i(1)|^2 + \int_0^1 m_1 \bar{w}f(z - \mu \hat{u} - v_i, \hat{u}_i) - \mu \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i \\
&\quad + \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i + \int_0^1 \eta_i |h_i|^2, \\
\int_0^1 |k_i|^2 &= -\gamma |k_i(1)|^2 + \int_0^1 m_2 \bar{w}g(z - \mu \hat{u} - v_i, \hat{u}_i) - \mu \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i \\
&\quad + \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i + \int_0^1 \eta_i |k_i|^2.
\end{align*}
\]

Adding the above two identities, we obtain
\[
\eta_i = \int_0^1 |h_i|^2 + \gamma |h_i(1)|^2 - \int_0^1 m_1 \bar{w}f(z - \mu \hat{u} - v_i, \hat{u}_i) - \mu \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i \\
&\quad + \hat{u} \bar{w}_i f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i + \int_0^1 m_1 \bar{w} f(z - \mu \hat{u} - v_i, \hat{u}_i) h_i k_i \\
&\quad + \int_0^1 |k_i|^2 + \gamma |k_i(1)|^2 - \int_0^1 m_2 \bar{w} g(z - \mu \hat{u} - v_i, \hat{u}_i) - \mu \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i \\
&\quad + \hat{u} \bar{w}_i g(z - \mu \hat{u} - v_i, \hat{u}_i) k_i + \int_0^1 m_2 \bar{w} g(z - \mu \hat{u} - v_i, \hat{u}_i) h_i k_i.
\]

Since the boundedness of \( \hat{u}_i, v_i, m_i, \mu_i \), it is easy to see that the imaginary part of the right hand side of the above identity is bounded. Hence \( \Im \eta_i \) is bounded. On the other hand, we can get \( \Re \eta_i \) is bounded from below by the equation of \( \eta_i \). Thus \( \eta_i \) is bounded as the assume \( \Re \eta_i \leq 0 \). By \( L^p \) estimates, \( \|h_i\|_{W^{2,2}} \), \( \|k_i\|_{W^{2,2}} \) are bounded. Hence we may assume \( h_i \to h, k_i \to k \).
in $H_0^1$ strongly. Since $\mu_i \hat{u}_i \to 0$, $\hat{u}_i \to w^*$, $v_i \to \Theta$ as $i \to \infty$, by letting $i \to \infty$ in (18), we see that $h$, $k$ satisfy the following equations weakly (then strongly):

$$
\begin{aligned}
&h_{\alpha} + m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*) h - m_1 w^* \tilde{f}_1(\tilde{z}-\Theta, w^*) k + \eta h = 0, \quad x \in (0, 1), \\
&k_{\alpha} + m_2 g_1(\tilde{z}-\Theta, \Theta) - \Theta g_2(\tilde{z}-\Theta, \Theta) + \Theta g_2(\tilde{z}-\Theta, \Theta) k + \eta k = 0, \quad x \in (0, 1), \\
&h_0 = h_1(1) + \gamma h(1) = 0, \\
k_0 = k_1(1) + \gamma k(1) = 0.
\end{aligned}
$$

(19)

If $k \equiv 0$, then $h$ satisfies

$$
\begin{aligned}
&h_{\alpha} + m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*) h + \eta h = 0, \quad x \in (0, 1), \\
h_0 = h_1(1) + \gamma h(1) = 0.
\end{aligned}
$$

The eigenvalue problem is just (12), thus $\eta$ is real and $\eta > 0$ from the proof of Theorem 3.1. But this is a contradiction.

If $k \not\equiv 0$, then the second equation of (19) implies that $\eta$ is real and

$$
\begin{aligned}
&h_{\alpha} + m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*) h + \eta h = m_1 w^* \tilde{f}_1(\tilde{z}-\Theta, w^*) k, \quad x \in (0, 1), \\
h_0 = h_1(1) + \gamma h(1) = 0.
\end{aligned}
$$

Since the assumption $\eta \leq 0$ and

$$
\lambda_1(-m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*)], 1) > 0,
$$

we have

$$
\lambda_1(-m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*)] - \eta, 1) > 0
$$

from Lemma 2.3. Therefore the operator

$$
d^2 \frac{d^2}{dx^2} + m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*)] + \eta
$$

is invertible and

$$
k = \left\{ d^2 \frac{d^2}{dx^2} + m_1 \tilde{f}(\tilde{z}-\Theta, w^*) + w^* \tilde{f}_2(\tilde{z}-\Theta, w^*)] + \eta \right\}^{-1} (m_1 w^* \tilde{f}_1(\tilde{z}-\Theta, w^*)].
$$

On the other hand, from the equation of $k$ in (19), and since all eigenvalues of $L_2$ are strictly negative from Remark 1.1, we get $\eta > 0$, which contradicts the assumption.

In the following, we establish the existence of a positive solution for $m_2 > \sigma_1$.

Since all positive solutions of (3) are non-degenerate, it follows from a simple compactness argument that there are at most finitely many positive solutions. If (3) has a positive solution denoted by $(u, v)$, then one can easily show that $\text{index}_W(A, (u, v)) = 1$ using Lemma 2.1 and the non-degeneracy and stability of $(u, v)$, where the operator $A$ has been given in Lemma 2.6.

(1) Case $\sigma_1 < m_2 < \hat{\sigma}_1$. For $m_1 \in (\hat{\lambda}_1, \sigma_1)$, suppose (3) has positive solutions $\{(u_i, v_i) : 1 \leq i \leq l\}$, where $l \geq 1$. It follows from Lemma 2.6 that $\text{index}_W(A, (0, 0)) = \text{index}_W(A, (0, \Theta)) = 0$, $\text{index}_W(A, (\theta, 0)) = 1$. Hence by the additivity of the fixed point index, we have

$$1 = \text{index}_W(A, D') = 1 + \sum_{i=1}^{l} \text{index}_W(A, (u_i, v_i)) = 1 + l \Rightarrow l = 0,$

which contradicts the supposition $l \geq 1$.

(2) Case $m_2 > \hat{\sigma}_1$. First, for $m_1 \in (\hat{\lambda}_1, \sigma_1)$, we know from Theorem 1.1 that (3) at least has a positive solution. Suppose the positive solutions of (3) be $\{(u_i, v_i) : 1 \leq i \leq l\}$, where $l \geq 1$. From Lemma 2.6 and the additivity of the fixed point index again, we have

$$1 = \text{index}_W(A, D') = 0 + \sum_{i=1}^{l} \text{index}_W(A, (u_i, v_i)) = l,$

which shows uniqueness in this case.

Finally, we consider $m_1 = \hat{\lambda}_1$.

By similar arguments as paper [16], for (17), we regard $m_1$ as the bifurcation parameter and can construct a positive solution branch from the semitrivial nonnegative solution branch $\{\hat{\lambda}_1, 0, \Theta\}$. Denote the positive bifurcation solution curve by $\gamma^*_s = \{(m_1(s), u(s), v(s)) = (m_1(s), s\hat{\phi}(s), \Theta - s\chi(s)), 0 < s \ll 1\}$, where $m_1(0) = \hat{\lambda}_1$, $\phi(0) = 0$, $\psi(0) = 0$, $\chi_1 = -L_2^{-1}(m_2 \mu \hat{\phi}_1(\zeta-\Theta, \Theta) \hat{\phi}_1) > 0$, and $L_2$ is given by Remark 1.1. Putting this positive
solution into the first equation of (17), dividing by \( s \) and differentiating with respect to \( s \), we can obtain that the derivative of \( m_1(s) \) at \( s = 0 \)

\[
m'_1(0) \left( \int_0^1 f(z - \theta, 0) \phi_1^2 \, dx \right) = \lambda_1 \int_0^1 \left[ (\mu \hat{\phi}_1 - \chi_1) \tilde{f}_1(z - \theta, 0) - \tilde{f}_2(z - \theta, 0) \hat{\phi}_1 \right] \phi_1^2 \, dx.
\]

As \( \beta_1 \to \infty \), i.e. \( \mu \to 0 \), we have

\[
m'_1(0) \to - \frac{\lambda_1 \int_0^1 \tilde{f}_2(z - \theta, 0) \phi_1^2 \, dx}{\int_0^1 f(z - \theta, 0) \phi_1^2 \, dx} > 0,
\]

since \( \chi_1 \to 0, \tilde{f}_2(z - \theta, 0) < 0 \), which implies the positive solution bifurcation branch is to the right. Furthermore, from similar arguments as paper [7], we can show that the positive bifurcation solution curve \( \Gamma^*_\mu \) joins with the semitrivial branch \((m_1, \beta_1 \theta) = 0 \) at \( m_1 = \hat{m}_1 \) which is given uniquely by \( m_2 = \lambda_1(0, g(z - \theta(\hat{m}_1), 0)) = \sigma_1(\theta(\hat{m}_1)) \). From the above conclusions it follows that for \( m_1 \in (\hat{\lambda}_1, \infty) \), \( \beta_1 \to \infty \), the unique positive solution is exactly on \( \Gamma^*_\mu \) and no positive solution curve can cover \( m_1 = \hat{\lambda}_1 \). Hence when \( m_1 = \hat{\lambda}_1 \), there is no positive solution of (3). The proof is complete. \( \square \)

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