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WHITNEY'S THEOREM FOR INFINITE GRAPHS

A.R. BEDNAREK
University of Florida, Gainesville, FL 32601, U.S.A.

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Whitney's theorem [4] asserts that any edge isomorphism of a finite connected graph of cardinality greater than four is induced by a vertex isomorphism. In his monograph on graphs [2], Ore proposed the extension of this theorem to infinite graphs. Needing this extension in connection with other investigations and unable to locate such in the literature, we came up with the argument that follows. If not the first proof of the extension, it may be of interest as yet another application of the powerful selection principle of Rado [3].

By a *graph* $G=(V, \mathscr{E})$ we mean a set V (*vertices*) and a collection $\mathscr{E}$ (*edges*) of subsets of V of cardinality two. Two graphs, $G=(V, \mathscr{E})$ and $G'=(V', \mathscr{E}')$, are (vertex) *isomorphic* if there is a bijection between their vertex sets that preserves adjacency. Similarly, the graphs are said to be *edge isomorphic* if there is a bijection between the sets of edges that preserves incidence. While vertex isomorphism induces an edge isomorphism, the converse need not be true. We state Whitney's theorem as follows.

Theorem 1 (Whitney). *If G is a finite connected graph with* card $V \geq 5$, *then any edge isomorphism of G is induced by a vertex isomorphism.*

As noted above, our extension argument will employ the following selection principle.

Theorem 2 (Rado). *Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a family of non-empty finite sets. For each non-empty finite subset A of Λ let there be given a (local) choice function $\phi_A : A \to U\{X_\lambda \mid \lambda \in A\}$. Then there exists a (global) choice function ϕ on Λ such that for any finite subset A of Λ there is a finite subset B of Λ satisfying* (i) $A \subset B$ *and* (ii) $\phi \mid A = \phi_B \mid A$.

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Mirsky [1] provides an extensive list of applications of this selection principle. Many of these applications, including the one that follows, require a slightly altered form of this theorem. Namely, rather than hypothesizing the existence of the local choice functions for all the non-empty finite subsets of Λ, one need only have a subfamily $\mathscr{D}$ with the property that if $D_1, D_2 \in \mathscr{D}$, then there exists a $D_3 \in \mathscr{D}$ such that $D_1 \cup D_2 \subset D_3$.

We now state and prove the extension.

Theorem 3. *Let $G = (V, \mathscr{E})$ and $G' = (V', \mathscr{E}')$ be infinite connected graphs and let $\theta: \mathscr{E} \to \mathscr{E}'$ be an edge isomorphism. Then θ is induced by a vertex isomorphism.*

Proof. Let $\mathscr{F}$ be a non-empty finite subset of $\mathscr{E}$ and let $V(\mathscr{F}) = \bigcup \{E \mid E \in \mathscr{F}\}$. Denote by $\mathscr{R}$ the family of non-empty finite subsets $\mathscr{F}$ of $\mathscr{E}$ satisfying (i) card $V(\mathscr{F}) \geq 5$ and (ii) the graph $G = (V(\mathscr{F}), \mathscr{F})$ is connected. It is clear that the family $\mathscr{R}$ has the property described following Theorem 2.

We note that $\theta \mid \mathscr{F}$ is an edge isomorphism of G and therefore, by Theorem 1, is induced by a vertex isomorphism $\Psi_{\mathscr{F}}$ on $V(\mathscr{F})$.

To apply Theorem 2, or rather the altered form noted following its statement, we let $\Lambda = \mathscr{E}$ and for each $\lambda \in \Lambda$, where $\lambda = \{a, b\}$, we let $X_\lambda = \{\{(a, a'), (b, b')\}, \{(a, b'), (b, a')\}\}$ where $\{a', b'\} = \theta(\lambda)$; that is $\{a', b'\}$ is the edge in G' corresponding to $\{a, b\} \in \mathscr{E}$.

Now for each $\mathscr{A} \in \mathscr{R}$ we let $\phi_{\mathscr{A}}$ be the choice function $\phi_{\mathscr{A}}: \mathscr{A} \to \bigcup \{X_\lambda \mid \lambda \in \mathscr{A}\}$ such that $\bigcup \{\phi(\lambda) \mid \lambda \in \mathscr{A}\} = \Psi_{\mathscr{A}}$. This is possible because of Theorem 1 and the fact that for finite connected graphs the vertex isomorphism $\Psi_{\mathscr{A}}$ that induces the edge isomorphism must satisfy $\{a, b\} \in \mathscr{E}$ if and only if $\{a', b'\} = \{\Psi_{\mathscr{A}}(a), \Psi_{\mathscr{A}}(b)\} \in \mathscr{E}'$. Let $\phi: \mathscr{E} \to \bigcup \{X_\lambda \mid \lambda \in \mathscr{E}\}$ be the global choice function given by Theorem 2.

We let $\Psi = \bigcup \{\phi(\lambda) \mid \lambda \in \mathscr{E}\}$ and claim that Ψ is a vertex isomorphism that induces θ.

In view of the connectivity of G and G', it is immediate that the domain of the relation Ψ is V and that its range is V'. Moreover, Ψ is a function since if (t, x), $(t, y) \in \Psi$ there are edges $\lambda_1, \lambda_2 \in \mathscr{E}$ such that $(t, x) \in \Psi(\lambda_1)$ and $(t, y) \in \phi(\lambda_2)$. There is an $\mathscr{A} \in \mathscr{R}$ such that $\{\lambda_1, \lambda_2\} \subset \mathscr{A}$. By Theorem 2 there is then a $\mathscr{B} \in \mathscr{R}$ such that $\mathscr{A} \subset \mathscr{B}$ and $\phi \mid \mathscr{A} = \phi_{\mathscr{B}} \mid \mathscr{A}$. Therefore $(t, x) \in \phi_{\mathscr{B}}(\lambda_1)$ and $(t, y) \in \phi_{\mathscr{B}}(\lambda_2)$ so that $(t, x), (t, y) \in \Psi_{\mathscr{B}}$ which is a function on $V(\mathscr{B})$. Consequently $x = y$, and thus Ψ is a function. In exactly the same way (see [1], p. 55) it is possible to show that if all of the local choice functions are injective, as is the case here, then so is the global choice function.

If $\{a, b\} = \lambda \in \mathscr{E}$, then $\phi(\lambda) = \{(a, a'), (b, b')\}$ or $\phi(\lambda) = \{(a, b'), (b, a')\}$. In either case $\{a', b'\} = \{\Psi(a), \Psi(b)\}$ so that Ψ induces the original edge isomorphism θ. $\square$

References

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