Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 322 (2006) 523–529
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Perturbations of eventually differentiable and eventually norm-continuous semigroups on Banach spaces

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Received 16 February 2005

Available online 21 October 2005

Submitted by J.A. Goldstein

Abstract

In this paper we discuss perturbations of eventually differentiable and eventually norm-continuous semigroups on a Banach space. Two kinds of new perturbation theorems are obtained.

Keywords: C_0 semigroup; Eventually differentiable; Eventually norm-continuous; Yosida approximation; Perturbation

1. Introduction

A C_0 semigroup $T(t)$ on a Banach space X is called eventually differentiable if there is a $t_0 \geqslant 0$ such that for every $x \in X$, $t \to T(t)x$ is differentiable for $t > t_0$. A C_0 semigroup $T(t)$ on a Banach space X is called eventually norm continuous if there is a $t_0 \geqslant 0$ such that $T(t)$ is continuous for $t > t_0$ in the uniform operator topology.

It is well known that if $T(t)$ is a C_0 semigroup with its generator A on a Banach space X and B is a bounded linear operator, the semigroup $S(t)$ generated by $A + B$ is a C_0 semigroup. But A. Pazy pointed out in [1] (1983) that "not all the properties of the semigroup $T(t)$ are preserved by a bounded perturbation on its infinitesimal generator" and that "if A is the infinitesimal generator of a semigroup $T(t)$ which is continuous in the uniform operator topology for $t \geqslant t_0 > 0$, or is differentiable for $t \geqslant t_0 > 0$ or is compact for $t \geqslant t_0 > 0$ then $S(t)$, the semigroup generated by $A + B$ where B is a bounded operator need not have the corresponding property." Q. Zheng said in [2] (1994) that "for a differentiable semigroup with the generator A whether the C_0 semi-

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.09.045

group $T_B(t)$ generated by $A+B$ is differentiable is not known so far, even if B is a bounded linear operator." In the past decade, the two perturbation problems of eventually differentiable and eventually norm continuous semigroups have got some development.

With respect to the perturbations of eventually differentiable semigroups, in 1995 M. Renardy [3] gave an example which shows that the semigroup $S(t)$ generated by $A+B$ is not differentiable, where A generates a differentiable semigroup $T(t)$ and B is a bounded linear operator. In 1997, B.D. Doytchinov et al. [4] showed that $S(t)$ is eventually differentiable if $\|AT(t)\|$ satisfies a certain condition for small t. In 1998, R. Nagel et al. [5] discussed the problem further by means of the concept of convolution.

With respect to the perturbations of eventually norm continuous semigroups, Ref. [6] gave a theorem which said that if $T(t)$ is eventually norm continuous with its generator A in a Banach space X and B is a compact operator, then the semigroup $S(t)$ generated by $A+B$ is eventually norm continuous. Ref. [2] had some discussion on norm continuous semigroups for $t>0$ on a Banach space. Recently, G.Q. Xu [7] and L.P. Zhang [8] gave some new perturbation theorems in Hilbert spaces, respectively.

In this paper, we discuss further the perturbations of the two semigroups. First, in Section 2 we recall the characterizations (Theorems 1 and 2) of the two semigroups by means of their generators' Yosida approximations and give some lemmas. Secondly, in Sections 3 and 4, using the above characterizations we obtain the perturbation theorems which are main results of the paper. Finally, we give the proof of the characterization theorems in Section 2 as in Appendix A to the paper.

2. Preliminaries

Let X be a Banach space and $T(t)$ be a C_0 semigroup on it with the generator A and $\|T(t)\| \leqslant Me^{\omega t}$, $\omega > 0$. The operator

$$A_\lambda =: \lambda AR(\lambda; A) = \lambda^2 R(\lambda; A) - \lambda I, \quad \lambda > \omega,$$

is the Yosida approximation of A, where $R(\lambda; A)$ is the resolvent operator of A at $\lambda \in \rho(A)$, the resolvent set of A. The Yosida approximation of A has properties

$$\lim_{\lambda \to +\infty} A_\lambda x = Ax, \quad x \in D(A), \tag{1}$$

where $D(A)$ is the domain of A, and

$$\lim_{\lambda \to +\infty} \lambda R(\lambda; A)x = x, \quad x \in X. \tag{2}$$

Theorem 1. [9] *Let $T(t)$ be a C_0 semigroup on a Banach space X with its infinitesimal generator A and $\|T(t)\| \leqslant Me^{\omega t}$ $(\omega > 0)$. Then $T(t)$ is differentiable for $t \geqslant t_0 > 0$ if and only if there is a constant M_1 such that*

$$\big\|A_\lambda T(t_0)\big\| \leqslant M_1, \quad \text{for } \lambda \geqslant \omega + 1, \tag{3}$$

where A_λ is the Yosida approximation of A.

Theorem 2. [9] *Let $T(t)$ be a C_0 semigroup on a Banach space X with its generator A and $\|T(t)\| \leqslant Me^{\omega t}$ $(\omega > 0)$. Then $T(t)$ is norm continuous for $t \geqslant t_0 > 0$ if and only if*

$$\lim_{\lambda \to +\infty} \frac{\|A_\lambda T(t_0)\|}{\lambda} = 0. \tag{4}$$

To obtain our main results, we need the following lemmas.

Lemma 3. *Let $T(t)$ be a C_0 semigroup on a Banach space X with its generator A and $\|T(t)\| \leqslant Me^{\omega t}$ $(\omega > 0)$, and B be a bounded linear operator on X commuting with $T(t)$. Then the C_0 semigroup $S(t)$ generated by $A + B$ has the expression*

$$S(t) = T(t)e^{Bt}$$

satisfying $\|S(t)\| \leqslant Me^{(\omega+M\|B\|)t}$ and the resolvent of $A + B$ has the expression

$$R(\lambda; A + B)S(t_0) = R(\lambda; A)T(t_0)e^{Bt_0} \sum_{n=0}^{\infty} \big[BR(\lambda; A)\big]^n, \quad \operatorname{Re}\lambda > \omega + M\|B\|.$$

Proof. We know that (see [1]) the semigroup $S(t)$ is determined via the integral equation

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\,ds, \quad x \in X.$$

Set $S_0(t) = T(t)$,

$$S_{n+1}(t) = \int_0^t T(t-s)BS_n(s)x\,ds, \quad x \in X,\ n = 0, 1, 2, \ldots.$$

When $T(t)$ satisfies $\|T(t)\| \leqslant Me^{\omega t}$ and $\operatorname{Re}\lambda > \omega + M\|B\|$,

$$S(t) = \sum_{n=0}^{\infty} S_n(t),$$

$$R(\lambda; A + B) = \sum_{n=0}^{\infty} R(\lambda; A)\big[BR(\lambda; A)\big]^n, \tag{5}$$

where the convergence in the above two series is in the uniform operator topology. In view of the commutation of B and $T(t)$, by the induction, we have

$$S_n(t) = T(t)\frac{t^n B^n}{n!}, \quad n = 0, 1, 2, \ldots.$$

Thus

$$S(t) = \sum_{n=0}^{\infty} T(t)\frac{(tB)^n}{n!} = T(t)e^{Bt}, \quad t \geqslant 0,$$

and $\|S(t)\| \leqslant Me^{(\omega+M\|B\|)t}$ clearly. Note that e^{Bt} commutes with $T(t)$ and $R(\lambda; A)$. So

$$R(\lambda; A + B)S(t_0) = R(\lambda; A)T(t_0)e^{Bt_0} \sum_{n=0}^{\infty} \big[BR(\lambda; A)\big]^n.$$

The proof is complete. □

Lemma 4. *Let the hypotheses of Lemma 3 hold. Then the Yosida approximation $(A + B)_\lambda$ of $A + B$ has property*

$$(A+B)_\lambda S(t_0) = A_\lambda T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n + B\lambda R(\lambda;A)T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n,$$
$$\lambda > \omega + M\|B\|. \tag{6}$$

Proof. From the definition of the Yosida approximation, Lemma 3 and (5), and noting the commutation of e^{Bt} and $T(t)$, $R(\lambda; A)$,

$$\begin{aligned}(A+B)_\lambda S(t_0) &= \lambda(A+B)R(\lambda;A+B)T(t_0)e^{Bt_0}\\ &= \lambda AR(\lambda;A)T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n\\ &\quad + \lambda BR(\lambda;A)T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n\\ &= A_\lambda T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n + B\lambda R(\lambda;A)T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\left[BR(\lambda;A)\right]^n,\end{aligned}$$

$\lambda > \omega + M\|B\|$. The proof is complete. □

Lemma 5. [2] *Let A be a densely defined linear operator on a Banach space X. If there is an $M \geqslant 1$ and an $\omega \in \mathbf{R}$ such that $(\omega, +\infty) \subset \rho(A)$ (the resolvent set of A) and if there is a strongly continuous family of operators $T(t)$ with $\|T(t)\| \leqslant Me^{\omega t}$ $(t \geqslant 0)$ such that $R(\lambda; A)x$ $(\lambda > \omega)$ is the Laplace transform of $T(t)x$ for every $x \in X$, $T(t)$ is a C_0 semigroup generated by A.*

3. Perturbation theorems for eventually differentiable semigroups

Theorem 6. *Let the following conditions hold*:

(i) *$T(t)$ is a C_0 semigroup on a Banach space X with its generator A and $\|T(t)\| \leqslant Me^{\omega t}$ $(\omega > 0)$.*
(ii) *$T(t)$ is differentiable for $t \geqslant t_0 > 0$.*
(iii) *B is a bounded linear operator on X commuting with $T(t)$.*

Then the C_0 semigroup $S(t)$ generated by $A + B$ is differentiable for $t \geqslant t_0$.

Proof. Because $T(t)$ is differentiable for $t \geqslant t_0 > 0$, from Theorem 1, (3) holds. Note that $\sum_{n=0}^{\infty}[BR(\lambda; A)]^n$ is bounded when $\lambda \geqslant \omega + M\|B\| + 1$ and $\lambda R(\lambda; A)$ is also bounded from (2). From (6), there is a constant M_2 such that

$$\left\|(A+B)_\lambda S(t_0)\right\| \leqslant M_2, \quad \lambda \geqslant \omega + M\|B\| + 1.$$

So $S(t)$ is differentiable for $t \geqslant t_0$ by Theorem 1. The proof is complete. □

Remark. Specially, if the operator B in Theorem 6 is $T(t_1)$ $(t_1 \geqslant 0)$ or $R(\lambda_1; A)$ $(\lambda_1 \in \rho(A))$, the conclusion of Theorem 6 holds.

Example. Let $X = \{f \mid f \in C[0,1],\ f(1) = 0\}$ with the sup-norm. On the Banach space X, we define for $t \geqslant 0$

$$\big(T(t)f\big)(s) = \begin{cases} f(s+t), & \text{for } s+t \leqslant 1, \\ 0, & \text{for } s+t > 1. \end{cases}$$

$T(t)$ is a strongly continuous contraction semigroup and differentiable for $t > 1$. Its infinitesimal generator is A with $D(A) = \{f \mid f, f' \in X\}$ and for $f \in D(A)$, $(Af)(s) = f'(s)$. Take $(Bg)(s) = \int_1^s g(\tau)\,d\tau$ for any $g \in X$. Then B is a bounded linear operator commuting with $T(t)$. From Theorem 6, $A + B$ generates a C_0 semigroup which is differentiable for $t > 1$.

Theorem 7. *Let the following conditions hold*:

(i) *$T(t)$ is a C_0 semigroup on a Banach space X with its generator A and $\|T(t)\| \leqslant Me^{\omega t}$ $(\omega > 0)$.*
(ii) *$T(t)$ is differentiable for $t \geqslant t_0 > 0$.*
(iii) *A linear operator B in X is relatively bounded with respect to A, i.e., $D(A) \subset D(B)$ and*

$$\|Bx\| \leqslant a\|Ax\| + b\|x\|, \quad \forall x \in D(A), \tag{7}$$

where a and b are nonnegative constants.
(iv) *B has properties $B(D(A)) \subset D(A)$ and $T(t)B \subset BT(t)$, where $B(D(A)) = \{y \mid y = Bx, x \in D(A)\}$.*

Then the C_0 semigroup $S(t)$ generated by $A + B$ is differentiable for $t \geqslant t_0$.

Proof. Take $\lambda_0 \in \rho(A)$, the resolvent set of A. From (7),

$$\big\|BR(\lambda_0; A)x\big\| \leqslant a\big\|AR(\lambda_0; A)x\big\| + b\big\|R(\lambda_0; A)x\big\|, \quad \forall x \in X.$$

Since $\lambda_0 - A$ is a closed operator, the linear operator $S =: (\lambda_0 - A)BR(\lambda_0; A)$ is bounded. Whence, $1 \in \rho(SR(\mu; A))$ as $\mu > \omega + M\|S\|$. Because for any bounded linear operators T_1, T_2,

$$\rho(T_1T_2) - \{0\} = \rho(T_2T_1) - \{0\}$$

(see [11, Problem 8, p. 281]), $1 \in \rho(BR(\mu; A))$. Set $U = I - BR(\mu; A)$. Obviously, $U(D(A)) \subset D(A)$. For any $y \in D(A) \subset R(U) = X$, where $R(U)$ is the range of U, there is an $x \in X$ such that $y = Ux = x - BR(\mu; A)x$. It yields that $x \in D(A)$ and $y = Ux \in U(D(A))$. That is $D(A) \subset U(D(A))$. So we have $U(D(A)) = D(A)$. Since $C =: (\mu - A)BR(\mu; A)$ is a bounded linear operator on X and commutes with $T(t)$, the C_0 semigroup $T_C(t)$ generated by $A + C$ is differentiable for $t \geqslant t_0 > 0$ by Theorem 6 and $\|T_C(t)\| \leqslant Me^{(\omega + M\|C\|)t}$. Note that on $D(A)$

$$\begin{aligned} &U(A+C)U^{-1} \\ &\quad = \big[I - BR(\mu; A)\big]\big[A + (\mu - A)BR(\mu; A)\big]U^{-1} \\ &\quad = \big[A + \mu BR(\mu; A) - ABR(\mu; A) - BR(\mu; A)A - B^2R(\mu; A)\big]U^{-1} \\ &\quad = \big\{A\big[I - BR(\mu; A)\big] + B\big[R(\mu; A)\mu - R(\mu; A)A - BR(\mu; A)\big]\big\}U^{-1} \\ &\quad = \{AU + BU\}U^{-1} = A + B. \end{aligned} \tag{8}$$

Thus, for $\lambda > \omega + M\|C\|$ and any $x \in X$

$$R(\lambda; A+B)x = UR(\lambda; A+C)U^{-1}x = \int_0^{+\infty} e^{-\lambda t}UT_C(t)U^{-1}x\,dt. \tag{9}$$

From Lemma 5, $A+B$ generates a C_0 semigroup

$$S(t)=UT_C(t)U^{-1}. \tag{10}$$

The Yosida approximation $(A+B)_\lambda$ of $A+B$ has the following expression by (8)–(10)

$$(A+B)_\lambda S(t_0)=\lambda(A+B)R(\lambda;A+B)S(t_0)=U\lambda(A+C)R(\lambda;A+C)T_C(t_0)U^{-1}.$$

From Theorem 1, $\lambda(A+C)R(\lambda;A+C)T_C(t_0)=(A+C)_\lambda T_C(t_0)$ is bounded for $\lambda\geqslant\omega+M\|C\|+1$. Therefore, for $\lambda\geqslant\omega+M\|C\|+1$, $(A+B)_\lambda S(t_0)$ is bounded and $S(t)$ is differentiable for $t\geqslant t_0$ by Theorem 1. The proof is complete. □

4. Perturbation theorems for eventually norm continuous semigroups

Theorem 8. *Let a C_0 semigroup $T(t)$ on a Banach space X be norm continuous for $t\geqslant t_0>0$ with its generator A and $\|T(t)\|\leqslant Me^{\omega t}$ $(\omega>0)$, and B be a bounded linear operator commuting with $T(t)$. Then the C_0 semigroup $S(t)$ generated by $A+B$ is norm continuous for $t\geqslant t_0$.*

Proof. From (6) in Lemma 4, for $\lambda>\omega+M\|B\|$,

$$\frac{1}{\lambda}(A+B)_\lambda S(t_0)=\frac{1}{\lambda}A_\lambda T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\big[BR(\lambda;A)\big]^n + BR(\lambda;A)T(t_0)e^{Bt_0}\sum_{n=0}^{\infty}\big[BR(\lambda;A)\big]^n.$$

Because (4) holds and $\lim_{\lambda\to+\infty}\|R(\lambda;A)\|=0$,

$$\lim_{\lambda\to+\infty}\frac{\|(A+B)_\lambda S(t_0)\|}{\lambda}=0.$$

Consequently, $S(t)$ is norm continuous for $t\geqslant t_0$ by Theorem 2. The proof is complete. □

Theorem 9. *Let a C_0 semigroup $T(t)$ on a Banach space X be norm continuous for $t\geqslant t_0>0$ with its generator A and $\|T(t)\|\leqslant Me^{\omega t}$ $(\omega>0)$, and B satisfy the hypotheses in Theorem 7. Then the C_0 semigroup $S(t)$ generated by $A+B$ is norm continuous for $t\geqslant t_0$.*

Proof. Similar to the proof of Theorem 7, the C_0 semigroup $T_C(t)$ generated by $A+C$ is norm continuous for $t\geqslant t_0$ by Theorem 8 and $A+B$ generates a C_0 semigroup $S(t)$ satisfying (8)–(10). It yields that

$$\frac{1}{\lambda}(A+B)_\lambda S(t_0)=U\frac{1}{\lambda}(A+C)_\lambda T_C(t_0)U^{-1}.$$

The operators C and U in the above statements are same as the proof of Theorem 7. Obviously, from Theorem 2,

$$\lim_{\lambda\to+\infty}\frac{1}{\lambda}\big\|(A+B)_\lambda S(t_0)\big\|=0,$$

and so $S(t)$ is norm continuous for $t\geqslant t_0$. The proof is complete. □

Acknowledgments

The author is very grateful to the Funds for Returned Research Workers of Shanxi Province, China for their support. The author also thanks the referees very much for the valuable revision suggestions.

Appendix A

Proof of Theorem 1. *Necessity*. Assume first that $T(t)$ is differentiable for $t \geqslant t_0 > 0$, then for $x \in X$ we see that $T(t_0)x \in D(A)$. Using (1) yields $\lim_{\lambda\to+\infty} A_\lambda T(t_0)x = AT(t_0)x$ for $x \in X$. Thus, there is a constant $M_x > 0$ such that $\|A_\lambda T(t_0)x\| \leqslant M_x$ for $\lambda \geqslant \omega + 1$. From the resonance theorem it follows that there exists a constant $M_1 > 0$ such that $\|A_\lambda T(t_0)\| \leqslant M_1$ for $\lambda \geqslant \omega + 1$.

Sufficiency. Now we assume that (3) holds. From (1),

$$\lim_{\lambda\to+\infty} A_\lambda T(t_0)x = AT(t_0)x, \quad x \in D(A). \tag{A.1}$$

Since (3) and $\overline{D(A)} = X$, (A.1) holds for every $x \in X$ and $AT(t_0)$ is a bounded linear operator. So it yields that for every $x \in X$, $T(t_0)x \in D(A)$. From this fact it is readily deduced that $T(t)$ is differentiable for $t \geqslant t_0$. The proof is complete. □

Proof of Theorem 2. Because for any $x \in X$ and $\lambda > \omega$,

$$\frac{1}{\lambda} A_\lambda T(t_0)x = \big(\lambda R(\lambda; A) - I\big)T(t_0)x = \lambda \int_0^{+\infty} e^{-\lambda s}\big(T(s+t_0)x - T(t_0)x\big)\,ds,$$

using the similar method of the proof for (a) ⇔ (e) of [10, Theorem 1] we obtain Theorem 2. The proof is complete. □

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