Toeplitz Operators with PQC Symbols on Weighted Hardy Spaces

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We prove a Fredholm criterion for Toeplitz operators with piecewise quasi-continuous symbols on weighted Hardy spaces, thus uniting part of the Gohberg-Krupnik and Sarason theories. The criterion established solved the problem of describing all the subsets $M$ of $(1, \infty)$ with the following property: there exists a Toeplitz operator which is Fredholm on $H^p$ if and only if $p$ belongs to $M$.

1. INTRODUCTION

Let $H^p$ ($1 \leq p \leq \infty$) denote the Hardy space of all functions in $L^p$ on the complex unit circle $\mathbb{T}$ whose negative Fourier coefficients vanish. We let $P$ stand for the Riesz projection of $L^p$ onto $H^p$ ($1 < p < \infty$), i.e., $P = (I + S)/2$, where $S$ is the Cauchy singular integral operator,

$$
(S\varphi)(t) = \frac{1}{\pi i} \int_{\mathbb{T}} \frac{\varphi(\tau)}{\tau - t} \, d\tau \quad (t \in \mathbb{T}).
$$

The Toeplitz operator $T(a)$ on $H^p$ ($1 < p < \infty$) generated by a function $a \in L^\infty$ is the (obviously bounded) operator acting on $H^p$ by the rule $T(a)\varphi = P(a\varphi)$ ($\varphi \in H^p$). The function $a$ is usually referred to as the symbol of $T(a)$.

Given $a \in L^\infty$, we denote by $\Phi(a)$ the set of all values $p \in (1, \infty)$ for which $T(a)$ is a Fredholm operator on $H^p$. The paper is devoted to the study of the subsets $\Phi(a)$ and their properties. Some results are new and others generalize some classical results.
which $T(a)$ is Fredholm on $H^p$ (that is, for which the image $T(a)H^p$ is closed and both the kernel and cokernel dimensions of $T(a)$ in $H^p$ are finite). We call $\Phi(a)$ the Fredholm domain of $a$. The set $\Phi(a)$ was introduced (under the name factorization domain) and studied in [23]; also see [13]. We clearly have $\Phi(a) = \bigcup \{ \Phi_k(a) : k \in \mathbb{Z} \}$, where $\Phi_k(a)$ denotes the collection of all $p \in (1, \infty)$ such that $T(a)$ is Fredholm of index $k$ on $H^p$ (recall that the index is the difference of the dimensions of the kernel and cokernel of $T(a)$ in $H^p$). It was shown in [23] that each $\Phi_k(a)$ is an open (possibly empty or infinite) interval, that $\Phi_k(a)$ is located on the left of $\Phi_l(a)$ whenever $k > l$, and that 1 and $\infty$ are the only possible accumulation points of the collection of the (centers of the) intervals $\Phi_k(a)$. Note that there is at most one infinite interval constituting a part of $\Phi(a)$: if $\Phi_{k_0}(a) = (p_0, \infty)$ for some $k_0 \in \mathbb{Z}$ and some $p_0 \in (1, \infty)$, then $\Phi_k(a) = \emptyset$ for all $k < k_0$.

The present paper addresses the following "inverse" problem. Given a family $\{ I_k \}_{k \in \mathbb{Z}}$ of open (possibly empty or infinite) subintervals of $(1, \infty)$ such that $I_k$ lies to the left of $I_l$ if $k > l$ and such that 1 and $\infty$ are the only possible condensing points of the family $\{ I_k \}$ (such families will be called admissible), is there a function $a \in L^\infty$ such that $\Phi_k(a) = I_k$ for all $k$?

If $a$ belongs to the Douglas algebra $C + H^\infty$, then either $\Phi(a) = (1, \infty)$ or $\Phi(a) = \emptyset$, depending on whether $a$ is invertible in $C + H^\infty$ or not. In the first case there is a single $k_0$ such that $\Phi(a) = \Phi_{k_0}(a)$.

Let PC denote the C*-algebra of all piecewise continuous functions on $\mathbb{T}$. The Gohberg-Krupnik theory of Toeplitz operators induced by PC symbols (see [9]) implies that if $a \in PC$, then $(1, \infty) \setminus \Phi(a)$ is either all of $(1, \infty)$ or an at most countable set condensing at most at 1 or $\infty$. Vice versa, it is not difficult to see that every set of such a kind is the Fredholm domain of some PC function (cf. [13, p. 209]).

One main result of this paper states that in fact for any admissible family $\{ I_k \}_{k \in \mathbb{Z}}$ there exists a function $a \in L^\infty$ such that $\Phi_k(a) = I_k$ for all $k \in \mathbb{Z}$. Moreover, we show that actually all possible Fredholm domains are produced by the class of PQC functions. Recall that PQC (the C*-algebra of piecewise quasicontinuous functions) is the smallest closed subalgebra of $L^\infty$ containing PC and the C*-algebra QC := $(C + H^\infty) \cap (C + H^\infty)$ of all quasicontinuous functions. The proof of the above result is based upon our second main result: we establish a Fredholm criterion for Toeplitz operators with PQC symbols on $H^p$. The latter result, extending the Gohberg-Krupnik criterion to PQC and Sarason's theorem (see Section 2.12) to the case $p \neq 2$, is of course of independent interest and gives an answer to a question which had been open for a long time.

Our program is as follows. The next section records some (more or less) well-known results on weighted Hardy spaces and Toeplitz operators acting on such spaces. Notwithstanding the excellent expositions of the $H^2$
theory of Toeplitz operators contained, e.g., in [6, 14] and despite the well-known monographs [5, 9], the recent treatises [2, 13] seem to be the only books paying due attention to the advanced $H^p$ theory of Toeplitz operators. This circumstance along with both the necessity and our desire of stating some things (certainly well known to specialists) in an explicit form led us to write Section 2. The Fredholm criterion for Toeplitz operators on $H^p$ with PQC symbols is formulated in Section 3. Note that this criterion is not surprising: it states that something is true which one would expect to be true. The proof, however, is unfortunately rather complicated and occupies the bulk of the rest of the paper. We divided the proof into two parts. The first part, constituting Section 4 and containing also a few results of independent interest, is closed by a lemma which says that the criterion is true in a certain limit case. The second part, making up Section 5, presents a construction for reducing the general case to the aforementioned limit case. In the final Section 6, we use the Fredholm criterion for Toeplitz operators on $H^p$ with PQC symbols in order to produce Toeplitz operators with arbitrary (admissible) Fredholm domains.

2. Preliminaries

2.1. The study of Toeplitz operators on the spaces $H^p$ leads naturally (and not only academically) to the problem of considering these operators on weighted Hardy spaces. Let $m$ denote Lebesgue measure on $\mathbb{T}$, and throughout what follows let $1 < p < \infty$ and $q = p/(p - 1)$. Given a non-negative function $w \in L^p$ which does not vanish identically, we define $L^p(w)$ as the Banach space of all $\varphi \in L^1$ for which

$$
\|\varphi\|_{p,w} := \left( \int_{\mathbb{T}} |\varphi w|^p \, dm \right)^{1/p} < \infty,
$$

and we denote by $H^p(w)$ (resp. $H^p(w)$) the closed subspace of $L^p(w)$ consisting of all functions whose negative (resp. positive) Fourier coefficients vanish.

Let $A_p$ denote the collection of all measurable functions $w$ on $\mathbb{T}$ obeying the following requirements: $w \in L^p$, $w^{-1} \in L^q$, $w \geq 0$ a.e., $w$ is not identically zero, and

$$
\sup_I \left( \frac{1}{|I|} \int_I w^p \, dm \right)^{1/p} \left( \frac{1}{|I|} \int_I w^{-q} \, dm \right)^{1/q} < \infty,
$$

the supremum over all subarcs $I$ of $\mathbb{T}$, $|I|$ denoting arc length.
2.2. The Hunt–Muckenhoupt–Wheeden Theorem [11, 8]. Let \( \mathcal{H} \in L^p, \mathcal{H} \geq 0 \), and suppose \( \mathcal{H} \) does not vanish identically. Then the Cauchy singular integral operator \( S \) (equivalently: the Riesz projection \( P \)) is bounded on \( L^p(\mathcal{H}) \) if and only if \( \mathcal{H} \in A_+. \)

2.3. The Helson–Szegö Theorem [10, 8]. Let \( \mathcal{H} \in L^2, \mathcal{H} \geq 0 \), and suppose \( \mathcal{H} \) is not the zero function. Then \( \mathcal{H} \in A_2 \) if and only if \( \mathcal{H} \) is of the form \( \mathcal{H} = e^{u+\tilde{v}} \), where \( u \) and \( v \) are real-valued \( L^\infty \) functions and \( ||v||_{\infty} < \pi/4 \); by \( \tilde{v} \) we denote the conjugate function (Hilbert transform) of \( v \), that is, \( \tilde{v} := -iSv - v_0 \), where \( v_0 \) is the zeroth Fourier coefficient of \( v \).

2.4. Theorem 2.2 ensures us that if \( \mathcal{H} \in A_p \), then the Toeplitz operator \( T(a) : \mathcal{H} \mapsto P(arp) \), \( P \) being the Riesz projection, is bounded on \( HP(\mathcal{H}) \) for every \( a \in L^p \). It is well known (Coburn’s theorem, see, e.g., [2]) that \( T(a) \) is Fredholm on \( HP(\mathcal{H}) \) if and only if there is a nonzero continuous function \( c \) on \( \mathbb{T} \) such that \( T(ac) \) is invertible on \( HP(\mathcal{H}) \); in that case the index of \( T(a) \) equals the winding number of \( c \) about the origin.

2.5. The Simonenko–Rochberg Invertibility Criterion. Let \( a \) be an invertible function in \( L^\infty \) and let \( \mathcal{H} \in A_p \). Then the following are equivalent:

(i) \( T(a) \) is invertible on \( HP(\mathcal{H}) \);

(ii) \( a = a_+ a_-, \) where \( a_+ \in H^1 \), \( a_- \in H^{-1} \), and \( \mathcal{H} | a_+^{-1} | \in A_p \);

(iii) \( a = a_+ a_- \), where \( a_+ \in H^2(\mathcal{H}^{-1}) \), \( a_- \in H^p(\mathcal{H}) \), \( a_+^{-1} \in H^p(\mathcal{H}^{-1}) \), \( a_-^{-1} \in H^p(\mathcal{H}) \), and \( \mathcal{H} | a_+^{-1} | \in A_p \);

(iv) \( a = |a| e^{ic} e^{i\theta} \), where \( c \in \mathbb{R} \), \( v \in BMO \) is real-valued, and \( \mathcal{H} e^{i\theta/2} \in A_p \).

The equivalences (i) \( \iff \) (ii) \( \iff \) (iii) (with the requirement that \( \mathcal{H} | a_+^{-1} | \) be in \( A_p \) replaced by the condition that \( S \) be bounded on \( L^p(\mathcal{H} | a_+^{-1} |) \)) are due to Simonenko [20, 21]. Proofs are also contained in [2, 5, 9, 13]. We remark that the factors \( a_\pm \) in (ii) and (iii) coincide. That (iv) is equivalent to (i) is a result of Rochberg [16]. Note that in the case \( p = 2 \) and \( \mathcal{H} = 1 \) the equivalence (i) \( \iff \) (iv) combined with the Helson–Szegö theorem yields the well-known Widom–Devinatz criterion. Also note that (ii) and (iv) are easily seen to be equivalent: if \( a \) is unimodular and satisfies (ii) then (iv) is fulfilled with \( v = 2 \log |a_+| \); on the other hand, if \( |a| = 1 \) a.e. and (iv) holds, then (ii) is true with \( a_- = e^{i(\theta - \pi)/2} \) and \( a_+ = e^{i\theta} e^{i(\theta + \pi)/2} \).

2.6. Let \( a \in L^\infty \), \( \mathcal{H} \in A_p \), \( \tau \in \mathbb{T} \). The Toeplitz operator \( T(a) \) is said to be locally Fredholm on \( HP(\mathcal{H}) \) at \( \tau \) if there is a function \( b \in L^\infty \) and an open arc \( U \subseteq \mathbb{T} \) containing \( \tau \) such that \( a|U = b|U \) and \( T(b) \) is Fredholm on \( HP(\mathcal{H}) \).

The Localization Theorem. Let \( a \in L^\infty \) and \( \mathcal{H} \in A_p \). Then \( T(a) \) is Fredholm on \( HP(\mathcal{H}) \) if and only if \( T(a) \) is locally Fredholm on \( HP(\mathcal{H}) \) at all points \( \tau \in \mathbb{T} \).
A result of this type was first established by Simonenko [19]. For the history of localization techniques and a proof of the preceding theorem see [2, 5, 6, 9, 13].

**The Extension Theorem.** Let \( a \in L^\infty \) and \( w \in A_p \). If \( T(a) \) is locally Fredholm on \( H^p(w) \) at \( \tau \in T \), then there are an open arc \( U \subset T \) containing \( \tau \) and a function \( b \in L^\infty \) which is continuous on the closure of \( T \setminus U \) such that \( a|_U = b|_U \) and \( T(b) \) is Fredholm on \( H^p(w) \).

A proof is in [22].

2.7. Again let \( a \in L^\infty \) and \( w \in A_p \). A representation \( a = a_- a_+ \) as in 2.5 (ii), (iii) is called a \( \Phi \)-factorization of \( a \) in \( L^p(w) \).

The notion of \( \Phi \)-factorizability in \( L^p(w) \) can be „localized” in a certain sense (see [4, 5, 13, 22]). We here confine our attention to the case where the weight \( w \) is of a special form. Namely, fix \( \tau \in T \) and let \( w(t) = |t - \tau|^\mu \), where \( \mu \in \mathbb{R} \). It is not difficult to check that \( w \in A_p \) if and only if \(-1/p < \mu < 1/q\). Suppose the latter condition is in force and denote \( L^p(w) \) by \( L^{p,\mu} \) and \( H^{p,\mu} \), respectively. Let \( \xi_{\mu,\tau}(z) \) and \( \eta_{\mu,\tau}(z) \), respectively, stand for the branches of the functions \((1 - \tau/z)^\mu \) and \((1 - z/\tau)^\mu \) which are analytic in \(|z| > 1 \) and \(|z| < 1 \) and take the value 1 at \( z = \infty \) and \( z = 0 \). We say that \( a \) admits a local factorization in \( L^{p,\mu} \) at \( \tau \) if there exist an open disk \( U \) centered at \( \tau \) and two functions \( a_- \) and \( a_+ \) analytic and nonzero in \( U_- := \{ z \in U : |z| > 1 \} \) and \( U_+ := \{ z \in U : |z| < 1 \} \), respectively, such that

\[
\xi_{\mu,\tau} a_- \in E^\varphi(U_-), \quad \xi_{\mu,\tau} a_+^{-1} \in E^\varphi(U_-),
\]

\[
\eta_{\mu,\tau} a_- \in E^\varphi(U_+), \quad \eta_{\mu,\tau} a_+^{-1} \in E^\varphi(U_+)
\]

and \( a = a_- a_+ \) a.e. on \( T := \mathbb{T} \cap U \); here \( E^\varphi(U_\pm) \) refers to the Smirnov–Hardy space over \( U_\pm \) (see [7, Chap. 10]).

A local factorization \( a = a_- a_+ \) of \( a \) in \( L^{p,\mu} \) at \( \tau \) is said to be a local \( \Phi \)-factorization of \( a \) in \( L^{p,\mu} \) at \( \tau \) if, in addition, the Cauchy singular integral operator

\[
(S_{\Gamma} \varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(z)}{z - t} \, dz \quad (t \in \Gamma = U \cap T)
\]

is bounded on \( L^p(T, |t - \tau|^\mu |a_-^{-1}(t)|) \).

2.8. The Clancey–Gohberg–Simonenko Equivalence Theorem. Let \( a \in L^\infty \) and \(-1/p < \mu < 1/q\). Then \( a \) admits a local \( \Phi \)-factorization in \( L^{p,\mu} \) at \( \tau \in T \) if and only if \( T(a) \) is locally Fredholm on \( H^{p,\mu} \) at \( \tau \).

For the evolution of this theorem and a proof see [4, 5, 13, 22].
2.9. **Theorem on the Stability of \( \Phi \)-Factorizations.** Let 
\( a \in L^\infty, \tau \in \mathbb{T}, \mu \in (-1/p, 1/q) \). If \( a = a_+ a_- \) is a (local) \( \Phi \)-factorization in \( L^{\infty,\mu} \) (at \( \tau \)), then \( a = a_- a_+ \) is also a (local) \( \Phi \)-factorization in \( L^{\infty,\mu} \) (at \( \tau \)) for all \( (\tau, \lambda) \in \mathbb{R}^2 \) in a sufficiently small open neighborhood of \( (p, \mu) \).

This theorem can be derived without difficulty from [13, 24] (where it is proved for \( \mu = 0 \)) and [25] (where one can find a proof for the case of fixed \( p \)).

2.10. Let \( \mathcal{A} \) be a \( C^* \)-subalgebra of \( L^\infty \) containing the constant functions. We denote by \( G \mathcal{A} \) the functions in \( \mathcal{A} \) which are invertible in \( L^\infty \) and thus in \( \mathcal{A} \), and we let \( M(\mathcal{A}) \) denote the maximal ideal space of \( \mathcal{A} \). We shall freely identify functions in \( \mathcal{A} \) with their Gelfand transform on \( M(\mathcal{A}) \). Thus, functions in QC and PQC may be thought of as continuous functions on \( M(\text{QC}) \) and \( M(\text{PQC}) \), respectively.

If \( \mathcal{A} \) and \( \mathcal{B} \) are two \( C^* \)-subalgebras of \( L^\infty \) such that \( \mathcal{A} \subseteq \mathcal{B} \) and \( \mathcal{B} \) contains the constants, then for each point (functional) \( \beta \in M(\mathcal{B}) \) the fiber \( M_\beta(\mathcal{A}) \) is defined as the set of all points (functionals) \( \alpha \in M(\mathcal{A}) \) such that \( \alpha(b) = \beta(b) \) for all \( b \in \mathcal{B} \). Note that \( M(\mathcal{A}) = \bigcup \{ M_\beta(\mathcal{A}) : \beta \in M(\mathcal{B}) \} \) is a partition ("fibration") of \( M(\mathcal{A}) \) into pairwise disjoint nonempty compact subsets.

Since \( M(C) \) may be naturally identified with \( \mathbb{T} \), for each \( \tau \in \mathbb{T} \) the fibers \( M_\tau(\text{QC}) \) are well defined.

2.11. **Sarason’s Theorem on \( M(\text{QC}) \) [18].** Each fiber \( M_\tau(\text{QC}) \) splits into three disjoint nonempty subsets \( M^0_\tau(\text{QC}), M^0_\tau(\text{QC}), M^\tau_\tau(\text{QC}) \) with the following properties. If \( \xi \in M^\tau_\tau(\text{QC}) \), then the fiber \( M_\xi(\text{PQC}) \) is a singleton, denoted by \( \{ (\xi, 0) \} \), and we have \( a(\xi, 0) = a(\tau - 0) \) for every function \( a \in \text{PC} \). Similarly, if \( \xi \in M^\tau_\tau(\text{QC}) \), then \( M_\xi(\text{PQC}) \) is a singleton, \( \{ (\xi, 1) \} \), and \( a(\xi, 1) = a(\tau + 0) \) for all \( a \in \text{PC} \). Finally, in case \( \xi \in M^0_\tau(\text{QC}) \), the fiber \( M_\xi(\text{PQC}) \) is a doubleton, \( \{ (\xi, 0), (\xi, 1) \} \), and \( a(\xi, 0) = a(\tau - 0), a(\xi, 1) = a(\tau + 0) \) for all \( a \in \text{PC} \).

2.12. **Sarason’s Theorem on Toeplitz Operators with PQC Symbols on \( H^2 \) [18].** (a) Let \( a \in G \text{PQC} \) and \( \tau \in \mathbb{T} \). Then \( T(a) \) is locally Fredholm on \( H^2 \) at \( \tau \) if and only if for each \( \xi \in M^0_\tau(\text{QC}) \) the closed straight line segment \( [a(\xi, 0), a(\xi, 1)] \) does not contain the origin.

(b) If \( a \in \text{PQC} \), then for \( T(a) \) to be Fredholm on \( H^2 \) it is necessary and sufficient that \( a \) be invertible in \( L^\infty \) (and thus in \( \text{PQC} \)) and that there be no \( \xi \in \bigcup \{ M^0_\tau(\text{QC}) : \tau \in \mathbb{T} \} \) such that the line segment \( [a(\xi, 0), a(\xi, 1)] \) contains the origin.

We remark that proofs of the preceding two theorems can be also found in [2].
3. The Main Result

3.1. Given two points \( z_1, z_2 \in \mathbb{C} \) and a number \( \lambda \in (1, \infty) \), we denote by \( \mathcal{A}_\lambda(z_1, z_2) \) the circular arc at the points of which the line segment \([z_1, z_2]\) is seen at the angle \( 2\pi/\max(\lambda, \lambda/(\lambda-1)) \) and which lies on the left (resp. right) of the segment \([z_1, z_2]\) directed from \( z_1 \) to \( z_2 \) for \( 1 < \lambda \leq 2 \) (resp. \( 2 < \lambda < \infty \)). A parametric representation of \( \mathcal{A}_\lambda(z_1, z_2) \) can be given by

\[
(1 - s_\lambda(\theta))z_1 + s_\lambda(\theta)z_2, \quad \theta \in [0, 1],
\]

where

\[
s_\lambda(\theta) := \frac{(\sin \alpha \theta/\sin \alpha)}{\exp(i\alpha(\theta - 1))}, \quad \alpha := \pi(1 - 2/\lambda).
\]

Note that \( \mathcal{A}_\lambda(z_1, z_2) \) is nothing but the segment \([z_1, z_2]\) itself.

Let \( a \in \mathbb{PQC} \) and \( \tau \in \mathbb{T} \). We shall write \( a \in \mathcal{A}_\lambda(a, \tau) \) if none of the arcs \( \mathcal{A}_\lambda(a(\xi, 0), a(\xi, 1)), \xi \) ranging over \( M_0^0(\mathbb{Q}C) \), contains the origin, that is, if

\[
(1 - s_\lambda(\theta))a(\xi, 0) + s_\lambda(\theta)a(\xi, 1) \neq 0 \quad \text{for all} \quad (\xi, \theta) \in M_0^0(\mathbb{Q}C) \times [0, 1].
\]

It is easy to see that \( a \in \mathcal{A}_\lambda(a, \tau) \) if and only if \( a(\xi, 0) \) and \( a(\xi, 1) \) are nonzero for all \( \xi \in M_0^0(\mathbb{Q}C) \) and none of the points \( a(\xi, 1)/a(\xi, 0) \) is located on the "critical ray" \( \{r\zeta : 0 < r < \infty\} \), where \( \zeta = e^{2\pi i(\lambda - 1)/\lambda} \). In particular, if \( a \) is unimodular, i.e., \( |a| = 1 \) a.e. on \( \mathbb{T} \), then \( a \in \mathcal{A}_\lambda(a, \tau) \) if and only if there is no \( \zeta \in M_0^0(\mathbb{Q}C) \) such that \( a(\zeta, 1)/a(\zeta, 0) \) assumes the "critical value" \( e^{2\pi i(\lambda - 1)/\lambda} \).

Throughout what follows we let \( \rho \) denote a weight of the form

\[
\rho(t) = \prod_{j=1}^n |t - t_j|^{\mu_j}, \quad t \in \mathbb{T},
\]

where \( t_1, \ldots, t_n \) are pairwise distinct points on \( \mathbb{T} \) and \( \mu_1, \ldots, \mu_n \) are real numbers subject to the condition \(-1/p < \mu_j < 1/q \) for all \( j \); the latter condition is equivalent to the inclusion \( \rho \in A_p \). Given such a weight, we associate with each point \( \tau \in \mathbb{T} \) a number \( \mu_\tau \) by the rule

\[
\mu_\tau = 0 \quad \text{if} \quad \tau \notin \{t_1, \ldots, t_n\}, \quad \mu_\tau = \mu_j \quad \text{if} \quad \tau = t_j.
\]

We are now in a position to state our main result.

3.2. Theorem. (a) Let \( a \in \mathbb{G}\mathbb{PQC} \) and \( \tau \in \mathbb{T} \). Then \( T(a) \) is locally Fredholm on \( L^p(\rho) \) at \( \tau \) if and only if \( a \in \mathcal{A}_\lambda(a, 1/(1/p + \mu_\tau)) \).

(b) If \( a \in \mathbb{PQC} \), then \( T(a) \) is Fredholm on \( L^p(\rho) \) if and only if \( a \in \mathbb{G}\mathbb{L}^\infty \) and \( a \in \mathcal{A}_\lambda(a, 1/(1/p + \mu_\tau)) \) for all \( \tau \in \mathbb{T} \).

In the case where \( p = 2 \) and \( \rho = 1 \), this theorem goes over into Sarason's Theorem 2.12. The "if" parts of Theorem 3.2 were first proved in \cite[Sec. 5.413]{2}; an independent proof (and the sufficiency portion of Theorem 3.2 for block Toeplitz operators) is also in \cite{3}. Therefore we here
focus our attention on the necessity part of Theorem 3.2, the proof of which will occupy the next two sections of this paper.

3.3. Here now are some consequences of Theorem 3.2. Given a function \( a \in \text{PQC} \), one can with each point \( \xi \in M(\text{QC}) \) associate a so-called local Toeplitz operator \( T_\xi(a) \) (depending on \( p \) and \( \rho \)); see [2, Sect. 5.24]. Once Theorem 3.2 is proved, it is a relatively easy matter to show that the spectrum of \( T_\xi(a) \) is a singleton for \( \xi \in M^-_\tau(\text{QC}) \) or \( M^+_\tau(\text{QC}) \) and that it equals the arc \( \mathcal{A}_{(1/\rho + \mu_\rho)}(a(\xi, 0), a(\xi, 1)) \) if \( \xi \in M^0_\tau(\text{QC}) \). Let \( \mathcal{F}_\rho^u(\rho) \) denote the Calkin image of the closed algebra \( \mathcal{T}_\rho^u(\rho) \) generated by all Toeplitz operators with PQC symbols on \( H^p(\rho) \). As the spectra of the local Toeplitz operators \( T_\xi(a) \) are known for all \( \xi \in M(\text{QC}) \), the reasoning of [2, Sects. 4.85-4.87 and 5.45-5.46] leads to an identification of the maximal ideal space and the Gelfand map for \( \mathcal{F}_\rho^u(\rho) \). Namely, the maximal ideal space is

\[
\mathfrak{M} := (M^-(\text{QC}) \times \{0\}) \cup (M^0(\text{QC}) \times [0, 1]) \cup (M^+(\text{QC}) \times \{1\}),
\]

where \( M^\pm(\text{QC}) := \bigcup \{ M^\pm_\tau(\text{QC}) : \tau \in \mathbb{T} \} \) and \( M^0(\text{QC}) := \bigcup \{ M^0_\tau(\text{QC}) : \tau \in \mathbb{T} \} \), the Gelfand topology is the one described by Sarason in [18] for the \( p = 2 \) and \( \rho = 1 \) case, and the Gelfand transform \( \mathcal{G}^u(a) \) of the coset \( T^u(a) \) of \( \mathcal{F}_\rho^u(\rho) \) containing the Toeplitz operator \( T(a) \) (\( a \in \text{PQC} \)) is given by

\[
(\mathcal{G}^u(a))(\xi, 0) = a(\xi, 0) \quad \text{for} \quad \xi \in M^-_\tau(\text{QC}),
\]

\[
(\mathcal{G}^u(a))(\xi, 1) = a(\xi, 1) \quad \text{for} \quad \xi \in M^+_\tau(\text{QC}),
\]

\[
(\mathcal{G}^u(a))(\xi, 0) = (1 - s_\tau(0))a(\xi, 0) + s_\tau(0)a(\xi, 1)
\]

for \( \xi \in M^0_\tau(\text{QC}) \),

where \( \tau := 1/(1/\rho + \mu_\rho) \). One can show that the Shilov boundary of \( \mathfrak{M} \) is all of \( \mathfrak{M} \), which implies that an operator \( A \in \mathcal{F}_\rho^u(\rho) \) is Fredholm on \( H^p(\rho) \) if and only if the Gelfand transform of the coset \( A^u \in \mathcal{F}_\rho^u(\rho) \) containing this operator does not vanish. This observation yields in particular a Fredholm criterion for block Toeplitz operators on \( H^p(\rho) \). More about the subject of this subsection will be published elsewhere.

4. PROOF OF THE MAIN RESULT: PART ONE.

The Local Factorization Domain

4.1. PROPOSITION. Let \( a \in L^\infty \) and \( \tau \in \mathbb{T} \). Let \( a = b_-b_+ \) and \( a = c_-c_+ \) be local factorizations at \( \tau \) in \( L^{p_1,\mu_1} \) and \( L^{p_2,\mu_2} \), respectively. Then there exists an open disk \( U \) centered at \( \tau \) and a function \( \chi \) analytic and nonzero on \( U \setminus \{\tau\} \) such that \( b_- = \chi^{-1}c_- \) on \( U_- := \{z \in U : |z| > 1\} \) and \( b_+ = \chi c_+ \) on
$U_+ := \{ z \in U : |z| < 1 \}$. At the point $\tau$ itself, there are three mutually excluding possibilities: either $\chi$ is analytic and nonzero at $\tau$, or $\chi$ is analytic at $\tau$ and has a simple zero there, or, thirdly, $\chi$ has a simple pole at $\tau$ (in which case $\chi^{-1}$ is analytic at $\tau$ and has a simple zero there). If $1/p_1 + \mu_1 \leq 1/p_2 + \mu_2$ then $\chi$ is analytic at $\tau$, and if $1/p_1 + \mu_1 \geq 1/p_2 + \mu_2$ then $\chi^{-1}$ is analytic at $\tau$.

Proof (the Following is a Slight Modification of the Argument Used to Prove [26, Theorem 2]). We define the functions $\xi_{\mu_1}$ and $\eta_{\mu_1}$ as in Section 2.7, but we now suppress the subscript $T$. From the definition of local factorizability we infer that there is a sufficiently small disk $U$ centered at $\tau$ such that, with $U_\pm$ as in the proposition,

$$
\xi_{-\mu_1} b_- \in E^{e_1}(U_-), \quad \xi_{\mu_1} b_-^{-1} \in E^{p_1}(U_-),
\xi_{-\mu_2} c_- \in E^{e_2}(U_-), \quad \xi_{\mu_2} c_-^{-1} \in E^{p_2}(U_-),
\eta_{\mu_1} b_+ \in E^{e_1}(U_+), \quad \eta_{-\mu_1} b_+^{-1} \in E^{p_1}(U_+),
\eta_{\mu_2} c_+ \in E^{e_2}(U_+), \quad \eta_{-\mu_2} c_+^{-1} \in E^{p_2}(U_+).
$$

For definiteness, let $p_1 \leq p_2$. Then $\xi_{\mu_2 - \mu_1} b_- c_-^{-1}$ belongs to $E'(U_-)$ for $1/r \geq 1/p_2 + 1/q_1$ and thus to $E'(U_-)$. Analogously, $\eta_{\mu_2 - \mu_1} b_+ c_+^{-1} \in E'(U_+)$. Write $\mu_2 - \mu_1 = m + \alpha$ with $m \in \mathbb{Z}$ and $0 \leq \alpha < 1$. We have $b_- c_-^{-1} = b_+ c_+$ on $\Gamma := \mathbb{T} \cap U$ and consequently,

$$
\xi_{1-z} \xi_{\mu_2 - \mu_1} b_- c_-^{-1} = \eta_{1-z} \eta_{\mu_2 - \mu_1} \phi b_+ c_+^{-1},
$$

where $\phi(z) := (-z/\tau)^{-m-1}$. The left-hand side of this equality is a function in $E'(U_-)$, while its right-hand side represents a function in $E'(U_+)$. We so deduce from a theorem by Carleman (see, e.g., [12, Theorem II.5.2]) that there is an analytic function $\psi$ on $U$ such that

$$
\xi_{1-z} \xi_{\mu_2 - \mu_1} b_- c_-^{-1} = \psi \quad \text{in } U_-,
\eta_{1-z} \eta_{\mu_2 - \mu_1} \phi b_+ c_+^{-1} = \psi \quad \text{in } U_+.
$$

Without loss of generality assume $\psi$ is nonzero in $U \setminus \{\tau\}$ (if necessary, replace $U$ by a smaller disk). Put $\chi = \xi_{\mu_2 + 1} \psi^{-1} \in U \setminus \{\tau\}$. The function $\chi$ is analytic and nonzero in $U \setminus \{\tau\}$, has at most a zero or a pole of a finite order at $\tau$, and we have $b_+ = c_+ \chi$ in $U_+$ and $b_- = c_- \chi^{-1}$ in $U_-$. Write $\chi(z) = (z - \tau)^n \chi_0(z)$, where $\chi_0$ is analytic and nonzero throughout $U$. Since

$$
\chi = b_+ c_+^{-1} = (\xi_{\mu_1} b_+)(\eta_{-\mu_2} c_+^{-1})\eta_{\mu_2 - \mu_1},
$$

it follows that the restriction of $\eta_{\mu_1 - \mu_2} \chi$ to $I$ belongs to $L'(I')$ for all $r$ such
that \(1/r \geq 1/p_1 + 1/q_2\). Because \(\eta_{\mu_1 - \mu_2}(z)\chi(z)\) behaves at \(r\) as \((z - \tau)^{\mu_1 - \mu_2}\), we conclude that
\[
(n + \mu_1 - \mu_2)(1/p_1 + 1/q_2)^{-1} > -1,
\]
whence
\[
n > -1 - (1/p_1 + \mu_1) - (1/p_2 + \mu_2).
\]
The equality \(\chi^{-1} = b_+^{-1}c_+ = (\eta^{-1}b_+^{-1})(\eta_{\mu_2}c_+)\eta_{\mu_1 - \mu_2}\) similarly yields that
\[
n < 1 + (1/p_2 + \mu_2) - (1/p_1 + \mu_1).
\]
Taking into account that \(0 < 1/p_i + p_j < 1\) and that \(n\) is an integer we find that \(n \in \{0, 1\}\) if \(1/p_2 + \mu_2 > 1/p_1 + \mu_1\) and that \(n \in \{0, -1\}\) in case \(1/p_2 + \mu_2 < 1/p_1 + \mu_1\), which implies all assertions of the proposition.

4.2. Two local factorizations \(a = b_- b_+\) and \(a = c_- c_+\) of \(a \in L^\infty\) at \(\tau \in \mathbb{T}\) in \(L^{p_1, \mu_1}\) and \(L^{p_2, \mu_2}\), respectively, are said to be equivalent if there is an open disk \(U\) centered at \(\tau\) and a function \(\chi\) analytic and nonzero throughout \(U\) such that \(a_- = \chi^{-1}b_-\) in \(U_-\) and \(a_+ = \chi b_+\) in \(U_+\).

Let \(\Pi\) be the parallelogram \(\Pi = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, \ 0 < x + y < 1\}\). Given \(a \in L^\infty\) and \(\tau \in \mathbb{T}\), denote by \(\Omega_\tau(a)\) the set of all \((x, y) \in \Pi\) such that \(a\) admits a local factorization in \(L^{1/x, y}\) at \(\tau\). Define an equivalence relation \(\sim\) on \(\Omega_\tau(a)\) by saying that \((1/p_1, \mu_1)\) is equivalent to \((1/p_2, \mu_2)\) if \(a\) admits equivalent local factorizations at \(\tau\) in both \(L^{p_1, \mu_1}\) and \(L^{p_2, \mu_2}\).

4.3. Proposition. The equivalence relation \(\sim\) divides \(\Omega_\tau(a)\) into at most two equivalence classes, each of which is convex. In case we have exactly two equivalence classes, they can be separated by a straight line of the form \(x + y = \text{constant}\).

Proof. Assume \(a = b_- b_+ = c_- c_+ = d_- d_+\) are three local factorizations at \(\tau\) which are not equivalent to one another. By Proposition 4.1, we have \(b_+ = \chi c_+\) and \(b_+ = \psi d_+\) with certain functions \(\chi\) and \(\psi\) analytic and nonzero in some open punctured neighborhood of \(\tau\). Since the factorizations \(b_- b_+\) and \(c_- c_+\) are not equivalent to each other, \(\chi\) must have a simple zero or a simple pole at \(\tau\). Analogously we conclude that \(\psi\) has either a simple zero or a simple pole at \(\tau\). Each of the four possibilities emerging leads to a contradiction: if both \(\chi\) and \(\psi\) possessed a simple zero or a simple pole at \(\tau\) then \(\chi^{-1}\psi\) were analytic and nonzero at \(\tau\) and so the factorizations \(c_- c_+\) and \(d_- d_+\) would be equivalent, and if \(\chi\) had a simple zero (resp. pole) and \(\psi\) a simple pole (resp. zero) then \(\chi\psi^{-1}\) would own a double zero (resp. pole) at \(\tau\), which contradicts Proposition 4.1. Thus we have shown that \(\sim\) divides \(\Omega_\tau(a)\) into at most two equivalence classes.
To prove that each equivalence class is convex, let \((l/p_1, p_1) \sim (l/p_2, p_2)\) and let \(a = b - b_+ \) and \(a = c - c_+ \) be local factorizations in \(L^{p_1, \mu_1}\) and \(L^{p_2, \mu_2}\), respectively. So \(b_- = \chi^{-1} c_- \) and \(b_+ = \chi c_+ \) with some function \(\chi\) analytic and nonzero in an open neighborhood of \(\tau\), implying that \(a = b - b_+ \) is not only a local factorization in \(L^{p_1, \mu_1}\) but also in \(L^{p_2, \mu_2}\). Therefore, \(\eta_\mu b_+ \in E^0(U_+)\) and \(\eta_\mu b_+ \in E^0(U_+)\) (recall 2.7), which entails that \(\eta_\mu b_+ \in E^0(U_+)\) for all \(\mu\) and \(p\) of the form \(\mu = \theta \mu_1 + (1 - \theta) \mu_2\), \(1/p = \theta/p_1 + (1 - \theta)/p_2\), \(\theta \in [0, 1]\). We analogously have \(\xi_\mu b_- \in E^0(U_-)\), \(\xi_\mu b_- \in E^0(U_-)\), \(\eta_\mu a_+^{-1} \in E^0(U_+)\) for the same values of \(p\) and \(\mu\). This proves the convexity of the equivalence classes.

Finally, suppose we have exactly two equivalence classes, \(\Omega_1(a)\) and \(\Omega_2(a)\). Let \(a = b - b_+ \) and \(a = c - c_+ \) be any local factorizations belonging to \(\Omega_1(a)\) and \(\Omega_2(a)\), respectively. Then \(b_- = \chi^{-1} c_- \) and \(b_+ = \chi c_+ \), where \(\chi\) is as in Proposition 4.1. Because the local factorizations \(a = b - b_+ \) and \(a = c - c_+ \) are not equivalent to each other, \(\chi\) must have a simple zero or a simple pole at \(\tau\), and since the situation occurring (zero or pole) is clearly independent of the particular choice of the factorizations from \(\Omega_1(a)\) and \(\Omega_2(a)\), we deduce from Proposition 4.1 that either \(1/p_1 + \mu_1 > 1/p_2 + \mu_2\) or \(1/p_1 + \mu_1 < 1/p_2 + \mu_2\) for all \((1/p_1, \mu_1) \in \Omega_i(a)\) \((i = 1, 2)\).

4.4. Let \(a \in L^\infty\) and \(\tau \in \mathbb{T}\). We denote by \(\Omega_\tau^\Phi(a)\) the set of all points \((x, y) \in \Pi\) for which \(a\) admits a local \(\Phi\)-factorization in \(L^{1/\nu, \nu}\) at \(\tau\). Since obviously \(\Omega_\tau^\Phi(a) \subset \Omega_\tau(a)\), the equivalence relation "\(~\)" induces an equivalence relation on \(\Omega_\tau^\Phi(a)\). By Proposition 4.3, \(\Omega_\tau^\Phi(a)\) splits into at most two equivalence classes, \(\Omega_{\tau 1}^\Phi(a)\) and \(\Omega_{\tau 2}^\Phi(a)\).

4.5. PROPOSITION. \(\Omega_\tau^\Phi(a)\) is an open subset of \(\Pi\) and \(\Omega_{\tau j}^\Phi(a)\) \((j = 1, 2)\) are open and convex subsets of \(\Pi\).

Proof. That \(\Omega_\tau^\Phi(a)\) and hence its components \(\Omega_{\tau j}^\Phi(a)\) are open follows immediately from Theorem 2.9. To prove the convexity of the equivalence classes we are in view of Proposition 4.3 left with verifying that if \(S_\tau\) is bounded on

\[
L^\mu (\Gamma, |t - \tau|^{-\mu} |a_+^{-1}(t)|) \quad \text{and} \quad L^{\nu 2} (\Gamma, |t - \tau|^{-\nu} |a_+^{-1}(t)|),
\]

then \(S_\tau\) is also bounded on \(L^\mu (\Gamma, |t - \tau|^{-\mu} |a_+^{-1}(t)|)\) whenever \(1/p = \theta/p_1 + (1 - \theta)/p_2\) and \(\mu = \theta \mu_1 + (1 - \theta) \mu_2\) for some \(\theta \in [0, 1]\). But this is a standard interpolation result (see, e.g., [1, Corollary 5.5.2]).

4.6. LEMMA. Let \(a \in GPQC\), \(\tau \in \mathbb{T}\), \(-\frac{1}{2} < \mu < \frac{1}{2}\). Then \(T(a)\) is locally Fredholm at \(\tau\) on \(H^{2, \mu}\) if and only if \(a \in \mathcal{B}_0^\tau(1/(1 + \mu))\).

Proof. As the sufficiency part of Theorem 3.2(a) is already proved, we
need only to show that if $T(a)$ is locally Fredholm at $\tau$ on $H^{2,\mu}$, then $a \in B_r(1/(1/2 + \mu))$. By virtue of the extension theorem cited in Section 2.6 we may assume that $T(a)$ is Fredholm on $H^{2,\mu}$. Define $\xi := \xi_{\mu, \tau}$ and $\eta := \eta_{\mu, \tau}$ as in Section 2.7. Put $\phi = \xi \eta^{-1}$. Note that $\phi$ is a piecewise continuous function with the property that $\phi(\tau - 0) = e^{-\pi i \mu}$ and $\phi(\tau + 0) = e^{\pi i \mu}$. The operators

$$T(\eta^{-1}) : H^2 \rightarrow H^{2,\mu}, \quad T(\xi) : H^{2,\mu} \rightarrow H^2$$

are easily seen to be invertible (see, e.g., [2, Lemma 5.62]), and hence

$$T(\xi) T(a) T(\eta^{-1}) : H^2 \rightarrow H^2$$

is Fredholm. But $T(\xi) T(a) T(\eta^{-1}) = T(\xi a \eta^{-1}) = T(a \phi)$, and so Theorem 2.12(b) implies that $a \phi \in B_r(\tau)$ or, equivalently, that $a \in B_r(1/(1/2 + \mu))$.

We take the opportunity to remark here that it was apparently Paatashvili [15] who for the first time used some kind of the above by now standard argument of removing a weight by multiplying the symbol by a piecewise continuous function.

4.7. Lemma. Let $a \in GPQC$ and $\tau \in \mathbb{T}$. If $T(a)$ is locally Fredholm at $\tau$ on $H^p$ and $a \in B_r(\tau)$ for all $r$ in some one-sided neighborhood of $p$, say $(p - \varepsilon, p)$, then $a \in B_r(p)$.

Proof. Because $a \in B_r(\tau)$ for all $r \in (p - \varepsilon, p)$, we obtain that $a \in B_r(1/(1/s + \mu))$ for all $(s, \mu)$ such that $(1/s, \mu) \in \Pi$ and $1/s + \mu = 1/r$. As the sufficiency part of Theorem 3.2(a) is already proved, we deduce that $T(a)$ is locally Fredholm at $\tau$ on $H^{p,\mu}$ for all $(s, \mu) \in \Pi$ satisfying $1/p < 1/s + \mu < 1/(p - \varepsilon)$. Hence, by Theorem 2.8, the stripe

$$K := \{(x, y) \in \Pi : 1/p < x + y < 1/(p - \varepsilon)\}$$

is a subset of $\Omega_{\tau}^p(a)$. Since $T(a)$ is supposed to be locally Fredholm at $\tau$ on $H^p$ (and thus, again by Theorem 2.8, to admit a local $\Phi$-factorization at $\tau$ in $L^p$), the point $(1/p, 0)$ also belongs to $\Omega_{\tau}^p(a)$. We so infer from Proposition 4.5 that the convex hull of the union of $K$ and some open neighborhood of $(1/p, 0)$ is contained in a single component of $\Omega_{\tau}^p(a)$, in $\Omega_{\tau}^{\Phi,1}(a)$, say. This in turn implies that, except for the endpoints, the line segment $\{(x, y) \in \Pi : x + y = 1/p\}$ is a subset of $\Omega_{\tau}^{\Phi,1}(a)$ as well. Hence in particular $(1/2, \mu) \in \Omega_{\tau}^{\Phi,1}(a)$, where $\mu$ satisfies the equality $1/2 + \mu - 1/p$. Now Theorem 2.8 shows that $T(a)$ is locally Fredholm at $\tau$ on $H^{2,\mu}$, and from Lemma 4.6 we conclude that $a \in B_r(1/(1/2 + \mu))$, which is equivalent to saying that $a \in B_r(p)$.
5. PROOF OF THE MAIN RESULT: PART TWO.

THE CONSTRUCTION

5.1. LEMMA. Let $b \in GPQC$ and $\tau \in \mathbb{T}$. Suppose $T(b)$ is locally Fredholm on $H^p$ at $\tau$, but $b \notin \mathcal{A}_\tau(p)$. Then there exists a unimodular function $a \in PQC$ with the following properties:

(i) $a$ is identically 1 on some arc $A := (e^{-i\delta} \tau, \tau)$ ($\delta > 0$);

(ii) there is a unimodular function $f \in GQC$ such that $a|\Gamma = f|\Gamma$ on some arc $\Gamma := (\tau, \tau e^{i\delta})$ ($\delta > 0$);

(iii) $a \notin \mathcal{A}_\tau(p)$;

(iv) $T(a)$ and $T(f)$ are invertible on $H^p$.

Proof. By the extension theorem quoted in Section 2.6, we may without loss of generality assume that $T(b)$ is Fredholm on $H^p$. Since $b \notin \mathcal{A}_\tau(p)$, there exists a $\xi_0 \in M^0_\tau(QQC)$ such that $0 \in A_p(b(\xi_0, 0), (\xi_0, 1))$. There is an $\varepsilon > 0$ such that $T(c)$ is Fredholm on $H^p$ whenever $c \in L^\infty$ and $\|c - b\|_\infty < \varepsilon$. Let $w$ be a finite sum $\sum u_i v_i$ such that $u_i$ is piecewise constant (with only finitely many jumps), $v_i$ belongs to QC, and $\|w - b\|_\infty < \varepsilon/2$. We clearly have $|w(\xi_0, j) - b(\xi_0, j)| < \varepsilon/2$ for $j = 0, 1$. Hence, there exists a function $d$ continuous on $\mathbb{T} \setminus \{\tau\}$ and constant on some left and right neighborhoods of $\tau$ such that $\|d\|_\infty < \varepsilon/2$ and

$$d(\tau - 0) + w(\xi_0, 0) = b(\xi_0, 0), \quad d(\tau + 0) + w(\xi_0, 1) = b(\xi_0, 1).$$

Consequently, if we put $c = w + d$, then $c \notin \mathcal{A}_\tau(p)$ and $\|c - b\|_\infty < \varepsilon$, implying that $T(c)$ is Fredholm on $H^p$.

Next, there exist $g, h \in QC$ and arcs $A = (\tau e^{-i\delta}, \tau)$, $\Gamma = (\tau, \tau e^{i\delta})$ such that $c|A = g|A$ and $c|\Gamma = h|\Gamma$. Without loss of generality assume $\delta = 2\pi/(2n)$ for some integer $n > 1$. We may also assume that $g$ and $h$ are bounded away from zero on $A$ and $\Gamma$, respectively. Define $h^*$ on $(\tau e^{i\delta}, \tau e^{2i\delta})$ by $h^*(\tau e^{i\delta}) = h(\tau e^{i(2\delta - \varphi)})$ ($0 < \varphi < \delta$) and then extend the function which is equal to $h^*$ on $(\tau e^{i\delta}, \tau e^{2i\delta})$ and equal to $h$ on $(\tau, \tau e^{i\delta})$ periodically to all of $\mathbb{T}$. From [18, Lemma 2] it is immediate that the function $h_0$ obtained in this way belongs to GQC. Thus, $c|\Gamma = h_0|\Gamma$ with some $h_0 \in GQC$. We analogously have $c|A = g_0|A$ with some $g_0 \in GQC$.

There is no loss of generality in assuming that $T(c)$, $T(h_0)$, $T(g_0)$ are invertible on $H^p$ if necessary, multiply $c, h_0, g_0$ by appropriate continuous functions which are identically 1 in some open neighborhood of $\tau$. Put $a_0 = g_0^{-1} c$ and $f_0 = g_0^{-1} h_0$. Then $a_0 \notin \mathcal{A}_\tau(p)$, $a_0|A = 1$, $a_0|\Gamma = f_0$, and $T(a_0)$ and $T(f_0)$ are invertible on $H^p$. Finally, the functions $a := a_0/|a_0|$ and $f := f_0/|f_0|$ meet all the requirements (i) to (iv) (note that neither the
property of being in $A_l(p)$ nor the invertibility of a Toeplitz operator is affected by dividing the function/symbol by its modulus).

5.2. Let $a \in \text{PQC}$ be a function subject to the conditions (i) to (iv) of the previous lemma.

By a theorem of Sarason [17], $f$ can be written in the form $f = e^{i\theta(v + w)}$, where $v$ and $w$ are real-valued continuous functions on $T$. For $\theta \in (0, 1)$, put $f^\theta := e^{i\theta(v + w)}$. Since $\theta(v + w) = (\theta v)^- + (\theta w)$, the afore-mentioned theorem by Sarason implies that $f^\theta$ belongs to GQC and that $T(f^\theta)$ is invertible on $H^p$ for all $\theta \in (0, 1)$.

Because $T(a)$ is invertible on $H^p$, Theorem 2.5 insures the existence of a real-valued function $u \in BMO$ and a number $c \in \mathbb{R}$ such that $a = e^{iu}e^{i\alpha}$ and $e^{-u/2} \in A_p$. Given a parameter $\theta \in (0, 1)$ and an integer $k \in \mathbb{Z}$, we define $a^{\theta,k} \in L^\infty$ by

$$a^{\theta,k} := e^{i\theta(c + \tilde{u} + 2k\pi)},$$

It is easy to see that $e^{-\theta u/2} \in A_p$ for all $\theta \in (0, 1)$, so that, again by Theorem 2.5, $T(a^{\theta,k})$ is invertible on $H^p$ for all $\theta \in (0, 1)$ and all $k \in \mathbb{Z}$.

5.3. LEMMA. There exist an integer $k_0$ and unimodular functions $g_0 \in \text{QC}$ such that $a^{\theta,k_0} | A = 1$ and $a^{\theta,k_0} | \Gamma = g_0 | \Gamma$ for all $\theta \in (0, 1)$.

Proof. We have $e^{i(\theta + w)} = e^{i(c + \tilde{u})}$ on $\Gamma$. Hence, if we put

$$\alpha = \frac{1}{2}[(v + i\tilde{v}) + i(w + i\tilde{w}) - (u + i\tilde{u})],$$
$$\beta = \frac{1}{2}[(v - i\tilde{v}) - i(w - i\tilde{w}) - (u - i\tilde{u}) + 2ic],$$

then $e^\phi = e^\theta$ on $\Gamma$.

We claim that $e^{\pm \phi} \in H^1$. To see this, note first that

$$e^{i\phi} = e^{i(\phi/2)(w - i\tilde{w})}e^{i(\phi/2)(w + i\tilde{w})}$$

is a $\Phi$-factorization, i.e., a factorization in the sense of Criterion 2.5(iii), of the (continuous!) function $e^{i\phi}$ in $L^p$, whence

$$e^{i(\phi/2)(w + i\tilde{w})} \in H^p, \quad e^{i(\phi/2)(w - i\tilde{w})} \in H^p,$$

and thus $|e^{\pm \phi/2}| \in L^p \cap L^q$. Since $e^{-u/2} \in A_p \subset L^p$ and $e^{u/2} \in A_q \subset L^q$, we conclude that

$$|e^{\pm \phi}| = e^{\pm u/2}e^{\mp \tilde{u}/2}e^{\pm u/2} \in L^1.$$
Because the negative Fourier coefficients of $e^{\pm z}$ clearly vanish, it follows that $e^{\pm z} \in H^1$. It can be shown similarly that $e^{\pm \beta} \in H^1$.

Once more having recourse to Carleman's theorem [12, Theorem III.E.2α] we now deduce that there are an open simply connected set $U \subset \mathbb{C}$ containing $\Gamma$ in its interior and a function $\phi$ analytic in $U$ such that

\[
\phi = e^\alpha \quad \text{on } \Gamma \cup U_+ \quad (U_+ := \{z \in U : |z| < 1\}),
\]

\[
\phi = e^\beta \quad \text{on } \Gamma \cup U \quad (U := \{z \in U : |z| > 1\});
\]

here we identify functions in $H^1$ and $\overline{H^1}$ with their analytic extensions into $\{|z| < 1\}$ and $\{|z| > 1\}$, respectively. Since $e^{-\alpha} = e^{-\beta}$ belongs to $H^1$, the function $\phi$ cannot have zeros on $\Gamma$, and so we may assume that $\phi$ is nonzero throughout $U$ (if necessary, lessen $U$).

Hence $\phi = e^\psi$ for some function $\psi$ analytic in $U$. It follows that there are integers $m, n \in \mathbb{Z}$ such that

\[
\alpha = \psi + 2m\pi i \quad \text{on } \Gamma \cup U_+, \quad \beta = \psi + 2n\pi i \quad \text{on } \Gamma \cup U_-(U).
\]

In particular, $\theta\alpha = \theta\psi + 2m\pi i \theta$ and $\theta\beta = \theta\psi + 2n\pi i \theta$ on $\Gamma$, so

\[
e^{-2m\pi i \theta} e^{\theta\alpha} = e^{-2n\pi i \theta} e^{\theta\beta} \quad \text{on } \Gamma,
\]

whence

\[
e^{2(n - m)\pi i \theta} e^{i(\theta + w)} e^{-i(\theta + \theta)} = 1 \quad \text{on } \Gamma.
\]

In a completely analogous fashion (replacing $f$ by the function identically 1) one can show that there is an $l \in \mathbb{Z}$ such that

\[
e^{-2l\pi i \theta} e^{-i(\theta + \theta)} = 1 \quad \text{on } \Delta.
\]

Consequently, if we put $k_0 = l$, then $a^{\theta, k_0}|_{\Delta} = 1$ and

\[
e^{2(n - m)\pi i \theta} f^0 / a^{0, k_0} = e^{-2k_0\pi i \theta} \quad \text{on } \Gamma,
\]

which gives the assertion with $g_\theta = e^{2(n - m + k_0)\pi i \theta} f^0$.

5.4. Abbreviate $a^{\theta, k_0}$ to $a^\theta$ and $e^{(c + 2k_0\pi i) \theta}$ to $\gamma^\theta$.

5.5. LEMMA. Let $0 < \theta < \sigma < \min(p/2, q/2)$. Then $a^\theta = h_\theta s_\theta$, where $h_\theta$ belongs to $GH^{\infty}$ (i.e., $h_\theta^{-1} \in H^{\infty}$) and $s_\theta \in GL^{\infty}$ is a function whose (essential) range is contained in the sector

\[
\{z \in \mathbb{C} : |\text{Im } z| < \tan(\theta n/(2\delta)) \text{ Re } z\}.
\]
Proof. By Hölder's inequality,

\[ \int_I e^{-\sigma u} \, dm \leq |I|^{1-2\alpha/p} \left( \int_I e^{\alpha u/2} \, dm \right)^{2\alpha/p} \]

\[ \int_I e^{\sigma u} \, dm \leq |I|^{1-2\sigma/q} \left( \int_I e^{\sigma u/2} \, dm \right)^{2\sigma/q} \]

whence

\[ \frac{1}{|I|} \left( \int_I (e^{-\sigma u/2})^2 \, dm \right)^{1/2} \left( \int_I (e^{\sigma u/2})^2 \, dm \right)^{1/2} \]

\[ \leq \left[ \frac{1}{|I|} \left( \int_I (e^{-u/2})^p \, dm \right)^{1/p} \left( \int_I (e^{u/2})^q \, dm \right)^{1/q} \right] \sigma \]

and since \( e^{-u/2} \in A_\sigma \), it follows that \( e^{-\sigma u/2} \in A_2 \). The Helson–Szegö Theorem 2.3 now shows that there are real-valued \( L^\infty \) functions \( \alpha \) and \( \beta \) such that \( -\sigma u/2 = \alpha + \beta \) and \( \|\beta\|_\infty < \pi/4 \). Put \( \mu = -2\alpha/\sigma \) and \( \nu = 2\beta/\sigma \). Then \( u = \mu - \nu \) and thus \( \tilde{u} = \mu + \nu + \delta \), where \( \delta \) is some constant. Consequently, if \( \theta \in (0, \sigma) \), then

\[ a^\theta = \gamma^\theta e^{i\theta \delta} = \gamma^\theta e^{i\theta \delta} e^{i\theta(\mu + \nu)} = [\gamma^\theta e^{i\theta \delta} e^{i\theta\mu + i\theta \nu}] e^{i\theta \mu + i\theta \nu}. \]

Since \( \mu \in L^\infty \), the expression in the first brackets belongs to \( GH^\infty \), and because \( \|\theta \|_\infty = (2\theta/\sigma) \|\beta\|_\infty < (\theta \pi)/(2\sigma) \), the range of the term in the second brackets belongs to the asserted sector.

5.6. Lemma (Maybe Well Known). The set \( M^0_\tau(QC) \) is a connected subset of \( M(QC) \).

Proof. Let \( SO[0, 1) \) denote the \( C^* \)-algebra of all bounded continuous functions on \([0, 1)\) which are slowly oscillating near the endpoint 1 (see [18, p. 820]), and let \( C_0[0, 1) \) stand for the continuous functions on \([0, 1)\) that tend to 0 at 1. Sarason showed that the quotient algebra \( SO[0, 1)/C_0[0, 1) \) is isometrically isomorphic to the restriction algebra \( QC|M^0_\tau(QC) \) (see [18, p. 823]). To show that \( M^0_\tau(QC) \) is connected, it suffices therefore to show that the spectrum of the coset \( \varphi + C_0[0, 1) \) is connected for every \( \varphi \in SO[0, 1) \). But the spectrum of \( \varphi + C_0[0, 1) \) is nothing but the intersection over all \( r \in (0, 1) \) of the closures of the connected sets \( \varphi([r, 1)) \) and is thus connected.

5.7. Lemma. Let \( \theta \) and \( \sigma \) be as in Lemma 5.5 and let \( g_\theta \) be the function introduced in Lemma 5.3. Then the set \( g_\theta(M^0_\tau(QC)) \) is a closed arc contained in the arc of length \( 2\pi \theta/\sigma \) centered at 1.
Proof. For simplicity, let $\tau = 1$. For $\beta \in (-\frac{1}{2}, \frac{1}{2})$, define $\varphi_\beta \in \mathcal{P}$ by $\varphi_\beta(x^{it}) = e^{it\beta}$ (see [6, Proposition 7.18] or [2, Theorem 2.17]). From Theorem 2.12(b) we deduce that
\[ T(\varphi_\beta a^\theta) \text{ is invertible on } \mathcal{H}_2, \]
whence, by Lemma 5.3,
\[ g_0(\xi)/e^{2\pi i \beta} \neq e^{2\pi i} \text{ for all } \xi \in M^0_1(QC), \]
i.e., $g_0(\xi) \neq e^{i(\pi + 2\pi \beta)}$ for all $\xi \in M^0_0(QC)$. But if $2|\beta| < 1 - \theta/\sigma$, then $\pi \theta/\sigma < \pi + 2\pi \beta < 2\pi - \pi \theta/\sigma$, which shows that the set $g_0(M^0_1(QC))$ is contained in the arc $(e^{-\pi \theta/\sigma}, e^{\pi \theta/\sigma})$. Lemma 5.6 finally implies that $g_0(M^0_1(QC))$ itself is an arc.

5.8. The preceding lemma says in particular that if $\theta > 0$ is sufficiently small, then the arc
\[ \{a^\theta(\xi, 1)/a^\theta(\xi, 0) : \xi \in M^0_1(QC)\} = \{g_0(\xi) : \xi \in M^0_1(QC)\} \]
does not contain the "critical value" $\zeta := e^{2\pi i/\theta}$, i.e., then $a^\theta \not\in \mathcal{B}_1(p)$ (recall 3.1). Let $\theta_0$ be the supremum of all $\theta > 0$ such that $a^\theta \not\in \mathcal{B}_1(p)$ for all $\lambda \in (0, \theta)$. By what has just been said, $\theta_0 > 0$. Our hypothesis 5.1(iii) shows that $\theta_0 \leq 1$.

5.9. Lemma. $a^{\theta_0} \not\in \mathcal{B}_1(p)$.

Proof. We must show that the "critical value" $\zeta$ belongs to $g_{\theta_0}(M^0_1(QC))$. Assume the contrary and denote by $d > 0$ the distance between $\zeta$ and $g_{\theta_0}(M^0_1(QC))$. If $0 < \sigma < \min(p/2, q/2)$ and $0 < \theta < (d\sigma)/(2\pi)$, then, by Lemma 5.7, $g_0(M^0_1(QC))$ lies on the arc of length $(2\pi \theta)/\sigma \leq d/2$ centered at 1. From the definition of $a^\theta$ it is obvious that $a^{\theta_1 + \theta_2} = a^{\theta_1} a^{\theta_2}$ whenever $\theta_1 > 0$, $\theta_2 > 0$, $\theta_1 + \theta_2 < 1$, and so Lemma 5.3 shows that also $g_{\theta_1 + \theta_2} = g_{\theta_1} g_{\theta_2}$ for $\theta_1 > 0$, $\theta_2 > 0$, $\theta_1 + \theta_2 < 1$. Consequently,
\[ g_{\theta_0 + \theta}(M^0_1(QC)) \subset g_{\theta_0}(M^0_1(QC)) g_{\theta}(M^0_1(QC)) \]
The arc on the right of this inclusion is contained in a $d/2$-neighborhood of $g_{\theta_0}(M^0_1(QC))$, which implies that $\zeta$ does not belong to $g_{\theta_0 + \theta}(M^0_1(QC))$ for all $\theta \in [0, (d\sigma)/(2\pi))$, contradicting the definition of $\theta_0$. 1
5.10. **Lemma.** If \( 0 < \theta < \theta_0 \), then \( a^\theta \in \mathcal{B}_\tau(p) \), i.e., \( \zeta \notin g_\theta(M^0_\tau(QC)) \), but the distance between \( \zeta \) and \( g_\theta(M^0_\tau(QC)) \) does not exceed \( \pi(\theta_0 - \theta)/\sigma \), where \( \sigma \) is any number between 0 and \( \min(p/2, q/2) \).

**Proof.** That \( \zeta \) does not belong to \( g_\theta(M^0_\tau(QC)) \) is immediate from the definition of \( \theta_0 \). To prove the distance estimate, note first that \( g_{\theta_0} = g_\theta g_{\theta_0 - \theta} \) (see the previous proof). Since, by Lemma 5.9, \( \zeta \in g_{\theta_0}(M^0_\tau(QC)) \), it follows that

\[
\zeta \in g_\theta(M^0_\tau(QC)) g_{\theta_0 - \theta}(M^0_\tau(QC)).
\]

The arc \( g_{\theta_0 - \theta}(M^0_\tau(QC)) \) is contained in the arc of length \( 2\pi(\theta_0 - \theta)/\sigma \) centered at 1; this results from Lemma 5.7 in case \( \theta_0 - \theta \geq \sigma \) and is trivial if \( \theta_0 - \theta < \sigma \). Hence \( \zeta \) belongs to some \( \pi(\theta_0 - \theta)/\sigma \)-neighborhood of \( g_\theta(M^0_\tau(QC)) \).

5.11. **Proof of Theorem 3.2.** Let \( a \in \mathcal{G}_{PQC} \) and \( \tau \in \mathbb{T} \). Assume \( T(a) \) is locally Fredholm on \( H^p \) at \( \tau \) but \( a \notin \mathcal{B}_\tau(p) \). In view of Lemma 5.1 we may a priori assume that \( a \) is a unimodular function in \( PQC \) subject to the conditions 5.1(i) to (iv).

Let \( \chi \in PC \) be a function which is continuous on \( \mathbb{T} \setminus \{ \tau \} \), whose range lies on an arc of \( \mathbb{T} \) of length \( 2\pi \) centered at 1, and which is identically 1 on \( \Delta \) and identically \( e^{i\alpha_0} \) on \( \Gamma \); the value of \( \alpha_0 \in (-\pi, \pi) \) will be specified below.

We have \( \chi = e^{i\psi} \), where \( \psi \) is real-valued and \( \| \psi \|_\infty \leq \alpha \). Define \( \chi^\theta \) as \( e^{i\theta \psi} \) for \( \theta \in (0, 1) \) and let \( \mu_0 := a^{\mu, k_0} \) be defined by Section 5.4. We claim that \( T(\chi^\theta a^\theta) \) is invertible on \( H^p \) for all \( \theta \in (0, 1) \). To this end, set \( \phi = -\psi \) and note first that \( \chi^\theta a^\theta \) is a constant multiple of \( e^{i\theta (a + \phi)} \). By Theorem 2.5, we need only show that \( e^{-\mu \psi^2} \in A_p \). Because \( e^{-\mu \psi^2} \in A_p \), there exist \( C > 0 \) and \( \delta > 0 \) such that

\[
\left( \frac{1}{|I|} \int_I e^{-(1 + \delta) \mu \psi^2} dm \right)^{1/(p+\delta)} \leq C \left( \frac{1}{|I|} \int_I e^{(1 + \delta) \mu \psi^2} dm \right)^{1/(p(1+\delta))}
\]

for all subarcs \( I \) of \( \mathbb{T} \) (see [8, Corollary VI.6.10]). Let \( r := (1 + \delta)/\delta \). Using Hölder's inequality we obtain that

\[
\left( \frac{1}{|I|} \int_I e^{-\mu \psi^2} dm \right)^{1/p} \leq C \left( \frac{1}{|I|} \int_I e^{\mu \psi^2} dm \right)^{1/(pr)} \times \left( \int_\mathbb{T} e^{\mu \psi^2} dm \right)^{1/(p+1)}
\]

for all subarcs \( I \) of \( \mathbb{T} \) (see [8, Corollary VI.6.10]). Let \( r := (1 + \delta)/\delta \). Using Hölder's inequality we obtain that

\[
\left( \frac{1}{|I|} \int_I e^{\mu \psi^2} dm \right)^{1/p} \leq C \left( \frac{1}{|I|} \int_I e^{\mu \psi^2} dm \right)^{1/(pr)} \times \left( \int_\mathbb{T} e^{\mu \psi^2} dm \right)^{1/(p+1)}
\]
and the latter two integrals are finite if only
\[ \| \phi \|_\infty \leq \alpha < \pi/(r \max(p, q)) \]
(see [12, Theorem V.D.1°]). Hence, if \( \alpha \) is chosen small enough, then \( e^{-(\alpha + \varphi)/2} \in A_\infty \), which proves our claim.

Since \( \chi^0 | A = 1 \) and \( \chi^0 | \Gamma = e^{\varphi_0 g_\theta} \), we deduce from Lemma 5.3 that \( \chi^0 a^\theta | A = 1 \) and \( \chi^0 a^\theta | \Gamma = e^{\varphi_0 g_\theta | \Gamma} \). Choose any \( \sigma \) between 0 and \( \min(p/2, q/2) \), define \( \theta_0 \) as in Section 5.8, and put
\[ \lambda := \frac{\pi/\sigma}{\alpha/2 + \pi/\sigma} \theta_0. \]

Clearly \( 0 < \lambda < \theta_0 \), and Lemma 5.10 implies that
\[ 0 < \text{dist}(\zeta, g_\lambda(M_0^0(QC))) \leq (\pi/\sigma)(\theta_0 - \lambda) = \alpha \lambda/2. \]

Hence, there is an \( \zeta_0 \in [-\pi/2, \pi/2] \) such that \( \zeta \) is an endpoint of the (closed) arc \( e^{i\zeta_0 \lambda} g_\lambda(M_0^0(QC)) \). So
\[ \zeta = e^{2\pi i/q} \in e^{i\zeta_0 \lambda} g_\lambda(M_0^0(QC)), \]
i.e., \( \chi^0 a^\lambda \notin \mathcal{B}_\zeta(p) \), whereas
\[ e^{2\pi i/q'} \notin e^{i\zeta_0 \lambda} g_\lambda(M_0^0(QC)) \]
for all \( q' \) in some one-sided neighborhood of \( q \), which means that \( \chi^0 a^\lambda \notin \mathcal{B}_\zeta(p') \) for all \( p' \) in some one-sided neighborhood of \( p \), say \( (p - \varepsilon, p) \).

Since \( T(\chi^0 a^\lambda) \) was shown to be invertible on \( H^p \), this contradicts Lemma 4.7 and completes the proof of part (a) of Theorem 3.2 for \( p = 1 \).

To extend the result for \( p = 1 \) to the case where a weight is present one may proceed as in the proof of Lemma 4.6. Finally, since \( a \in G_{PQC} \) whenever \( a \in PQC \) and \( T(a) \) is Fredholm on \( H^p(p) \), (the necessity portion of) part (b) is an immediate consequence of part (a). 

6. Toeplitz Operators with Prescribed Fredholm Domain

6.1. Let \( M \) be a (finite or infinite) subset of \( \mathbb{Z} \) consisting of consecutive integers. Suppose we are given a family \( \{I_n\}_{n \in M} \) of disjoint nonempty open subintervals of \( (1, \infty) \) such that \( I_n \) lies on the left of \( I_m \) for \( n < m \) and such that 1 and \( \infty \) are the only possible accumulation points of the family \( \{I_n\} \).

Also, let \( \{\kappa_n\}_{n \in M} \) be a family of integers with the property that \( \kappa_n > \kappa_m \) whenever \( n < m \).

6.2. Theorem. Let \( \{I_n\}_{n \in M} \) and \( \{\kappa_n\}_{n \in M} \) be as in Section 6.1. Then there exists a function \( a \in PQC \) such that \( \Phi(a) = \bigcup_{n \in M} I_n \) and the index of \( T(a) \) on \( H^p \) equals \( \kappa_n \) for \( p \in I_n \).
Proof. Given any closed subarc $G$ of $\mathbb{T}$ and a point $\tau \in \mathbb{T}$, there exists a unimodular function $g \in QC$ such that $g(M^0_{\tau}(QC)) = G$. To see this recall the proof of Lemma 5.6 and note that there is a unimodular function $\psi \in SO[0, 1]$ such that the spectrum of $\psi + C_0[0, 1)$ equals $G$. Then the function $g$ defined by $g(e^{ix}) = \psi(1 - |x|/\pi)$ $(0 < |x| < \pi)$ belongs to $QC$ and $g(M^0_{\tau}(QC))$ coincides with $G$ (see [18, p. 823]). Using the function $g$ just constructed it is easy to see that if $\Gamma = \{ e^{ix} : \alpha < x < \beta \}$ $(\beta - \alpha < 2\pi)$ is any open and $G$ any closed subarc of $\mathbb{T}$, there is a unimodular function $f \in PQC$ such that $f|\Gamma$ is continuous, $f|\Gamma = h|\Gamma$ for some $h \in QC$, $\lim_{x \to \beta - 0} h(e^{ix}) = 1$, $h(M^0_{\tau}(QC)) = G$.

To avoid unessential complications, assume that $M = \{1, 2, 3, \ldots\}$ and that the left endpoint of $I_1$ is not 1; all other possible cases can be treated similarly. Let $I_n := (\gamma_n, \delta_n)$ and put $\Omega_0 = [1, \gamma_1]$, $\Omega_n = [\delta_n, \gamma_{n+1}]$ $(n \geq 1)$. Then denote by $G_n$ $(n \geq 0)$ the closed arc of $\mathbb{T}$ consisting of all points $e^{2\pi i/q}$ such that $p \in \Omega_n$ $(1/p + 1/q = 1)$, and let $\Delta_n$ $(n \geq 1)$ denote the open subarc of $\mathbb{T}$ constituted by the points $e^{2\pi i/q}$ for which $p \in I_n$. Finally, choose any points $\tau_n \in \Delta_n$ $(n \geq 1)$, put $\tau_0 = 1$, and let $\Gamma_n$ $(n \geq 0)$ stand for the open arc between $\tau_n$ and $\tau_{n+1}$.

By what has been said in the first paragraph of the proof, there exists a unimodular function $b \in PQC$ with the following properties: for all $n \geq 0$, $b$ is continuous in $\Gamma_n$, $b(t) \to 1$ as $t \to \tau_{n+1} - 0$, there is a function $h_n \in QC$ such that $b|\Gamma_n = h_n|\Gamma_n$ and $h_n(M^0_{\tau_n}(QC)) = G_n$.

It is immediate from Theorem 3.2(b) that $\Phi(b) = I_1 \cup I_2 \cup \cdots$. Let $c \in PC$ be any unimodular function subject to the following conditions: for all $n \geq 0$, $c$ is continuous in $\Gamma_n$, $c(\tau_{n+1} - 0) = 1$, $c(\tau_n + 0) \in G_n$. Using the sufficiency portion of Theorem 3.2(b) it is easy to see that $b$ and $c$ are homotopic to each other within the class of Fredholm Toeplitz operators on $H^p$ for all $p \in \Phi(b)$. Hence $T(b)$ and $T(c)$ have the same index on $H^p$ for all $p \in \Phi(b)$. Simple application of Gohberg–Krupnik theory gives that there is an integer $\mu_0$ such that the index of $T(c)$ (and thus the one of $T(b)$) equals $\mu_0 - n$ for $p \in I_n$ $(n \geq 1)$.

Once again using Gohberg–Krupnik theory, it is not difficult to produce a function $d \in PC$ owning the following properties: $\Phi(d) \subset \Phi(d)$, the index of $T(d)$ on $H^p$ $(p \in I_n, n \geq 1)$ equals $\kappa_n + n - \mu_0$ (note that $\kappa_n + n - \mu_0 \geq \kappa_{n+1} + (n+1) - \mu_0$), and $d$ is continuous at $\tau_n$ $(n \geq 0)$. It follows that $a = bd$ is in $PQC$, that $\Phi(a) = I_1 \cup I_2 \cup \cdots$, and that the index of $T(a)$ on $H^p$ $(p \in I_n, n \geq 1)$ equals $(\mu_0 - n) + (\kappa_n + n - \mu_0) - \kappa_n$.

References