Linear Algebra and its Applications 432 (2010) 173–179

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Joint numerical range and its generating hypersurface

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ARTICLE INFO

Article history:
Received 18 September 2008
Accepted 27 July 2009
Available online 22 August 2009

Submitted by R.A. Brualdi

AMS classification:
15A60
32S25

Keywords:
Joint numerical range
Hypersurface
Singular point

ABSTRACT

In this paper, we study the joint numerical range of m-tuples of Hermitian matrices via their generating hypersurfaces. An example is presented which shows the invalidity of an analogous Kippenhahn theorem for the joint numerical range of three Hermitian matrices.

1. Introduction

Let T be an $n \times n$ complex matrix. The classical numerical range of T is defined as the set

$$W(T) = \{\xi^* T\xi : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}.$$

The numerical range $W(T)$ provides various information on the structure of the matrix T and localization of the eigenvalues of T (cf. [8]). One of the important generalizations of classical numerical range is the joint numerical range. Suppose that n and m are positive integers and $(H_1, H_2, \ldots, H_m)$ is an ordered m-tuple of $n \times n$ Hermitian matrices. The *joint numerical range* of $H_1, H_2, \ldots, H_m$ is defined as the set

$$W(H_1, H_2, \ldots, H_m) = \{(\xi^* H_1\xi, \xi^* H_2\xi, \ldots, \xi^* H_m\xi) \in \mathbf{R}^m : \xi \in \mathbf{C}^n, \xi^*\xi = 1\}.$$

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1 Supported in part by Taiwan National Science Council.

0024-3795/$ - see front matter
doi:10.1016/j.laa.2009.07.034

The joint numerical range is not necessarily convex (cf. [6,11–13]). If $m = 2$ then the range is convex (cf. [7,15]). It is also known that if $m = 3$ and $n \geqslant 3$, the range $W(H_1, H_2, H_3)$ is convex too(cf. [2,3]). The structure of the joint numerical range $W(H_1, H_2, (H_1 + iH_2)^*(H_1 + iH_2))$ is closely related with the so called q-numerical range of the $n \times n$ matrix $T = H_1 + iH_2$ (cf. [5]).

We introduce a homogeneous polynomial

$$\begin{aligned} F(y_0, y_1, y_2, \ldots, y_m) &= F(y_0, y_1, y_2, \ldots, y_m : H_1, H_2, \ldots, H_m) \\ &= \det(y_0 I_n + y_1 H_1 + y_2 H_2 + \cdots + y_m H_m) \end{aligned} \tag{1}$$

associated with the m-tuple of Hermitian matrices $(H_1, H_2, \ldots, H_m)$. This form is *hyperbolic* with respect to the point $(1, 0, 0, \ldots, 0) \in \mathbf{R}^{m+1}$ (cf. [1]), that is,

(i) $F(1, 0, \ldots, 0) \neq 0$.
(ii) Every root of the equation $F(t, y_1, y_2, \ldots, y_n) = 0$ in t is real for an arbitrary fixed $(y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$.

Suppose that

$$F(y_0, y_1, y_2, \ldots, y_m) = \prod_{j=1}^{p} F_j(y_0, y_1, y_2, \ldots, y_m)^{m_j},$$

and

$$F_0(y_0, y_1, y_2, \ldots, y_m) = \prod_{j=1}^{p} F_j(y_0, y_1, y_2, \ldots, y_m), \tag{2}$$

are respectively the irreducible decomposition and reduced polynomial of the form F in the polynomial ring $\mathbf{C}[y_0, y_1, \ldots, y_m]$, where F_j are mutually distinct irreducible factors and m_j are their multiplicities. It is known that each factor F_j has a non-zero scalar c_j for which $c_j F_j$ is a real polynomial. Hence we may assume that $F_j^{\prime} s$ are real polynomials. It is also known that all factors F_j are hyperbolic with respect to $(1, 0, \ldots, 0)$. We consider the algebraic variety

$$S_F = S_{F_0} = \{[(y_0, y_1, \ldots, y_m)] \in \mathbf{CP}^m : F_0(y_0, y_1, \ldots, y_m) = 0\},$$

where $[(y_0, y_1, \ldots, y_m)]$ is the equivalence class containing $(y_0, y_1, \ldots, y_m) \in \mathbf{C}^{m+1} - (0, \ldots, 0)$ under the relation $(y_0, y_1, \ldots, y_m) \sim (z_0, z_1, \ldots, z_m)$ if $(y_0, y_1, \ldots, y_m) = k(z_0, z_1, \ldots, z_m)$ for some nonzero complex number k. The dual surface $S_F^{\wedge}$ of the form F (1) is the set of points $(x_0, x_1, x_2, \ldots, x_m) = (1, x_1, x_2, \ldots, x_m)$ for which the hyperplane $y_1 x_1 + y_2 x_2 + \cdots + y_m x_m + y_0 x_0 = 0$ is tangent to S_F at a non-singular point of S_F. The real affine part $S_F^{\wedge}(\mathbf{R})$ of $S_F^{\wedge}$ is called the *boundary generating hypersurface* of $W(H_1, H_2, \ldots, H_m)$. The main aim of this paper is the treatment of the joint numerical range $W(H_1, H_2, \ldots, H_m)$ via the hypersurface S_F and the boundary generating hypersurface $S_F^{\wedge}$.

2. Boundary generating hypersurface

Let $F(y_0, y_1, y_2, \ldots, y_m)$ be an irreducible homogeneous polynomial. A point $(1, x_1, \ldots, x_m) \in \mathbf{R}^{m+1}$ is a singular real point of the surface S_F if

$$F(1, x_1, x_2, \ldots, x_m) = F_{y_i}(1, x_1, x_2, \ldots, x_m) = 0,$$

where F_{y_i} denotes the partial derivative of F with respect to y_i, $i = 0, 1, \ldots, m$.

At first, we obtain a result for $m = 2$.

Theorem 2.1. *Let $F_0(y_0, y_1, y_2)$ be the reduced real form (2) for $m = 2$. If (a_0, a_1, a_2) is a real singular point of the curve S_{F_0} then the curve S_{F_0} is expressed as the union of analytically parametrized arcs*

$$(a_0^{(j)}, a_1^{(j)}, a_2^{(j)}) = (f_j(t), g_j(t), h_j(t)), \quad j = 1, 2, \ldots, \ell, \ |t| < \epsilon$$

near the points (a_0, a_1, a_2) *satisfying*

$$(f_j(0), g_j(0), h_j(0)) = (a_0, a_1, a_2) \quad \text{and} \quad (f_j'(0), g_j'(0), h_j'(0)) \neq (0, 0, 0).$$

Proof. We prove this theorem by the Newton–Puiseux method. By a real transformation, we may assume that $(a_0, a_1, a_2) = (a_0, 1, 0)$ and $F(y_0, 1, 0) = (y_0 - a_0)^\ell (y_0 - t_1) \cdots (y_0 - t_q)$ for some real numbers $t_j \neq a_0$, $j = 1, \ldots, q$. By the Newton–Puiseux method, we solve the equation

$$F(y_0, 1, y_2) = y_0^{\ell+q} + f_1(y_2) y_0^{\ell+q-1} + \cdots + f_{\ell+q}(y_2) = 0$$

in y_0. Then the solutions are expressed in Puiseux series. We are interested in ℓ solutions corresponding to $y_0 = a_0$ for $y_2 = 0$. Each of the ℓ solutions is expressed as a fractional power series

$$y_0 = g_j(y_2) = a_0 + b_1^{(j)} y_2^{1/p} + b_2^{(j)} y_2^{2/p} + b_3^{(j)} y_2^{3/p} + \cdots, \tag{3}$$

where p is a natural number and $b_k^{(j)}$ are real coefficients (cf. [16, pp. 98–106]). As a function of $y_2^{1/p}$, the series (3) converges absolutely on some disc $|y_2^{1/p}| < \epsilon$. We assume that the greatest common divisor of $\{k \in \mathbf{N} : b_k^{(j)} \neq 0\} \cup \{p\}, j = 1, 2, \ldots, \ell$, is 1. If $p = 1$ for every $1 \leqslant j \leqslant \ell$, then by taking the variable $t = y_2$, we have nothing to prove. We assume that $p \geqslant 2$ for some j. Then we have the following equation for every pth root η of 1:

$$F(a_0 + b_1^{(j)} \eta t + b_2^{(j)} \eta^2 t^2 + b_2^{(j)} \eta^3 s^3 + \cdots, 1, t^p) = 0$$

(cf. [16, p. 107]). By the hyperbolicity of F, the series

$$a_0 + b_1^{(j)} \eta t + b_2^{(j)} \eta^2 t^2 + b_2^{(j)} \eta^3 s^3 + \cdots$$

takes real value for every $t \in \mathbf{R}$. By repeating differentiation of this relation with respect to t, it implies that $b_k^{(j)} \exp(i\, 2kh\pi/p) \in \mathbf{R}$ for every $k \in \mathbf{N}$ and $h \in \mathbf{Z}$. Hence $2k$ is a multiple of p for every k with $b_k^{(j)} \neq 0$. We set $\zeta = 1$ if p is odd and $\zeta = 2$ if p is even. Then the above relation implies that p/ζ is a common divisor of $\{k \in \mathbf{N} : b_k^{(j)} \neq 0\}$. By the assumption on the coefficients $b_k^{(j)}$, we have $p = 2$ and $b_{2k-1}^{(j)} \neq 0$ for some $k \in \mathbf{N}$. Under this condition, we obtain that

$$F(a_0 + b_1^{(j)} t + b_2^{(j)} t^2 + b_3^{(j)} t^3 + \cdots, 1, t^2) = 0,$$
$$F(a_0 + i b_1^{(j)} t - b_2^{(j)} t^2 - i b_3^{(j)} t^3 + \cdots, 1, (it)^2) = 0$$

for every $t \in \mathbf{R}$. By the hyperbolicity of F, we have $b_k^{(j)} \in \mathbf{R}$ and $i^k b_k^{(j)} \in \mathbf{R}$ for every k, and hence $b_{2k-1} = 0$, a contradiction. Thus we conclude that $p = 1$ for every $1 \leqslant j \leqslant \ell$. $\square$

Theorem 2.1 is related to Rellich's theorem (cf. [9,14]). However the above proof does not depend on the properties of Hermitian matrices.

For general m and n, we consider the convex hull $\operatorname{conv}(W(H_1, H_2, \ldots, H_m))$ of the compact set $W(H_1, H_2, \ldots, H_m)$. By the separation theorem for compact convex sets, we have that

$$\begin{aligned} &\operatorname{conv}(W(H_1, H_2, \ldots, H_m)) \\ &\quad = \{(x_1, x_2, \ldots, x_m) \in \mathbf{R}^m : c_1 x_1 + c_2 x_2 + \cdots + c_m x_m \leqslant g(c_1, c_2, \ldots, c_m), \\ &\qquad (c_1, c_2, \ldots, c_m) \text{ is a unit vector in } \mathbf{R}^m\}, \end{aligned}$$

where

$$\begin{aligned} &g(c_1, c_2, \ldots, c_m) \\ &\quad = \max\{c_1 y_1 + c_2 y_2 + \cdots + c_m y_m : (y_1, y_2, \ldots, y_m) \in W(H_1, H_2, \ldots, H_m)\} \\ &\quad = \max\{\xi^* (c_1 H_1 + c_2 H_2 + \cdots + c_m H_m) \xi : \xi \in \mathbf{C}^n, \xi^* \xi = 1\} \\ &\quad = \max \sigma(c_1 H_1 + c_2 H_2 + \cdots + c_m H_m). \end{aligned}$$

The dual set of the convex hull of the joint numerical range is defined and denoted as

$$\begin{aligned}&\mathrm{conv}(W(H_1,H_2,\ldots,H_m))^\wedge\\&\quad=\{(y_1,y_2,\ldots,y_m)\in\mathbf{R}^m : x_1y_1+x_2y_2+\cdots+x_my_m+1\geqslant 0,\\&\qquad(x_1,x_2,\ldots,x_m)\in W(H_1,H_2,\ldots,H_m)\}\\&\quad=\{(y_1,y_2,\ldots,y_m)\in\mathbf{R}^m : \xi^*(y_1H_1+y_2H_2+\ldots+y_mH_m+I_n)\xi\geqslant 0,\quad \xi\in\mathbf{C}^n\}\\&\quad=\{(y_1,y_2,\ldots,y_m)\in\mathbf{R}^m : y_1H_1+y_2H_2+\ldots+y_mH_m+I_n \text{ is positive semidefinite}.\end{aligned}$$

This dual set is a closed convex set, and every point $(y_1,y_2,\ldots,y_m)$ on the boundary of $\mathrm{conv}(W(H_1,H_2,\ldots,H_m))^\wedge$ satisfies

$$\det(I_n+y_1H_1+y_2H_2+\ldots+y_mH_m)=0. \tag{4}$$

We consider the open set

$$\Omega=\{(y_1,y_2,\ldots,y_m)\in\mathbf{R}^m : F(1,y_1,y_2,\ldots,y_m : H_1,H_2,\ldots,H_m)\neq 0\}.$$

The interiors of $\mathrm{conv}(W(H_1,H_2,\ldots,H_m))^\wedge$ are contained in Ω, and by Corollary 3.2 in [1], the set of the interiors coincides with the connected component Ω_0 of Ω containing the origin $(0,\ldots,0)$. Moreover we have that

$$\mathrm{conv}(W(H_1,H_2,\ldots,H_m))^\wedge=\mathrm{closure}(\Omega_0)=\mathrm{conv}(\partial\Omega_0), \tag{5}$$
$$\mathrm{conv}(W(H_1,H_2,\ldots,H_m))=\mathrm{conv}(\partial\Omega_0)^\wedge. \tag{6}$$

These facts provide an algebraic method to determine all supporting hyperplanes of $\mathrm{conv}(W(H_1,\ldots,H_m))$.

Theorem 2.2. *Let $F_0(y_0,y_1,y_2,\ldots,y_m)$ be the reduced real form (2). Suppose that $(a_0,a_1,\ldots,a_m)\in\mathbf{R}^{m+1}$ is a non-singular real point of S_{F_0} with $(a_1,\ldots,a_m)\neq(0,\ldots,0)$ and $\alpha_0y_0+\alpha_1y_1+\cdots+\alpha_my_m=0$ is the equation of the tangent hyperplane of S_{F_0} at this point. Then this hyperplane does not pass through any point of Ω_0.*

Proof. Suppose, on the contrary, the hyperplane passes through a point of Ω_0. Since the form F_0 is hyperbolic with respect to every point of Ω_0 (cf. [1, p. 133]), we may assume that the hyperplane passes through the point $(1,0,\ldots,0)$ by using a real projective transformation. We may also assume that $(a_0,a_1,a_2,\ldots,a_m)=(1,1,0,\ldots,0)$. The line joining the two points $(a_0,a_1,\ldots,a_m)=(1,0,\ldots,0)$ and $(1,1,0,\ldots,0)$ is contained in the tangent hyperplane. Define a polynomial $f(t,y_2,\ldots,y_m)$ by

$$f(t,y_2,\ldots,y_m)=F_0(t,1,y_2,\ldots,y_m).$$

It is obvious that $f(1,0,\ldots,0)=0$ and $f_t(1,0,\ldots,0)=0$. Since $(a_0,a_1,\ldots,a_m)=(1,1,0,\ldots,0)$ is a non-singular point of S_{F_0}, it follows that $f_{y_j}(1,0,\ldots,0)\neq 0$ for some $2\leqslant j\leqslant m$. By using a rotation, we may assume that $j=2$. Then the ternary form $\widetilde{F}(t,y_1,y_2)=F_0(t,y_1,y_2,0,\ldots,0)$ is hyperbolic with respect to $(1,0,0)$, and the point $(a_0,a_1,a_2)=(1,1,0)$ is a non-singular point of $S_{\widetilde{F}}$ and the tangent line of $S_{\widetilde{F}}$ at this point is $y_2=0$. Set $n=\deg(F_0)$. By the hyperbolicity, the equation

$$\tilde{f}(t,y)=F_0(t,1,y,0,\ldots,0)=0$$

in t has n real solutions counting multiplicity for every $y\in\mathbf{R}$. It implies geometrically that the real affine algebraic curve $\tilde{f}(t,y)=0$ and the real line $y=y_0$ interset at n points counting multiplicity for every $y_0\in\mathbf{R}$. By Theorem 2.1, even if the line $y=0$ has singular points of the curve $\tilde{f}(t,y)=0$, the real affine curve $\tilde{f}(t,y)=0$ is expressed as the union of analytic arcs near the singular points. So we can treat such a case in the same fashion. By the assumption

$$\tilde{f}(t,0)=(t-\alpha_1)^{m_1}(t-\alpha_2)^{m_2}\cdots(t-\alpha_k)^{m_k},$$

where $\alpha_1,\alpha_2,\ldots,\alpha_k$ are distinct real numbers and $\alpha_1=1$, the numbers $m_1,m_2,\ldots,m_k$ are positive integers satisfying $m_1\geqslant 2$, $m_1+m_2+\cdots+m_k=n$. If $(t_j,0)$ is a non-singular point of the curve

$\tilde{f}(t,y)=0$, we define two numbers $N_+(m_j)=1$ and $N_-(m_j)=1$ for odd number $m_j, j=1,\ldots,k$. If m_j is even, the implicit function $y=y_j(t)$ defined by $\tilde{f}(t,y)=0$ near $t=\alpha_j$ satisfies

$$\frac{d^{m_j}y}{dt^{m_j}}(\alpha_j)>0, \quad \text{or} \quad \frac{d^{m_j}y}{dt^{m_j}}(\alpha_j)<0.$$

If $\frac{d^{m_j}y}{dt^{m_j}}(\alpha_j)>0$, we set $N_+(m_j)=2$ and $N_-(m_j)=0$. If $\frac{d^{m_j}y}{dt^{m_j}}(\alpha_j)<0$, we set $N_+(m_j)=0$ and $N_+(m_j)=2$. In the case $(t_j,0)$ is a singular point of the curve $\tilde{f}(t,y)=0$, we express the curve $\tilde{f}(t,y)=0$ as the union of analytic arcs, and set $\tilde{m}_j$ the multiplicity of the intersection of the arc and the line $y=0$ at $(t_j,0)$. We define $N_+(m_j), N_-(m_j)$ as the sum of the numbers $\widetilde{N}_+(\tilde{m}_j), \widetilde{N}_-(\tilde{m}_j)$ defined for each arc as in the above fashion. Then the number of the intersection points of the curve $\tilde{f}(t,y)=0$ and the curve $y=y_0$ is

$$N_+(m_1)+N_+(m_2)+\cdots+N_+(m_k) \tag{7}$$

if $y_0>0$ is sufficiently small, and

$$N_-(m_1)+N_-(m_2)+\cdots+N_-(m_k) \tag{8}$$

if $y_0<0$ and $|y_0|$ is sufficiently small. One of the numbers (7) and (8) is strictly less than n, a contradiction to the hyperbolicity of $\widetilde{F}$. □

By the argument used in the proof of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. *Let $F_0(y_0,y_1,y_2)$ be the reduced real form (2) for $m=2$. If (a_0,a_1,a_2) is a non-singular real point of the curve S_{F_0} then any tangent line of the curve S_{F_0} at (a_0,a_1,a_2) does not pass through any point of Ω_0.*

In particular, we improve the result of Theorem 2.2 for $m=2$.

Theorem 2.4. *Let $F_0(y_0,y_1,y_2)$ be the reduced real form (2) for $m=2$. Suppose that $\alpha_0y_0+\alpha_1y_1+\alpha_2y_2=0$ is the common real tangent of a pair of imaginary non-singular point (a_0,a_1,a_2) of the curve S_{F_0} and its conjugate, or $\alpha_0y_0+\alpha_1y_1+\alpha_2y_2=0$ is a real tangent of imaginary singular point (a_0,a_1,a_2) and its conjugate. Then the tangent line does not pass through any point of Ω_0.*

Proof. We may assume that the tangent line passes through the point $(y_0,y_1,y_2)=(1,0,0)$. By a real transformation, we assume that the equation of the tangent line is given by $y_1=0$ and the point of S_{F_0} is $(a_0,0,a_2)$. If $a_0=0$, then the point is given by $(0,0,1)$ which is real, contradicting the assumption. Thus we have that $a_0\neq 0$. Since $F_0(1,0,0)\neq 0$, the coordinate a_2 does not vanish. So we may assume that $a_2=1$ and a_0 is imaginary. But this implies that the equation $F_0(t,0,1)=0$ in t has an imaginary solution which contradicts the hyperbolicity of F_0. □

Corollary 2.5. *Let $F(y_0,y_1,y_2)$ be the polynomial (1) for $m=2$. If $(x_0,x_1,x_2)=(1,x_1,x_2)$ is a real affine point of $S_F^{\wedge}$ for the form $F(y_0,y_1,y_2)$ then the point (x_1,x_2) belongs to the numerical range $W(H_1,H_2)$.*

Proof. If the point (x_1,x_2) does not belong to the compact convex set $W(H_1,H_2)$, then by the duality of the closed convex sets, there exists a point $(\tilde{y_1},\tilde{y_2})$ of the closure of the convex set Ω_0 such that

$$x_1\tilde{y_1}+x_2\tilde{y_2}+1<0.$$

Further, the point $(y_1^{(0)},y_2^{(0)})=(0,0)\in\Omega_0$ satisfies

$$x_1y_1^{(0)}+x_2y_2^{(0)}+1=1.$$

By the convexity of the open set Ω_0, there exists a point $(\hat{y_1},\hat{y_2})$ in the line segment joining the above two points satisfying

$$x_1\hat{y_1} + x_2\hat{y_2} + 1 = 0.$$

The point $(\hat{y_1}, \hat{y_2})$ belongs to Ω_0, which contradicts Corollary 2.3 and Theorem 2.4. □

The result of Corollary 2.5 was obtained by Kippenhahn in [10]. However, its proof is rather intuitive. The proof provided here is more rigorous.

We come back to a general situation. For each irreducible form F_j, we consider the linear reduction of variables. If

$$F_j(y_0, y_1, \ldots, y_{m-1}, y_m) = F_j(y_0 + \alpha_0 y_m, y_1 + \alpha_1 y_m, \ldots, y_{m-1} + \alpha_{m-1} y_m, 0)$$

or equivalently

$$\partial F_j/\partial y_m = \alpha_0 \partial F_j/\partial y_0 + \cdots + \alpha_{m-1} \partial F_j/\partial y_{m-1}.$$

Then the number of essential variables for F_j is less than m. We consider whether there exist non-zero coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_m)$ for which

$$\alpha_0 \partial F_j/\partial y_0 + \cdots + \alpha_m \partial F_j/\partial y_m = 0.$$

Such a reduction example actually appeared in [6]. The dual algebraic object $S_F^\wedge$ of S_F is defined as the union of the dual algebraic varieties $S_{F_j}^\wedge$. Each algebraic variety

$$S_{F_j}^\wedge = \{[(x_0, x_1, \ldots, x_m)] \in \mathbf{CP}^m : G_{j,k}(x_0, x_1, \ldots, x_m) = 0,\ k = 0, 1, 2, \ldots, \ell_j\}$$

is characterized by irreducible form $G_{j,0}$ and linear forms $G_{j,1}, \ldots, G_{j,\ell_j}$ satisfying

$$G_{j,k}(a_0, a_1, \ldots, a_m) = 0, \quad k = 0, 1, \ldots, \ell_j$$

for every tangent hyperplane $a_0 y_0 + a_1 y_1 + \cdots + a_m y_m = 0$ at a non-singular point of S_{F_j}.

Theorem 2.6. *Let $F(y_0, y_1, y_2, \ldots, y_m)$ be the polynomial (1). Suppose that $(x_0, x_1, \ldots, x_m) = (1, x_1, \ldots, x_m)$ is a real affine point of $S_F^\wedge$ for which $x_1 y_1 + \cdots + x_m y_m + 1 = 0$ is a tangent hyperplane of S_F at some non-singular real point of S_F. Then the point $(x_1, \ldots, x_m)$ belongs to the convex hull of $W(H_1, \ldots, H_m)$.*

Proof. If the point $(x_1, \ldots, x_m)$ does not belong to the convex hull of $W(H_1, \ldots, H_m)$, by the duality of the closed convex sets, there exists a point $(\tilde{y_1}, \ldots, \tilde{y_m})$ of the closure of the convex set Ω_0 such that

$$x_1\tilde{y_1} + \cdots + x_m\tilde{y_m} + 1 < 0.$$

Further, the point $(y_1^{(0)}, \ldots, y_m^{(0)}) = (0, \ldots, 0) \in \Omega_0$ satisfies

$$x_1 y_1^{(0)} + \cdots + x_m y_m^{(0)} + 1 = 1.$$

By the convexity of the open set Ω_0, there exists a point $(\hat{y_1}, \ldots, \hat{y_m})$ in the line segment joining the above two points satisfying

$$x_1\hat{y_1} + \cdots + x_m\hat{y_m} + 1 = 0.$$

The point $(\hat{y_1}, \ldots, \hat{y_m})$ belongs to Ω_0, which contradicts Theorem 2.2. □

3. Example

If the polynomial F is a non-linear irreducible form and the hypersurface S_F has no singular point, then $S_F^\wedge$ is defined by a single form $G \in \mathbf{C}[x_0, x_1, \ldots, x_m]$ and its degree is $n(n-1)^{m-1}$ (cf. [4, p. 253]). In this case, the multiplicity of the maximal eigenvalue of the Hermitian matrix $y_1 H_1 + y_2 H_2 + \cdots + y_m H_m$ is 1 for every unit vector $(y_1, y_2, \ldots, y_m) \in \mathbf{R}^m$. Provided that $n \geqslant 3$, it implies the convexity of the joint numerical range $W(H_1, H_2, \ldots, H_m)$ by Theorem 5.1 in [6] and the range coincides with the closed domain surrounded by the boundary generating hypersurface $S_F^\wedge(\mathbf{R})$.

Suppose that $(x_0, x_1, x_2, x_3) = (1, x_1, x_2, x_3)$ is a real affine point of $S_F^\wedge$ and (x_0, x_1, x_2, x_3) lies on the hyperplane $y_1x_1 + y_2x_2 + y_3x_3 + y_0x_0 = 0$ where (y_0, y_1, y_2, y_3) is a real singular point of S_F. It may occur that the point (x_1, x_2, x_3) does not belong to the convex hull of $W(H_1, H_2, H_3)$. We provide such an example in the below, which shows an analogous property of Kippenhahn's result does not hold for $m = 3$.

Example. Let

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \text{and} \quad H_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the form $F(y_0, y_1, y_2, y_3)$ (1) associated to these Hermitian matrices is given by

$$F(y_0, y_1, y_2, y_3) = y_0^3 + y_0^2y_3 - 2y_0y_1^2 - y_0y_2^2 - y_1^3 - y_1^2y_3 + y_1y_2^2.$$

The cubic surface S_F has a biplanar double point at $(y_0, y_1, y_2, y_3) = (0, 0, 0, 1)$ and an ordinary double point at $(y_0, y_1, y_2, y_3) = (1, -1, 0, -1/2)$ (cf. [4]). The boundary of the joint numerical range has a flat portion on the plane

$$-x_1 - \frac{1}{2}x_3 + 1 = 0$$

and its projection on the (x_1, x_2) plane is the elliptical disc bounded by the curve

$$20x_1^2 - 32x_1 + x_2^2 + 12 = 0.$$

However the plane $x_3 = 0$ supports the range $W(H_1, H_2, H_3)$ and its intersection with the range is not a single point, it is a line segment

$$\{(x_1, 0, 0) : -1 \leqslant x_1 \leqslant 1\}.$$

One end point $(1, 0, 0)$ of the above line belongs also to the flat portion. The equation of the dual surface of the cubic surface S_F is given by

$$\begin{aligned} G(1, x_1, x_2, x_3) = {} & 20x_3^4 - 8x_1x_3^3 - 24x_3^3 + 4x_1^2x_3^2 + 8x_2^2x_3^2 + 8x_1x_3^2 \\ & + 4x_3^2 - 4x_1x_2^2x_3 - 4x_2^2x_3 + x_2^4. \end{aligned}$$

Since $G(1, x_1, x_2, 0) = x_2^4$, the line $x_2 = 0$ on the plane $x_3 = 0$ is contained in the quartic surface $S_F^\wedge(\mathbf{R})$. Thus this surface contains a point $(x_1, x_2, x_3) = (2, 0, 0) \notin W(H_1, H_2, H_3)$.

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