Almost Primes in Short Intervals

H.-Q. Liu

206-10, Bao Guo Street, Harbin, 150066, People's Republic of China

Communicated by R. C. Vaughan

Received June 18, 1994; revised February 20, 1995

It is shown that the number of integers \( n \) for which \( n \in (x - x^\theta, x] \) and \( n \) having at most two prime factors (counting multiplicity) is greater than \( cx^\theta \log x \), here \( x \) is a sufficiently large positive number, \( c \) is some effective and positive constant, and \( \theta = \frac{1}{2} \).

INTRODUCTION

Let \( x \) be a large positive number. To find an as small as possible number \( \theta > 0 \) such that the interval \( (x - x^\theta, x] \) always contains an almost prime \( P \) is one of the important problems connected with the application of sieve methods, and it has been attacked, chronologically, in Wang [13], Jurkat and Richert [7], Richert [12], Chen [1], and Laborde [8] before the 1980s. In 1981, Halberstam et al. [3] made advances in both the exponential sum method and the sieve method by showing that \( \theta = 0.455 \) is admissible. Iwaniec and Laborde [6] sharpened this result to \( \theta = 0.45 \) by more powerful exponential sum methods. In 1985 Halberstam and Richert [4] further attained \( \theta = 0.4476 \) by putting the exponential sum result of [6] into the Greaves weighted sieve. Since the research of [3], a crucial role to this problem is played by an estimate for \( S(H, M, N) \) to the effect that

\[
S(H, M, N) \ll MNx^{-\theta},
\]

with \( MN \) as large as possible, where

\[
S(H, M, N) = \sum_{h-H} \sum_{m-M} \sum_{n-N} a_m b_n e \left( \frac{hx}{mn} \right),
\]

\( |a_m| \leq 1, \ |b_n| \leq 1, \ e(\xi) = \exp(2\pi i \xi) \) for a real \( \xi, \ M, N, H \geq 1, \ H \leq MNx^{-\theta+\delta} \) is a sufficiently small positive number, \( h \sim H \) means \( 1 < h/H < 2 \), and so on. The sum \( S(H, M, N) \) comes from the bilinear form remainders in the Rosser–Iwaniec sieve (cf. [5]). Applying an inequality of Bombieri and Iwaniec (cf. Lemma 1.7), Fouvry and Iwaniec gave a new treatment for \( S(H, M, N) \) in 1989, this enabled Fouvry [2] to get \( \theta = 0.4436... \). Replacing Laborde's weighted sieve used in [2] by the Greaves

303
weighted sieve and giving refined estimates for exponential sums, Wu [14] succeeded with $\theta = 0.44$.

We prove the following better result.

**Theorem.** $\theta = 0.436$ is admissible.

The saving comes entirely from a new estimate for $S(H, M, N)$, as compared with [14]. To deduce our new estimate, we base our argument on the method in the recent paper [11]. The innovation here is to apply Theorem 2 of [10] and its analogy (cf. Lemmas 1.8 and 1.1) simultaneously, after an appeal to Lemma 1.7. We note that our result can be improved if the term $Q^{8/3}$ in Lemma 1.8 can be reduced.

1. **An Estimate for $S(H, M, N)$**

In this section we prove

**Lemma 1.** Suppose $M \leq x^{\frac{1}{4}}$ and $N \leq (y^{22} x^{-25})^{1/70} x^{-\frac{1}{4}}$, $y = x^{0.436}$, then $S(H, M, N) \leq MN^{-\frac{1}{4}}$.

We first prove the following analogy of Theorem 2 of [10].

**Lemma 1.1.** Let $B(M, N, r, \Lambda)$ be the number of lattice points $(m, m_1, n, n_1)$ such that $m, m_1 \sim M, n, n_1 \sim N$ and

$$|t(m, n) - t(m_1, n_1)| < \Lambda,$$

where $t(m, n) = m^n(n + r)^{\beta} - (n - r)^{\beta}$, $r$ is a fixed positive integer with $r \sim R \leq x^{1/4}, \alpha$ and $\beta$ are real with $\alpha \beta (\beta - 1) \neq 0$, $\Lambda < 1$ and $T = M^3 N^{3/2} R$.

Then

$$B(M, N, r, \Lambda) \leq (MN + AM^2 N^2 + M^2 R^{2/3})(\log(4MN))^4,$$

the implied constant relying at most on $\alpha, \beta$, and $\epsilon$.

**Proof.** Let $B_1(M, N, r, A_1)$ be the number of lattice points $(m, m_1, n_1)$ such that $m, m_1 \sim M, n_1 \sim N$ and

$$\|An_1 + Br^2 n_1^{-1} + Cr^4 n_1^{-3}\| \leq \epsilon^{-1} A_1, \quad A_1 = AN + R^6 N^{-5},$$

where $\|\xi\| = \min_{n \in \mathbb{Z}} |\xi - n|$ for a real $\xi$, and, for $\gamma = \alpha / (\beta - 1)$,

$$A = (m_1 m^{-1})^{\gamma}, \quad B = \alpha \gamma \gamma^2 (m_1 m^{-1})^{\gamma} - (mm_1^{-1} \gamma),$$

$$C = \alpha \gamma^2 (m_1 m^{-1})^{2\gamma} + \delta_{2}^4 (mm_1^{-1} \gamma) - (mm_1^{-1} \gamma).$$
where \( d_1 \) and \( d_2 \) are the coefficients given by the Taylor expansion
\[
\frac{(1 + u)^\beta - (1 - u)^\beta}{2\beta u} = 1 + d_1 u^2 + d_2 u^4 + \cdots, \quad 0 < u < 1.
\]

Then we claim that
\[
B(M, N, r, A) \ll B_1(M, N, r, A_1). \tag{1.1}
\]

To verify (1.1) we can assume that \( AN \) is small, for otherwise the required estimate in Lemma 1.1 follows at once from
\[
|\tau(m, n) - \tau(m_1, n_1)| < AT. \tag{1.2}
\]

From (1.2) and the Taylor expansion we get
\[
n\left(1 + d_1 \left(\frac{r}{m}\right)^2 + d_2 \left(\frac{r}{m}\right)^4\right) - \left(\frac{m_1}{m}\right) n_1 \left(1 + d_1 \left(\frac{r}{n_1}\right)^2 + d_2 \left(\frac{r}{n_1}\right)^4\right) \ll A_1. \tag{1.3}
\]

From (1.3) we have
\[
n = \left(\frac{m_1}{m}\right)^\gamma n_1 (1 + O(AN + R^2 N^{-2})), \tag{1.4}
\]

and
\[
n - \left(\frac{m_1}{m}\right)^\gamma n_1 + d_1 r^2 \left(n^{-1} - \left(\frac{m_1}{m}\right)^\gamma n_1^{-1}\right) = O(AN + R^4 N^{-3}). \tag{1.5}
\]

By substituting (1.4) into (1.5) we get a more precise expansion
\[
n = \left(\frac{m_1}{m}\right)^\gamma n_1 - d_1 r^2 n_1^{-1} \left(\frac{m}{m_1}\right)^\gamma - \left(\frac{m_1}{m}\right)^\gamma\right) + O(AN + R^4 N^{-3}). \tag{1.6}
\]

Now we can use (1.4) to expand \( d_1 r^2 n^{-1} \) and use (1.6) to expand \( d_1 r^2 n_1^{-1} \), thereby obtaining, in view of (1.3), the estimate
\[
n - An_1 - Br^2 n_1^{-1} - Cr^4 n_1^{-3} \ll A_1,
\]

and (1.1) follows in view of the fact that \( A_1 \) is small. Let \( A_2 = AN + N^{-1} R^{2/3} \). Clearly we have
\[
B_1(M, N, r, A_1) \leq B_1(M, N, r, A_2). \tag{1.7}
\]
By virtue of the identity (\([S]\) is the integral part of \(S\) and \(\{S\} = S - [S]\))

\[
\sum_{|s| < S} \left(1 - \frac{|s|}{S}\right) e(s\xi) = \frac{1 - \{S\}}{S} \left(\frac{\sin \pi_s [S]}{\sin \pi_s}\right)^2 + \frac{\{S\}}{S} \left(\frac{\sin \pi_s [S + 1]}{\sin \pi_s}\right)^2,
\]

it is easy to see that for each given pair \((m, m_1)\), the number of lattice points counted in \(B_1(M, N, r, A_2)\) is

\[
es^{-1} \sum_{1 \leq r < S} \sum_{1 \leq s < N} e(Asn + Br^2sn^{-1} + Cr^3sn^{-3}) = A_2N, \tag{1.8}
\]

where \(S = e(4A_2)^{-1}\). By our assumption, the innermost sum in (1.8) is

\[
I = \int_{\mathbb{R}^{2N}} e(\pm \|As\| \xi + Br^2s\xi^{-1} + Cr^3s\xi^{-3}) d\xi + O(1) = I + O(1), \tag{1.9}
\]

by the truncated Possion summation. If \(\|As\| \geq 3s |B| N^{-2}r^2\), then

\[
I \ll \|As\|^{-1}, \tag{1.10}
\]

by a partial integration; and if \(\|As\| < 3s |B| N^{-2}r^2\), we use the second derivative estimate to get

\[
I \ll (R^2 |B| sN^{-3})^{-1/2}, \tag{1.11}
\]

where we have observed the fact that \(|C| \ll |B|\). Gathering (1.8) to (1.11) we conclude that

\[
B_1(M, N, r, A_2) \ll A_2NM^2 + E_1(M, N, A_2) + E_2(M, N, A_2), \tag{1.12}
\]

where

\[
E_1(M, N, A_2) = A_2 \sum_{1 \leq r < S} \sum_{m, m_1 - M} \min \left(N, \frac{1}{\|As\|}\right),
\]

\[
E_2(M, N, A_2) = A_2 \sum_{1 \leq r < S} \sum_{m, m_1 - M} \min \left(N, (sR^3 |B|)^{-1/2} N^{3/2}\right).
\]

Then, similarly as (12), (13) of [10], we can deduce that

\[
E_1(M, N, A_2) \ll (MN + M^2)(\log (4MN))^4, \tag{1.13}
\]

\[
E_2(M, N, A_2) \ll (MN + RM^2 N^{-1/2} A_2^{-1/2})(\log (4MN))^4. \tag{1.14}
\]

Lemma 1.1 follows from (1.1), (1.7), (1.12), (1.13), and (1.14).
The following lemmas will also be used in proving Lemma 1.

**Lemma 1.2.** Let \( M > 0, N > 0, u_m > 0, v_n > 0, A_m > 0, B_n > 0 \) for \( 1 \leq m \leq M, 1 \leq n \leq N \), and let \( Q_1 \) and \( Q_2 \) be given non-negative numbers, \( Q_1 < Q_2 \). Then there is an \( q \) such that \( Q_1 < q < Q_2 \) and

\[
\sum_{m=1}^{M} A_m q^u + \sum_{n=1}^{N} B_n q^v \leq \sum_{m=1}^{M} \left( A_m^u B_n^v \right)^{1/(u+v)} + \sum_{m=1}^{M} A_m Q_1^u + \sum_{n=1}^{N} B_n Q_2^v.
\]

**Lemma 1.3.** Let \( I \) be a subinterval of \((Y, 2Y)\) and \( J \) be a positive integer. Then for any complex numbers \( z_n(Y < n \leq 2Y) \), there holds

\[
\left| \sum_{n \in I} z_n \right|^2 \leq 2(1 + YJ^{-1}) \sum_{1 \leq |s| + 1 < J} (1 - (2 |s| + 1) J^{-1}) \sum_{n \in I} z_{n+s} z_{n-s}.
\]

**Lemma 1.4.** Let \( M \leq N < N_1 \leq M_1 \), and let \( a_n \) be any complex numbers. Then

\[
\left| \sum_{N < n \leq N_1} a_n \right| \leq \int_{-\infty}^\infty K(t) \left| \sum_{M < n \leq M_1} a_n e(tn) \right| dt,
\]

where \( K(t) = \min(M_1 - M + 1, (\pi |t|)^{-1}, (\pi t)^{-2}) \), and

\[
\int_{-\infty}^\infty K(t) \ dt \leq 3 \log(2 + M_1 - M).
\]

**Lemma 1.5.** Suppose that \( 0 < a < b \leq 2a \) and \( R \) is an open convex set in \( C \) containing the real segment \([a, b]\). Suppose further that \( f(z) \) is analytic on \( R \), \( f(x) \) is real for real \( x \) in \( R \), \( |f''(z)| \leq M \) for \( z \in R \) and there is a constant \( k > 0 \) such that \( f''(x) \leq -kM \) for all real \( x \) in \( R \). Let \( \alpha = f'(b), \beta = f'(a) \), and define \( x_v \) for each integer \( v \) in the range \( \alpha < v < \beta \) by \( f(x_v) = v \). Then

\[
\sum_{\alpha < x < \beta} e(f(x_v)) = e(-1/8) \sum_{\alpha < x < \beta} |f''(x_v)|^{-1/2} e(f(x_v) - vx_v) + O(M^{-1/2} \log(2 + (b - a)M)).
\]

**Lemma 1.6.** Let \( n, n' \) be positive integers with \( (n, n') = 1 \) and \( n, n' \sim N = N/d \). Let \( Q = aH/bN \geq 100 \), here \( a, b, d \) are positive integers, and \( a, b, H, N \ll x' \), \( c \) is some positive number. For \( \psi(\xi) = \xi - [\xi] - \frac{1}{2} (\xi \text{ is any}}
real number and \([ \xi ] \) is its integral part), any integer \( r \) with \( |r| \leq 8HN \), and any real \( t \), we define

\[
\omega_d(n, n'; r) = \sum_{h=1}^{\infty} \sum_{h'=-H}^{H} \psi \left( \frac{OhN}{dhH} \right) 10^{-1/2},
\]

\[
\tilde{\omega}_d(n, n'; t) = \sum_{|m| \leq 8HN} \omega_d(n, n'; m) e(-mt).
\]

Then

\[
\omega_d(n, n'; r) = \int_0^1 \tilde{\omega}_d(n, n'; t) e(tr) \, dt,
\]

\[
\int_0^1 |\tilde{\omega}_d(n, n'; t)| \, dt \ll \left( 1 + \frac{H}{N^r} \right) (\log x)^3.
\]

**Lemma 1.7.** Let

\[
\omega_{\phi}(X, Y) = \sum_{r} \sum_{x} \phi_r \psi_r e(x, y),
\]

where \( X = (x_r) \), \( Y = (y_s) \) be finite sequences of real numbers with

\[
|x_r| \leq P, \quad |y_s| \leq Q.
\]

and \( \phi_r, \psi_s \) be complex numbers. Then

\[
|\omega_{\phi}(X, Y)|^2 \leq 20(1 + PQ) \omega_d(X, Q) \omega_d(Y, P),
\]

where

\[
\omega_d(X, Q) = \sum_{|x| \leq Q^{-1}} |\phi_r|,
\]

and \( \omega_d(Y, P) \) being defined similarly.

**Lemma 1.8.** Let \( Q \geq 1, m \sim M, q \sim Q \), let \( \alpha(\neq 0, 1) \) be a real number, \( t(m, q) = (m + q)^2 - (m - q)^2 \), \( T = M^{-1}Q \), and let \( B(M, Q, A) \) be the number of lattice points \((m, m_1, q, q_1)\) such that \( m, m_1 \sim M, q, q_1 \sim Q \) and

\[
|t(m, q) - t(m_1, q_1)| < A T.
\]

If \( Q < \varepsilon M^{3/4} \), then

\[
B(M, Q, A) \ll (MQ + A M^{2}Q^{2} + Q^{3/2})(\log(4MN))^{4},
\]

the implied constant relying at most on \( \varepsilon \) and \( \alpha \).
For an account of Lemmas 1.2, 1.3, 1.4, 1.5, and 1.7, the reader is referred to Lemmas 1, 2, 3, 4, and 10 of [9]. Lemma 1.6 is Lemma 4 of [11]. Lemma 1.8 is Theorem 2 of [10]. We note that although the use of Lemma 1.2 can be avoided in choosing parameters, it can really simplify the practical calculations. Lemmas 1.1 and 1.8 hold also if \( t \) is replaced by \( t' \), here \( t \equiv T \) means \( 1 \leq t/T \leq 1 \).

Now we are ready to prove Lemma 1. Suppose \( Q = aH/bN \geq 100 \), \( a \) and \( b \) are positive integers, \( Q \) is viewed as a parameter. Let

\[
T_q = \left\{ (h, n) \mid h \sim H, n \sim N, \frac{Q - 1}{Q} \cdot \frac{H}{N} 10^{1/2} < \frac{h}{n} \leq \frac{QH}{QN} 10^{1/2} \right\}, \quad 1 \leq q \leq Q.
\]

As in [11], Cauchy’s inequality gives

\[
|S(H, M, N)|^2 \leq QM \sum_{1 \leq q \leq Q} \sum_{m - M} \left| \sum_{(h, n) \in T_q} b_n e\left( \frac{hx}{mn} \right) \right|^2
\]

\[
= QM \sum_{1 \leq q \leq Q} \sum_{(h, n), (h', n') \in T_q} b_n b_{n'} \sum_{m - M} e\left( \frac{x}{m} \frac{h}{n} - \frac{h'}{n'} \right)
\]

\[
\times \sum_{m - M} e\left( \frac{x}{m} \frac{h}{n} - \frac{h'}{n'} \right) S\left( \frac{h}{n}, \frac{h'}{n'} \right), \tag{1.15}
\]

where

\[
S\left( \frac{h}{n}, \frac{h'}{n'} \right) = \sum_{\max(1, (HM+H)) 10^{-1/2}, (QH/NH) 10^{-1/2} \leq q < \min(Q, 1 + QH/NH) 10^{-1/2}, 1 + QH/NH) 10^{-1/2}}
\]

It suffices to treat the portion of the sum of (1.15) with \( hn' \geq nh' \), as the portion with \( hn' < nh' \) can be treated similarly. Now we have

\[
S\left( \frac{h}{n}, \frac{h'}{n'} \right) = \left[ 1 + Q \frac{hN}{nH} 10^{-1/2} \right] - \left[ Q \frac{hN}{nH} 10^{-1/2} \right]
\]

\[
= 1 + 10^{-1/2} Q \frac{N}{H} \left( \frac{h}{n} - \frac{h'}{n'} \right) - \psi \left( Q \frac{hN}{nH} 10^{-1/2} \right)
\]

\[
+ \psi \left( Q \frac{hN}{nH} 10^{-1/2} \right).
\]
thus from (1.15) we get

$$|S(H, M, N)|^2 \ll QM(S_1 + S_2 + S_3 + S_4),$$

(1.16)

where

$$S_i = \sum_{m_1} \sum_{h,n} e \left( \frac{x}{m} \left( \frac{h}{n} - \frac{h'}{n'} \right) \right) P \left( \frac{h}{n} \right),$$

$$\Sigma_i$$ means a summation over lattice points \((h, h', n, n')\) with

$$h, h' \sim H, \quad n, n' \sim N, \quad \frac{|h - h'|}{n} \leq \frac{H}{NQ} 10^{1/2},$$

and

$$P_1 = 1, \quad P_2 = \frac{Q}{H} 10^{-1/2} \left( \frac{h'}{n'} - \frac{h}{n} \right),$$

$$P_3 = \psi \left( \frac{Q}{nH} 10^{-1/2} \right), \quad P_4 = -\psi \left( \frac{Q}{nH} 10^{-1/2} \right).$$

We give an estimate for \(QMS_3\) as follows. By a familiar reduction,

$$x^{-\varepsilon}QMS_3 \ll QM \sum_{d \leq D} \sum_{n,n' \in N'} \sum_{r \leq R} \sum_{m} \omega_d(n, n'; r) \sum_{m} e \left( \frac{-rx}{dmm'} \right) + QNM^2$$

(1.17)

holds for some pair \((D, R)\) with \(1 \leq D \leq N\) and \(1 \leq R \leq HN/DQ\). \(N' = N/d, \omega_d(n, n'; r)\) is defined in Lemma 1.6. We can transform the summation over \(m\) to a summation over \(u\) via Lemma 1.5, here \(u \geq U = RDx(MN)^{-1}\) \((u \geq U\) means that \(C_1 \leq u \leq C_2\) for some absolute constants \(C_1\) and \(C_2\), and so on). Then we exchange the order of summation to get

$$x^{-\varepsilon}QMS_3 \ll \frac{QNM^{5/2}}{(RDx)^{1/2}} \sum_{d \leq D} \sum_{n,n' \in N'} \sum_{u \leq U} \sum_{r \in I} \omega_d(n, n'; r)$$

$$\times e \left( C_3 \left( \frac{urx}{dmm'} \right)^{1/2} \right) + E,$$

where \(E = Q(M^3 N^2 r^{-1})\), \(I\) is some subinterval of \([ R, 2R] \), which depends on the variables \(d, n, n'\) and \(u\) outside the absolute symbol, \(|C(r)| \leq 1\). We can use Lemma 1.4 to relax the condition \(r \in I\), and obtain
By Cauchy’s inequality and Lemma 1.6, we have

\[ x^{-3}Q_{NS_1} \leq \frac{Q_{NM}^{5/2}}{(RD^2)^{3/2}} \sum_{d-D} \sum_{n \in \mathbb{N}} \sum_{n' \in \mathbb{N}} \sum_{(n,n') = 1} |\sum_{dU} \omega_d(n, n'; r)| \times e\left(C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) |\tilde{C}(r)| + E, \]

where \( \tilde{C}(r) = e(r) C(r) \), \( t \) is some real number, which is independent of variables. By Cauchy’s inequality we have

\[ x^{-6c}(Q_{NS_1})^2 \leq \frac{Q_{NM}^3 N^2}{D} \sum_{d-D} \sum_{n \in \mathbb{N}} \sum_{n' \in \mathbb{N}} \sum_{(n,n') = 1} |\sum_{dU} \tilde{C}(r) \omega_d(n, n'; r)| \times e\left(C_3 \left(\frac{ux}{dW}\right)^{1/2}\right)^2 + E^2. \] (1.18)

By Cauchy’s inequality and Lemma 1.6, we have

\[ \left| \sum_{r \in R} \tilde{C}(r) \omega_d(n, n'; r) e\left(C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) \right|^2 \]

\[ \leq \left( \int_0^1 |\tilde{\omega}_d(n, n'; \xi)| \, d\xi \right) \left( \int_0^1 |\tilde{\omega}_d(n, n'; \xi)| \, d\xi \right) \]

\[ \times \left| \sum_{r \in R} \tilde{C}(r) e\left(r\xi + C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) \right|^2 \, d\xi \]

\[ \leq (1 + HN^{-1}D)(\log x)^{3/2} \left( \int_0^1 |\tilde{\omega}_d(n, n'; \xi)| \, d\xi \right) \]

\[ \times \left| \sum_{r \in R} \tilde{C}(r) e\left(r\xi + C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) \right|^2 \, d\xi. \] (1.19)

By Lemma 1.3, for a parameter \( Q_1 \) we have

\[ \left| \sum_{r \in R} \tilde{C}(r) e\left(r\xi + C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) \right|^2 \]

\[ \leq 2R^2 Q_1^{-1} + 2(1 + RQ_1^{-1}) \sum_{1 \leq |q| \leq (Q_1 - 1/2)} (1 - 2|q| + 1) Q_1^{-1} S(q_1). \] (1.20)

where

\[ S(q_1) = \sum_{r + q_1 \in R} \frac{\tilde{C}(r - q_1) \tilde{C}(r + q_1)}{(r - q_1) \tilde{C}(r + q_1)} \]

\[ \times e(-2q_1 \xi) e\left(C_3 \left(\frac{ux}{dW}\right)^{1/2}\right) (r, q_1). \]
where \( t(r, q_1) = (r - q_1)^{1/2} - (r + q_1)^{1/2} \). From (1.18) to (1.20), and Lemma 1.6, we get
\[
x^{-7}(QMS)_2^2 \leq (1 + HN^{-1}D)^2 \left( \frac{xR^3N^2Q^2M}{DQ_1} + \frac{RQ^2M^2N^2}{DQ_1} \right) \sum_{1 \leq |q_1| \leq Q_1} S_i(q_1) + E^2,
\]
(1.21)
where
\[
S_i(q_1) = \sum_{d \in D} \sum_{\nu \in \mathbb{N}^*} \left| \sum_{w \in U \cap R(q_1)} Z(r, q_1, e \left( C_3 \left| \frac{ux}{w} \right| \right)^{1/2} t(r, q_1) \right|,
\]
\( R(q_1) = \{ r | r + q_1 = R, r - q_1 = R \} \), \( Z(r, q_1) = \overline{C(r - q_1)} \overline{C(r + q_1)} \); we have assumed that \( Q_1 \in (100, \varepsilon R^{3/4}) \), and \( Q_1 \) depends only on \( R, x, M, N, D \) and \( Q \). Let \( W = N^2D^{-1} \), then it is clear that
\[
S_i(q_1) \leq x^e \sum_{w \in W} \sum_{u \in U \cap R(q_1)} Z(r, q_1, e \left( C_3 \left| \frac{ux}{w} \right| \right)^{1/2} t(r, q_1) \right| = x^e \sum_{w \in W} \sum_{u \in U \cap R(q_1)} Z(r, q_1, e \left( C_3 \left| \frac{ux}{w} \right| \right)^{1/2} t(r, q_1) \right|,
\]
where \( \lambda(w, q_1) \) depends only on the variables \( w \) and \( q_1 \), and \( |\lambda(w, q_1)| \leq 1 \). Assume that \( Q_2 \) is a parameter with \( Q_2 \in (100, \varepsilon W^{1/4}) \). By Lemma 1.3,
\[
x^{-3}(S_i(q_1))^2 \leq (URW)^2 Q_2^{-1} + URWQ_2^{-1} S_2(Q_3),
\]
(1.22)
where \( Q_3 \) is some number with \( Q_3 \in (1, Q_2) \), and
\[
S_2(Q_3) = \sum_{q_2 - Q_3} \sum_{w \in W} \sum_{w \in U \cap R} e(C_3u^{1/2}(r, q_1)((w - q_2)^{-1/2} - (w + q_2)^{-1/2}) \chi^{1/2} \right|.
\]
Lemma 1.7 gives the estimate
\[
(S_2(Q_3))^2 \leq \left( 1 + \frac{xD}{MN^{-1}W} |q_1| Q_3 \right) AB \leq \frac{xD}{MN^{-1}W} |q_1| Q_3 AB, \quad (1.23)
\]
where $A$ is the number of lattice points $(w, q_2, \tilde{w}, \tilde{q}_2)$ with
\[ w, \tilde{w} \equiv W, \quad q_2, \tilde{q}_2 \sim Q, \quad |g(w, q_2) - g(\tilde{w}, \tilde{q}_2)| \ll x^{-1/2}U^{-1/2} |q_1|^{-1} R^{1/2}, \]
g(w, q_2) = (w - q_2)^{-1/2} - (w + q_2)^{-1/2}$, and $B$ is the number of lattice points 
$(u, r, \tilde{u}, \tilde{r})$ with
\[ u, \tilde{u} \equiv U, \quad r, \tilde{r} \sim R, \quad |u^{1/2}(r, q_1) - \tilde{u}^{1/2}(\tilde{r}, q_1)| \ll x^{-1/2}Q^{-1}W^{1/2}. \]
Thus Lemmas 1.8 and 1.1 give respectively the estimates
\[
A \ll (WQ_1 + W^2Q_2 |q_1|^{-1} x^{-1/2}D^{-2}MN^4 + Q_3^{4/3})(\log x)^4 \tag{1.24}
\]
and
\[
B \ll (UR + |q_1|^{-1} D^{-2}x^{-1}M^{-2}N^{-2}Q_1^{-1}U^2R^2 + U^2 |q_1|^{-2/3})(\log x)^4 \tag{1.25}
\]
in (1.24) we have used the fact that $MN^6 \ll x$. From (1.22) to (1.25) we can deduce that
\[
x^{-4}\left((S_1(q_1))^2 \ll R^4x^3M^{-4}Q_2^{-1}
+ R^4x^3M^{-9}N^{-1}D^{3/2}(x^{1/2} |q_1|^{5/8} + MN |q_1|^{1/2})
+ R^4x^3M^{-9}N^{-4}Q_3^{4/3}(D^2 |q_1|^{5/8} x^{1/2} + D^{3/2} |q_1|^{1/2} MN).\right.
\]
Clearly this estimate holds also for $Q_2 \ll 1$. By Lemma 1.2 there is an $Q_2 \in (0, xW^{3/4})$ for which
\[
x^{-4}\left(\tilde{S}_2(q_1) \ll (R^8x^{22}D^{12}M^{-47}N^{-24} |q_1|^{5/6})^{1/11}
+ (R^8x^{22}M^{-41}N^{-18} |q_1|^{3} D^{2})^{1/11}
+ (R^{46}x^{8}M^{-16}N^{-6}D^{2})^{1/4}
+ R^4x^3M^{-9}N^{-3}D^{3/2}(x^{1/2} |q_1|^{5/6} + MN |q_1|^{1/2})\right). \tag{1.26}
\]
From (1.21) and (1.26) we get
\[
x^{-9}(QMS_1)^2 \ll (1 + HN^{-1}D^2) \left(xR^3MN^2Q_1^{-1}D^{-1}
+ (R^{60}D^{-10}x^{25}M^{19}N^{20}Q_3^{1})^{1/22}
+ (R^{60}D^{-13}M^{23}N^{26}x^{22}Q_2^{1})^{1/22}
+ (R^{90}x^{15}M^{N}N^4D^{-3}Q_1^{1/12} + R^3xMN^{3/2}D^{-5/8}
+ (R^{40}x^{4}M^{5}N^4D^{-4}Q_1)^{1/4}) Q^2 + M^6N^4y^2Q^2. \tag{1.27}
\]
Obviously (1.27) holds also for \( Q_i \ll 1 \). By Lemma 1.2 there is an \( Q \in (0, e^{R^{1/4}}) \) for which

\[
x^{-y_1}(QMS_i)^2 \ll (1 + HN^{-1}D^2)Q^2(\epsilon R^{0.4}MN^2D^{-1})
\]

\[
+ (x^{30}R^{25}M^{24}N^{30}D^{-15})^{1/27} + xRMN^{5.4}D^{-5.8}
\]

\[
+ (x^{25}R^{66}M^{28}N^{32}D^{-16})^{1/25} + (x^{20}R^{43}M^{14}N^{16}D^{-8})^{1/17}
\]

\[
+ (x^{5}R^{13}M^{6}N^{6}D^{-2}1/5) + M^6N^4y^{-2}Q^2
\]

\[
\ll (x^{-9}M^{13}M^{13}/Q^{1/4} + x^{30}M^{29}N^{4}Q^{-1})
\]

\[
+ (x^{10}M^{33}N^{60}Q^{-7})^{1/9}
\]

\[
+ (x^{25}M^{97}N^{170}Q^{-19})^{1/25}
\]

\[
+ (x^{y}M^{59}N^{160}Q^{-11})^{1/17}
\]

\[
+ (x^{z}M^{19}N^{32}Q^{-3})^{1/5} + M^6N^4y^{-2}Q^2 \right) x^{3i}
\]

\[
= x^{3i}E_i(Q), \quad \text{say.}
\]

Similarly and more easily, we can also deduce that

\[
x^{-12}(QMS_i)^2 \ll E_i(Q), \quad \text{for } i = 1, 2 \text{ and } 4.
\]

Thus from (1.16) we obtain

\[
|S[H, M, N]|^4 \ll x^{12}E_i(Q). \quad \text{(1.28)}
\]

By Lemma 1.2 there is an \( \tilde{Q} \) with \( \tilde{Q} \in [100, x^{100}] \) such that

\[
E_i(\tilde{Q}) \ll (M)^4((x^{3}y^{-20}M^{-4}N^{20})^{1/9}
\]

\[
+ (x^{20}y^{-8}M^{2}N^{13}C^{13}D^{13})^{1/3} + (x^{20}y^{-68}M^{8}N^{48})^{1/25}
\]

\[
+ (x^{10}y^{-176}M^{32}N^{140}1/69 + (x^{40}y^{-112}M^{4}N^{76})^{1/45}
\]

\[
+ (x^{10}y^{-32}M^{4}N^{24})^{1/13} + x^{-1000})
\]

\[
\ll (M)^4x^{-1000}, \quad \text{(1.29)}
\]

as \( M \leq yx^{-\sqrt{2}}, N \leq (y^{72}x^{-25})^{1/70} x^{-\sqrt{2}} \) and \( M \geq (y^{-2}x^{25})^{1/70} \) (note that \( MN \geq yx^{-1} \)). Now we choose \( \tilde{Q} = (H(1 + \lfloor N \rfloor)/N(1 + \lfloor H \rfloor)) \tilde{Q} \right) 1000, \)

then clearly

\[
E_i(Q) \ll E_i(\tilde{Q}),
\]

and Lemma 1 follows from (1.28) and (1.29).
2. AN AUXILIARY TRIPLE EXPONENTIAL SUM

Let

\[ S_1(H, M, N) = \sum_{h=H}^{H+1} \sum_{n=N}^{N+1} \sum_{m=M}^{M+1} a_m e \left( \frac{hx}{nm} \right), \]

where \(|a_m| \leq 1, H, M, N \geq 1, H \leq M N y^{-1} x^s, y = x^{0.436}\). In this section we present an estimate for \( S_1(H, M, N) \). Lemma 1.8 again plays a central role here. We have

**Lemma 2.** Suppose \( y = x^{0.436}, M N \leq (xy)^{1/2} x^{-\sqrt{7}}, N \leq x^{1/2} - \sqrt{7} \) and \( M \leq (x^{-3} y^{10})^{1/6} x^{-\sqrt{7}} \). Then

\[ S_1(H, M, N) \ll M N x^{-s}. \]

**Proof.** First we transform the summation over \( n \) to a summation over \( u \) by using Lemma 1.5, here \( u = H x (H x M)^{-1} \). Then we exchange the order of summation to get

\[ S_1(H, M, N) \ll (MN x^{-1})^{1/2} \sum_{h=H}^{H+1} \sum_{u \leq U/M} \left| \sum_{m \in I(h, u)} b_m e \left( \frac{hx u}{m} \right) \right| + M N x^{-10s}, \]

where \( I(h, u) \) is some subinterval of \([M, 2M]\), \(|b_m| \leq 1\). Apply Lemma 1.4 to relax the condition \( m \in I(h, u) \), we have

\[ x^{-s} S_1(H, M, N) \ll (MN x^{-1})^{1/2} \sum_{h=H}^{H+1} \sum_{u \leq U/M} \left| \sum_{m \in M} c_m e \left( \frac{hx u}{m} \right) \right| + M N x^{-10s}, \]

\[ \ll x^{-s} \sum_{u \leq U/M} \left| \sum_{m \in M} c_m e \left( \frac{hx u}{m} \right) \right| (MN x^{-1})^{1/2} + M N x^{-10s}, \]

(2.1)

where \(|c_m| \leq 1, W = H^2 x M^{-1} N^{-2}\). Cauchy's inequality and Lemma 1.3 give

\[ \left( \sum_{u \leq U/M} \left| \sum_{m \in M} c_m e \left( \frac{hx u}{m} \right) \right| \right)^2 \ll (WM)^2 Q^{-1} + WMQ^{-1} \sum_{1 \leq q \leq Q} \sum_{m \in M} |S(m, q)| \]

\[ \ll (WM)^2 Q^{-1} + WMQ^{-1} x^s \sum_{q \leq Q} \sum_{m \in M} |S(m, q)|, \]

(2.2)

ALMOST PRIMES IN SHORT INTERVALS
where \( Q \in (100, \varepsilon M^{1/4}) \) is a parameter, and \( Q_1 \) is some number with \( 1 \leq Q_1 \leq Q \).

\[
S(m, q) = \sum_{w \geq W} e(C_d(xw)^{1/2} t(m, q)) , \quad t(m, q) = (m - q)^{-1/2} - (m + q)^{-1/2}.
\]

We use Lemma 1.5 to transform the summation over \( w \) to a summation over \( v \), here \( v \geq V = Q_1(xM^{-1}W^{-1})^{1/2} \). Then we get, after an application of Lemma 1.4, the estimate

\[
S(m, q) \ll (W^3 M^3 x^{-1} Q_1^{-2})^{1/4} \left| \sum_{v \in J(m, q)} d_v e(C_{\delta} x v^{-1} i^2(m, q)) \right| + (xH^3 N^{-3} Q_1^{-1})^{1/2} + \log x
\]

\[
\ll x^2 (W^3 M^3 x^{-1} Q_1^{-2})^{1/4} \left| \sum_{v \geq V} d_v e(C_{\delta} x v^{-1} i^2(m, q)) \right| + (xH^3 N^{-3} Q_1^{-2})^{1/2} + \log x,
\]

(2.3) where \( |d_v| \leq 1, J(m, q) \) is some subinterval of \( [C_6 V, C_7 V] \), and \( D_v = e(tv) d_v \), here \( t \) is some real number which is independent of variables. We proceed to estimate \( S_2(Q_1, M, V) \), where

\[
S_2(Q_1, M, V) = \sum_{q \ll Q_1} \sum_{m \ll M} \left| \sum_{v \geq V} D_v e(C_{\delta} x v^{-1} i^2(m, q)) \right|.
\]

By Cauchy's inequality and Lemma 1.3, we have, with a parameter \( Q_2, Q_2 \in (100, \varepsilon V^{3/4}), \) the estimate

\[
|S_2(Q_1, M, V)|^2 \ll (Q_1 MV)^2 Q_2^{-1} + Q_1 MVQ_2^{-1} \times \sum_{1 \leq q_1 \leq Q_2} \sum_{v \ll V} \sum_{q \ll Q_1} \sum_{m \ll M} e(\delta(g))
\]

\[
\ll (Q_1 MV)^2 Q_2^{-1} + Q_1 MV x Q_2^{-1} S_3(Q_3),
\]

(2.4) where \( g = g(m, q, v, q_1) = C_{\delta} x v^2(m, q)((v - q_1)^{-1} - (v + q_1)^{-1}) \), \( Q_3 \) is some number with \( 1 \leq Q_3 \leq Q_2 \), and

\[
S_3(Q_3) = \sum_{q_1 - Q_1} \sum_{v \ll V} \left| \sum_{q - Q_1} \sum_{m \ll M} e(g(m, q, v, q_1)) \right|.
\]
To simplify the calculation we assume the following inequality

\[ M^{3/4}N^3 \ll xH^3. \]  \hspace{1cm} (2.5)

By Lemma 1.7 we get

\[ (S_3(Q_3))^2 \ll WQ_3AB, \]  \hspace{1cm} (2.6)

where \( A \) is the number of lattice points \((m, q, \tilde{m}, \tilde{q})\) such that

\[ r^2(m, q) - r^2(\tilde{m}, \tilde{q}) \ll x^{-1}Q_3^{-1}V^2, \quad m, \tilde{m} \sim M, \quad q, \tilde{q} \sim Q_1, \]

and \( B \) is the number of lattice points \((v, q_1, \tilde{v}, \tilde{q}_1)\) such that

\[ F(v, q_1) - F(\tilde{v}, \tilde{q}_1) \ll x^{-1}(Q_1M^{-3/2})^{-2}, \quad v, \tilde{v} \equiv V, \quad q_1, \tilde{q}_1 \sim Q_3, \]

and

\[ F(v, q_1) = (v - q_1)^{-1} - (v + q_1)^{-1}. \]

Thus by Lemma 1.8 and (2.5) we have

\[ A \ll (MQ_1 + M^3Q_1^3Q_3^{-1}x^{-1}N^2H^{-2} + Q_1^6)(\log x)^4, \]  \hspace{1cm} (2.7)

\[ B \ll (VQ_3 + VQ_1N^2Mx^{-1}H^{-2}V + Q_1^6)(\log x)^4 \ll (VQ_3 + Q_1^6)(\log x)^4. \]  \hspace{1cm} (2.8)

Gathering (2.6) to (2.8) we obtain an estimate for \( S_3(Q_3) \), which, in conjunction with (2.4), gives

\[ x^{-2\varepsilon} |S_3(Q_1, M, V)|^2 \ll (NQ_1^2H^{-1})^2 Q_2^{-1} \]

\[ + (H^{-1}xQ_1^4M^{-1}N)^{1/2} + (H^{-3}x^3Q_1^3M^{-6}N^3)^{1/6} \]

\[ + (x^3Q_1^3Q_1^3)^{1/6} + (x^3M^{-3}Q_1^6)^{1/6} \]

\[ + (MN^3H^{-3}Q_1^7)^{1/2} + MNH^{-1}Q_1^3Q_2^{-3/2}. \]

Evidently this estimate holds also for \( Q_2 \ll 1 \). By Lemma 1.2 we can choose an \( Q_2 \in (0, eV^{3/4}) \) for which

\[ x^{-2\varepsilon} |S_3(Q_1, M, V)|^2 \ll (H^{-1}xQ_1^4M^{-1}N)^{1/2} \]

\[ + (H^{-3}x^3M^{-6}N^3Q_1^6)^{1/6} + (MN^3H^{-3}Q_1^7)^{1/2} \]

\[ + (N^{10}Q_1^7H^{-10}x^4)^{1/11} + (N^{10}Q_1^9H^{-10}x^3M^{-3})^{1/11} \]

\[ + (M^3N^5H^{-5}Q_1^{13})^{1/4}. \]  \hspace{1cm} (2.9)
From (2.2), (2.3) and (2.9) we obtain

\[
x^{-4e} \sum_{m \geq x} \left| \sum_{i \in M} c_i e \left( \frac{Hx}{m} \right)^{1/2} \right|^2
\]

\[
\ll (H^2xN^{-2})^2 M^{-1} + \sqrt[3]{H^{12}x^3M^{-1}N^{-13}}
\]

\[
+ \sqrt[4]{H^9x^5M^{-6}N^{-9}Q^3} + \sqrt[5]{H^7x^7N^{-6}Q^4}
\]

\[
+ \sqrt[6]{H^5x^5M^{-3}N^{-6}Q^4} + \sqrt[7]{H^{13}x^{12}M^1N^{-23}}Q
\]

\[
+ \sqrt[8]{H^{11}x^6MN^{-11}Q + (x^7N^7M^2)^{1/2} + xH^2MN^{-2}}.
\]  

Plainly (2.10) holds also for \( Q \). By Lemma 1.2 we can choose an \( Q \) for which

\[
x^{-4e} \left| \sum_{m \geq x} \left( \sum_{i \in M} c_i e \left( \frac{Hx}{m} \right)^{1/2} \right) \right|^2
\]

\[
\ll (H^2xN^{-2})^2 M^{-3/4} + \sqrt[3]{H^{12}x^3M^{-1}N^{-13}}
\]

\[
+ \sqrt[4]{H^9x^5N^{-6}M^1} + \sqrt[5]{H^7x^7N^{-9}Q^3}
\]

\[
+ \sqrt[6]{H^5x^5M^{-3}N^{-6}Q^4} + \sqrt[7]{H^{13}x^6MN^{-11}Q}
\]

\[
+ \sqrt[8]{H^{11}x^6MN^{-11}Q + (x^7N^7M^2)^{1/2} + xH^2MN^{-2}}.
\]  

From (2.1) and (2.11) we find that

\[
x^{-5S}(H, M, N) \ll \frac{8}{9}H^{12}N^{-4}x^4M + \frac{8}{9}H^7x^8M^1N^{-1}
\]

\[
+ \frac{5}{9}H^9x^{11}N^{-2} + \frac{4}{9}H^{11}x^{17}M^2N^{-3}
\]

\[
+ \frac{8}{9}H^9x^{11}N^{-2} + \frac{18}{10}H^7x^7M^{18}
\]

\[
+ \frac{10}{9}H^7x^7M^5 + (H^2N^{-1}xM^4)^{1/4}
\]

\[+ (HM^2N^3)^{1/2} + MNx^{-10e}, \]

\[\ll MNx^{-10e}, \]

by our assumptions. Thus we have verified Lemma 2 when (2.5) holds. Assume \( M^{3/4}N^3 > xH^2 \). If \( M > x^{5/7} \), by the exponent pair \((\frac{3}{4}, \frac{5}{7})\) we have

\[
S(H, M, N) \ll \frac{M}{x^3}(HxM^{-1}N^{-2}) \ll MNx^{1/2}H^{-1}x^{-1}
\]

\[
\ll (H^xMN^{-1})^{1/2} + MNx^{-10e}
\]

\[
\ll MNx^{1/2} + MNx^{-100e} \ll MNx^{-100e},
\]

by our assumptions. Thus we have verified Lemma 2 when (2.5) holds. Assume \( M^{3/4}N^3 > xH^2 \). If \( M > x^{5/7} \), by the exponent pair \((\frac{3}{4}, \frac{5}{7})\) we have

\[
S(H, M, N) \ll \frac{M}{x^3}(HxM^{-1}N^{-2}) \ll MNx^{1/2}H^{-1}x^{-1}
\]

\[
\ll (H^xMN^{-1})^{1/2} + MNx^{-10e}
\]

\[
\ll MNx^{1/2} + MNx^{-100e} \ll MNx^{-100e},
\]
and if $M \leq x^{1/2}$ the exponent pair $(2/7, 4/7)$ yields

$$S_1(H, M, N) \leq HM((HxM^{-1}N^{-2})^{2/7}N^{4/7} + MN^{2}(Hx)^{-1})$$

$$\leq \sqrt[n]{H^2x^2M^7 + MNX^{100c}}$$

$$\leq MN(x^{-1}N^2x^{1/4}\sqrt{7})^{1/7} + MNX^{100c} \leq MNX^{100c}.$$  

This completes the proof of Lemma 2.

From the context of the above proof, it is seen that the condition on $M$, that is, $M \leq (x^{-1}N^{1/6}x^{1/4})$, is not the best available, and it can be enlarged a bit. But such an improvement does not lead to an improvement on our Theorem.

3. PROOF OF THEOREM

We use the Greaves weighted sieve of [4]. Let $T$, $U$, $V$, $E$ and $D$ be a set of parameters with

$$E_0 \leq V \leq \frac{1}{3}, \quad E_0 = \max(E, (1 - T)/3),$$

$$\frac{1}{2} \leq U \leq T < 1, \quad U + 3V \geq 1, \quad D \geq 3.$$ 

Let

$$H(\alpha, D^V, D^T) = \sum_{\alpha \in \mathfrak{A}} \mu((a, P(D^T))),$$

$$P(z) = \prod_{p < z} p, \quad \mathfrak{A} = \{n; x - y < n \leq x\}, \quad y = x^{0.436},$$

$$\mu(n) = \left\{1 - \sum_{p \mid n} (1 - W(p))\right\}^+, \quad \{z\}^+ = \max(0, z)$$

for a real $z$, and $W(p) = 0$, for $p \notin [D^V, D^T], \quad W(p) = \begin{cases} \frac{1}{T - E} \left(\frac{\log p}{\log D} - E\right), & \text{for } D^{1/4} \leq p < D^T, \\ \frac{1}{T - E} \left(\frac{\log p}{\log D} - E_0\right), & \text{for } D^V \leq p < D^{1/4}. \end{cases}$$

For $D = x^\theta$, $\theta_1^{-1} \leq 2T + E$, $T < \min(1, (\theta + 1)/2\theta_1)$, an inequality

$$H(\alpha, D^V, D^T) \geq y^3 \log x$$
would imply that the interval \((x - y, x]\) contains \(\gg y/\log x\) \(P_2\) numbers. And we have

\[
H(\mathcal{A}, D^V, D^U) \gg H(\mathcal{A}, D^V, D^U) - \sum_{D^U < p < D^V} (1 - W(p)) S(\mathcal{A}_p, D^V),
\]

(3.1)

where \(S(\mathcal{A}_p, D^V)\) is the usual sieve function. Moreover, for

\[
E = V = V_0 = 0.074368..., \quad \alpha = 0.1505528..., \quad \beta = 0.876950..., 
\]

and \(MN = D, M > D^U, N > 1\), we have

\[
H(\mathcal{A}, D^V, D^U) \geq \frac{2y}{(T - E) \log D} \left( T \log \left( \frac{1}{T} \right) + (1 - T) \log \left( \frac{1}{1 - U} \right) \right.
\]

\[
- \log 3 + \alpha - E \log 3 - E_0 \beta + 0(\epsilon)
\]

\[-(\log D)^{1/4} \sup_{m < M, n < N, mn \mid P(D^U)} a_m b_n r(\mathcal{A}, mn),
\]

(3.2)

where \(r(\mathcal{A}, d) = \sum_{\xi < x, \xi \equiv d, 1 - y/d, \text{the supremum is taken over all sequences}} a_m b_n r(\mathcal{A}, mn)\), with \(|a_m| \leq 1, |b_n| \leq 1\). If we choose

\[
M = x^{\theta_1 - \sqrt{\varepsilon}}, \quad N = x^{1/2(\theta_1 - 25)/70 - \sqrt{\varepsilon}}, \quad D = MN,
\]

\[
U = \log M \log D / \varepsilon, \quad T = \frac{1}{2} (\theta_1^{-1} - E) + \varepsilon, \quad \theta_0 = 0.436,
\]

then by a familiar reduction, our Lemma 1 implies that the last error term of (3.2) is \(0(x^{\theta_1 - (1/2)\varepsilon})\); as now we have

\[
\theta_1 = (142\theta - 25)/70 - 2 \sqrt{\varepsilon}, \quad U = \theta_0 \theta_1^{-1} + 0(\sqrt{\varepsilon}) \quad \text{and} \quad E_0 = E,
\]

from (3.2) we get

\[
H(\mathcal{A}, D^V, D^U) \geq \frac{2y}{(T - E) \log D} (0.04521).
\]

(3.3)

To estimate the second term of (3.1) we use the Rosser–Iwaniec sieve of [5] to obtain an upper bound for each sieve function \(S(\mathcal{A}_p, D^V)\). To choose the level of distribution for a single function \(S(\mathcal{A}_p, D^V)\) and to estimate the total contribution of the resulting error terms, we can benefit from the following lemma.
Lemma 3. Let

\[ R(D, K) = \sum_{k \leq K} \sum_d A(k) \lambda(d) r(\sigma d, dk), \]

where \( A(k) \) is the von Mangoldt function, \( \lambda(d) \) is a well-factorable function of level \( D \), \( y = x^{0.436} \), \( K \geq yx^{-2} \sqrt{z} \) and \( K^4 D^6 \leq y^8 x^{-2} \). Then

\[ R(D, K) \ll yx^{-\sqrt{2}}. \]

Proof. This lemma extends the range of validity for \( y \) given by Proposition 4 of [14], which requires \( y \geq x^{7/16} = x^{0.4375} \), thus it can not be applied directly to our case \( y = x^{0.436} \). To verify Lemma 3 we take the same way as in [14], namely, we decompose \( A(k) \) by Vaughan identity and then estimate the resulting type I and type II sums. The type II sum can be estimated exactly the same as in [14]. The type I sum can also be estimated as in [14], except that we must use our Lemma 2 to replace Proposition 2 and Lemma 12 there in the argument. To save space we do not include the routine details here, the reader is referred to [14] for a complete treatment. The proof of Lemma 3 is thus finished.

By Lemma 3 and the Rosser–Iwaniec sieve of [5], after a routine procedure of calculation, we get, as in (5.3) of [14], the estimate

\[ \sum_{D^\alpha \leq p < D^\beta} (1 - W(p)) S(\sigma p, D^\gamma) \leq \frac{2y}{(T - E) \log D} \left( \frac{9U - 8T}{12U} \log \left( \frac{9U - 8T}{U} \right) + \frac{2T}{3U} \log \frac{T}{U} + O(\sqrt{u}) \right) \leq \frac{2y}{(T - E) \log D} (0.04519). \] (3.4)

Our Theorem follows from (3.1), (3.3), and (3.4).

References