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Special Bi-Axial Monogenic Functions

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In this paper we extend our recent work on axial monogenic functions in $\mathbf{R}^{m+1}$ to functions which are monogenic in bi-axially symmetric domains of $\mathbf{R}^{p+q}$. We show that an integral transform of a wide class of holomorphic functions of a single complex variable gives monogenic functions of this type. It is demonstrated that these integral transforms are related to plane wave monogenic functions. A bi-axial monogenic exponential function is defined using the exponential function of a complex variable and bounds are obtained on its modulus. Bi-axially symmetric monogenic generating functions are used to define generalisations of Gegenbauer polynomials and Hermite polynomials. Finally, bi-axial power functions are constructed using the above integral transform. © 1994 Academic Press, Inc.

1. INTRODUCTION

We have considered previously [1, 2] functions defined over $(m+1)$-dimensional space taking values in a complex Clifford algebra $\mathscr{A}$. The generating vectors of the Cliffford algebra $\mathscr{A}$ are $\{e_l; l=1, 2, ..., m\}$ satisfying the defining relations

$$e_l e_j + e_j e_l = -2\delta_{jl} e_0, \qquad j, l = 1, ..., m, \tag{1.1}$$

where e_0 is the unit element of the algebra. We denote

$$x \equiv x_0 e_0 + \vec{x}, \qquad \vec{x} \equiv \sum_{l=1}^{m} x_l e_l, \tag{1.2}$$

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and it corresponds to a point in $\mathbf{R}^{m+1}$. There exist corresponding cylindrical coordinates $\{x_0, \rho, \vec{\omega}\} \in \mathbf{R} \times R_+ \times S^{m-1}$ defined by

$$x = x_0 e_0 + \rho\vec{\omega};\ \vec{\omega} \equiv \vec{e}_\rho \equiv \frac{\vec{x}}{|\vec{x}|};\ \rho = |\vec{x}| = \left[\sum_{l=1}^{m} x_l^2\right]^{1/2}. \tag{1.3}$$

We can write the Clifford algebra

$$\mathscr{A} = \left\{\sum_A a_A e_A;\ a_A \in \mathscr{C}\right\}, \tag{1.4}$$

where

$$e_A \equiv e_{\alpha_1} e_{\alpha_2} \cdots e_{\alpha_l} \tag{1.5}$$

and $A \equiv \{\alpha_1, \alpha_2, ..., \alpha_l\}$ is an ordered subset of $\{1, 2, ..., m\}$.

The function f defined on an open set Ω of $\mathbf{R}^{m+1}$ is said to be left monogenic when

$$(\partial_{\vec{x}} + \partial_{x_0}) f = 0, \qquad \forall x \in \Omega, \tag{1.6}$$

where $\partial_{\vec{x}} = \sum_{j=1}^{m} e_j (\partial/\partial x_j)$. Sommen [1] considered the sub-class of these functions which have the form

$$f_k(x) = [A_k(x_0, \rho) + \vec{e}_\rho B_k(x_0, \rho)]\ P_k(\vec{\omega}), \qquad k = 0, 1, 2, ..., \tag{1.7}$$

where the $P_k(\vec{x})$ are inner spherical functions of order k, i.e., the P_k's satisfy (1.6) and $P_k(\lambda\,\vec{x}) = \lambda^k P_k(\vec{x})$ for scalar λ. They have symmetry about the x_0-axis and are called axial monogenic functions. In our recent work [2] we showed how such functions can be constructed from holomorphic functions as given by the following result.

THEOREM 1. *Let f be an holomorphic function of complex variable Z on the line segment $\{Z = x_0 + i\rho t;\ -1 \leqslant t \leqslant 1\}$ for all $x = x_0 e_0 + \rho\vec{e}_\rho$ in a domain $\mathscr{D}$ of $\mathbf{R}^{m+1}$. Then*

$$f_k(x) \equiv \frac{\Gamma(k + m/2)}{\Gamma(k + m/2 - 1/2)\,\Gamma(1/2)} \times \int_{-1}^{1} (1 - t^2)^{k + m/2 - 3/2} f(x_0 + i\rho t)(1 - it\vec{e}_\rho)\, dt\, P_k(\vec{x}) \tag{1.8}$$

is an axial monogenic function of degree k for all $x \in \mathscr{D}$. ∎

Axially symmetric generalizations of the exponential and power functions were defined by taking $f(Z)$ to be of these forms and their properties were investigated. Hermite polynomials and Gegenbauer polynomials were also generalized using a generating function method used previously by Sommen [1], giving polynomials equivalent to those obtained by Cnops [3] when considering self-adjoint operators on Hilbert modules.

Monogenic functions in more general bi-axially symmetric domains have been introduced by Sommen [4] and are related to pseudo-analytic functions [5]. The approach is to consider the splitting $\mathbf{R}^m = \mathbf{R}^p + \mathbf{R}^q$ and to denote a general element $\vec{x}$ of $\mathbf{R}$ by $\vec{x} = \vec{x}_1 + \vec{x}_2 = \rho_1 \vec{\omega}_1 + \rho_2 \vec{\omega}_2$, where $\rho_1 = |\vec{x}_1|$, $\rho_2 = |\vec{x}_2|$, and $\vec{x}_1 \in \mathbf{R}^p$, $\vec{x}_2 \in \mathbf{R}^q$. Inner spherical monogenics $P_{k,l}(\vec{x}_1, \vec{x}_2)$ have been introduced in [5]. They are polynomials which are homogeneous of degrees k in $\vec{x}_1$ and l in $\vec{x}_2$ and which satisfy

$$\partial_{\vec{x}_1} P_{k,l} = \partial_{\vec{x}_2} P_{k,l} = 0, \tag{1.9}$$

where

$$\partial_{\vec{x}_1} \equiv \sum_{l=1}^{p} e_l \frac{\partial}{\partial x_l}; \qquad \partial_{\vec{x}_2} = \sum_{l=p+1}^{p+q} e_l \frac{\partial}{\partial x_l}. \tag{1.10}$$

One may then define monogenic functions of the form

$$\begin{aligned} f_{k,l}(\vec{x}) = [A_{k,l}(\rho_1, \rho_2) + \vec{\omega}_1 B_{k,l}(\rho_1, \rho_2) + \vec{\omega}_2 C_{k,l}(\rho_1, \rho_2) \\ + \vec{\omega}_1 \vec{\omega}_2 D_{k,l}(\rho_1, \rho_2)]\ P_{k,l}(\vec{\omega}_1 \vec{\omega}_2), \end{aligned} \tag{1.11}$$

with $A_{k,l}$, etc., scalar functions, and they are called bi-axial monogenic functions of degree (k, l). It is easy to show [5] that $f_{k,l}$ is monogenic if and only if

$$\left[\frac{\partial}{\partial \rho_1} - \frac{k}{\rho_1}\right] A_{k,l} + \left[\frac{\partial}{\partial \rho_2} + \frac{l+q-1}{\rho_2}\right] D_{k,l} = 0, \tag{1.12}$$

$$\left[\frac{\partial}{\partial \rho_1} + \frac{k+p-1}{\rho_1}\right] D_{k,l} - \left[\frac{\partial}{\partial \rho_2} - \frac{l}{\rho_2}\right] A_{k,l} = 0, \tag{1.13}$$

$$\left[\frac{\partial}{\partial \rho_1} + \frac{k+p-1}{\rho_1}\right] B_{k,l} + \left[\frac{\partial}{\partial \rho_2} - \frac{l+q-1}{\rho_2}\right] C_{k,l} = 0, \tag{1.14}$$

$$\left[\frac{\partial}{\partial \rho_1} - \frac{k}{\rho_1}\right] C_{k,l} - \left[\frac{\partial}{\partial \rho_2} - \frac{l}{\rho_2}\right] B_{k,l} = 0. \tag{1.15}$$

Taking monogenic functions of the form (1.11) means that we are basically working with functions of two real variables ρ_1, ρ_2, just as in the case of a holomorphic function of a complex variable $Z = X + iY$. In Section 2 we demonstrate how the two classes of functions are intimately related by showing how it is possible to transform a function f of complex variable Z into a bi-axial monogenic function of the form (1.11), thus generalising the result given by Theorem (1.1). Another class of monogenic functions which can be obtained from $f(Z)$ are the plane wave monogenics,

$$F(\vec{x}; \vec{u}, \vec{v}) = [\vec{u} + i\vec{v}]\, f(\langle \vec{x}, \vec{u}\rangle + i\langle \vec{x}, \vec{v}\rangle), \tag{1.16}$$

where $\vec{u} \in S^{p-1}$, $\vec{v} \in S^{q-1}$, and $\langle \vec{x}, \vec{u}\rangle = \sum_{l=1}^{p} u_l x_l$; $\langle \vec{x}, \vec{v}\rangle = \sum_{l=p+1}^{p+q} v_l x_l$. Here S^{p-1} and S^{q-1} are the unit spheres in $\mathbf{R}^p$ and $\mathbf{R}^q$, respectively.

The monogenicity of $F(\vec{x}; \vec{u}, \vec{v})$ follows from the Cauchy–Riemann equations satisfied by the real and imaginary parts of $f(Z)$. It is demonstrated in Section 2 that the bi-axial monogenic functions we constructed may also be obtained by integrating $F(\vec{x}; \vec{u}, \vec{v})$ with respect to $(\vec{u}, \vec{v})$ over suitable domains.

In Section 3 we construct the bi-axial exponential function $\mathscr{E}^{p,q}_{k,l}(\vec{x})$ corresponding to taking $f(Z) \equiv e^Z$ and then obtain an upper bound on its modulus. A topic discussed by Sommen [1] was the generalisation of Hermite polynomials $H_n(Z)$ to polynomials $H_{n,m,k}(\vec{x})$ using an axial monogenic generating function such that $\{H_{n,m,k}P_k; k, n \in N\}$ form an orthogonal basis for $L_2\{\mathbf{R}^m; \exp(-\rho^2/2)\}$. These polynomials were also considered in [2] and a similar method was used to construct generalisations of Gegenbauer polynomials, $C^{(\alpha)}_{n,m,k}(\vec{x})$ such that

$$\left\{C^{(\alpha)}_{n,m,k}\left(\frac{\vec{x}}{(1+\rho^2)^{1/2}}\right) P_k\left(\frac{\vec{x}}{(1+\rho^2)^{1/2}}\right); k, n \in N)\right\}$$

form a basis for $L_2\{\mathbf{R}^m; (1+\rho^2)^{-[\alpha+m/2+k+1/2]}\}$. In Section 4 we repeat the process by starting from a bi-axial monogenic generating function. We find that the polynomials we obtain are very closely related to those above and are orthogonal on $\mathbf{R}^p$.

Finally, in Section 5 we consider the bi-axial monogenic power functions starting from $f(Z) = Z^\alpha$ for real α. Because of the singularity that $f(Z)$ then has at $Z = 0$ when α is non-integer, we have to modify our definition of the corresponding integral transform by deforming the contour of integration. We then find expressions for these bi-axial monogenic power functions in terms of hypergeometric functions ${}_2F_1[a_1, a_2; b_1; Z]$.

2. A Biaxial Monogenic Transform of Holomorphic Functions

Our method for constructing bi-axial monogenic functions from holomorphic functions is contained in the following result.

THEOREM 2. *Let f be a scalar valued holomorphic function of complex variable Z on the rectangle $\{Z=\rho_1 t+i\rho_2 s;\ -1\leqslant s, t\leqslant 1\}$ for all $\vec{x}=\rho_1\vec{\omega}_1+\rho_2\vec{\omega}_2$ in a domain $\mathscr{D}$ of $\mathbf{R}^{p+q}$. Then*

$$f_{k,l}(\vec{x})=C_{k,l}^{p,q}\int_{-1}^{1}\int_{-1}^{1} f(\rho_1 t+i\rho_2 s)(1-t^2)^{k+(p-3/2)}(1-s^2)^{l+(q-3)/2} \times\{1+\vec{\omega}_1 t+i\vec{\omega}_2 s+i\vec{\omega}_1\vec{\omega}_2 st\}\,dt\,ds\,P_{k,l}(\vec{x}_1,\vec{x}_2) \tag{2.1}$$

is a bi-axial monogenic function of degree (k, l) for all $\vec{x}\in\mathscr{D}$, where $C_{k,l}^{p,q}$ is a scalar normalisation constant to be defined later.

Proof. Taking the normalisation constant to be understood for the time being, the function defined in (2.1) has the representation (1.11) with

$$A_{k,l}(\rho_1,\rho_2)=\rho_1^k\rho_2^l\int_{-1}^{1}\int_{-1}^{1} f(\rho_1 t+i\rho_2 s)(1-t^2)^{k+(p-3)/2} \times(1-s^2)^{l+(q-3)/2}\,ds\,dt \tag{2.2}$$

$$B_{k,l}(\rho_1,\rho_2)=\rho_1^k\rho_2^l\int_{-1}^{1}\int_{-1}^{1} f(\rho_1 t+i\rho_2 s)\,t(1-t^2)^{k+(p-3)/2} \times(1-s^2)^{l+(q-3)/2}\,ds\,dt \tag{2.3}$$

$$C_{k,l}(\rho_1,\rho_2)=\rho_1^k\rho_2^l\int_{-1}^{1}\int_{-1}^{1} f(\rho_1 t+i\rho_2 s)\,is(1-t^2)^{k+(p-3)/2} \times(1-s^2)^{l+(q-3)/2}\,ds\,dt \tag{2.4}$$

$$D_{k,l}(\rho_1,\rho_2)=\rho_1^k\rho_2^l\int_{-1}^{1}\int_{-1}^{1} f(\rho_1 t+i\rho_2 s)\,ist(1-t^2)^{k+(p-3)/2} \times(1-s^2)^{l+(q-3)/2}\,ds\,dt. \tag{2.5}$$

It is straightforward to show using direct substitution and integration by parts that these quantities satisfy (1.12) to (1.15) so long as $f(\rho_1 t+i\rho_2 s)$ is differentiable for all values of $\rho_1 t+i\rho_2 s$ corresponding to $\vec{x}$ in $\mathscr{D}$ and $-1\leqslant t,\ s\leqslant 1$. The function $f_{k,l}(\vec{x})$ is then monogenic for $(\vec{x})$ in this domain. ∎

In our previous study of axial monogenic functions [2] we showed that the integral transform (1.8) is obtained by integrating a plane wave monogenic over the unit sphere S^{m-1} in $\mathbf{R}^m$. Here we demonstrate the corresponding result for the integral transform (2.1).

Let $g(Z)$ have necessary differentiability properties and integrate the corresponding plane wave monogenic $[\vec{u}+i\vec{v}]\,g(\langle \vec{x}, \vec{u}\rangle + i\langle \vec{x}, \vec{v}\rangle)$ over the unit spheres S^{p-1} in $\mathbf{R}^p$ and S^{q-1} in $\mathbf{R}^q$ after multiplication from the right by $P_{k,l}(\vec{u}, \vec{v})$. This defines the integral transform

$$g_{k,l}(\vec{x}) = \int_{S^{p-1}} \int_{S^{q-1}} (\vec{u}+i\vec{v})\, g(\langle \vec{x}_1, \vec{u}\rangle + i\langle \vec{x}_2, \vec{v}\rangle) \times P_{k,l}(\vec{u}, \vec{v})\, d^{p-1}\vec{u}\, d^{q-1}\vec{v}. \tag{2.6}$$

Now $P_{k,l}(\vec{u}, \vec{v})$ are spherical harmonics of degree k in $\vec{u}$ and degree l in $\vec{v}$ while $\vec{u}P_{k,l}(\vec{u}, \vec{v})$ is a spherical harmonic of degree $k+1$ and l, respectively. Similarly, $\vec{v}P_{k,l}(\vec{u}, \vec{v})$ is a spherical harmonic of degree k in $\vec{u}$ and $l+1$ in $\vec{v}$. The Funk–Hecke theorem [6] may be applied to the RHS of (2.6) to give

$$\begin{aligned} g_{k,l}(\vec{x}) = \omega_{p-1}\omega_{q-1} \int_{-1}^{1}\int_{-1}^{1} & g(\rho_1 t + i\rho_2 s) \\ & \times [\mathbf{P}_{k+1,p}(t)\, \mathbf{P}_{l,q}(s)\, \vec{\omega}_1 + i\mathbf{P}_{k,p}(t)\, \mathbf{P}_{l+1,q}(s)\, \vec{\omega}_2] \\ & \times (1-t^2)^{(p-3)/2}\, (1-s^2)^{(q-3)/2}\, dt\, ds\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2), \end{aligned} \tag{2.7}$$

where $t = \langle \vec{u}, \vec{\omega}_1\rangle$, $s = \langle \vec{v}, \vec{\omega}_2\rangle$, and ω_{p-1}, ω_{q-1} are the areas of the unit spheres in $\mathbf{R}^p$, $\mathbf{R}^q$, respectively. The $\mathbf{P}_{k,p}(t)$ are Legendre polynomials in p-dimensions and are given by the Rodrigues formula,

$$\mathbf{P}_{k,p}(t) = \left(-\frac{1}{2}\right)^k \frac{[\Gamma(p-1)/2]}{\Gamma[k+(p-1)/2]} (1-t^2)^{(3-p)/2} \frac{d^k}{dt^k} (1-t^2)^{k+(p-3)/2}. \tag{2.8}$$

Substituting from (2.8) in (2.7) and integrating by parts, one arrives at

$$\begin{aligned} g_{k,l}(\vec{x}) = {} & \omega_{p-1}\omega_{q-1} N_{p,k} N_{q,l}\, i^l (-)^{k+l} \\ & \times \rho_1^k \rho_2^l \int_{-1}^{1}\int_{-1}^{1} g^{(k+l)}(\rho_1 t + i\rho_2 s)(1-t^2)^{k+(p-3)/2}\, (1-s^2)^{l+(q-3/2)} \\ & \times \{\vec{\omega}_1 t + i\vec{\omega}_2 s\}\, dt\, ds\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2), \end{aligned} \tag{2.9}$$

where

$$N_{p,k} = (-1/2)^k\, \Gamma(p-1/2)/\Gamma[k+(p-1)/2]. \tag{2.10}$$

We can similarly transform the function $(1+i\vec{u}\vec{v})\,g(\langle \vec{x}_1, \vec{u}\rangle + i\langle \vec{x}_2, \vec{v}\rangle)$ which is also a plane wave monogenic. Carrying through the same reasoning we find

$$G_{k,l}(\vec{x}) \equiv \int_{S^{p-1}} \int_{S^{q-1}} (1 + i\vec{u}\vec{v})\, g(\langle \vec{x}_1, \vec{u} \rangle + i\langle \vec{x}_2, \vec{v} \rangle)\, P_{k,l}(\vec{u}, \vec{v})\, d^{p-1}\vec{u}\, d^{q-1}\vec{v}$$
$$= \omega_{p-1}\omega_{q-1} N_{p,k} N_{q,l} i^l (-)^{k+l}$$
$$\times \rho_1^k \rho_2^l \int_{-1}^{1} \int_{-1}^{1} g^{(k+l)}(\rho_1 t + i\rho_2 s)(1-t^2)^{k+(p-3)/2}$$
$$\times (1-s^2)^{l+(q-3)/2} \{1 + i\vec{\omega}_1 \vec{\omega}_2 st\}\, dt\, ds\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2). \tag{2.11}$$

Adding (2.9) to (2.11) we obtain a bi-axial monogenic function of the form (2.1) with $f(Z) = g^{(k+l)}(Z)$. We have demonstrated therefore that the integral transform (2.1) may be obtained by averaging a monogenic plane wave.

3. Bi-Axial Monogenic Exponential Function

As in the axial monogenic case, we now define a bi-axial monogenic exponential function $\mathscr{E}_{k,l}^{p,q}$ by substituting $f(Z) = e^Z$ in (2.1):

$$\mathscr{E}_{k,l}^{p,q}(\rho_1, \rho_2)\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2)$$
$$\equiv C_{k,l}^{p,q} \rho_1^k \rho_2^l \int_{-1}^{1} \int_{-1}^{1} e^{\rho_1 t + i\rho_2 s}(1-t^2)^{k+(p-3)/2}(1-s^2)^{l+(q-3)/2}$$
$$\times \{1 + \vec{\omega}_1 t + i\vec{\omega}_2 s + i\vec{\omega}_1\vec{\omega}_2 st\}\, dt\, ds\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2). \tag{3.1}$$

The integrals over s and t in (3.1) may be separated. Using the standard integral representations for Bessel functions and modified Bessel functions given by

$$J_\nu(\rho) = \frac{(\rho/2)^\nu}{\Gamma(\nu + 1/2)\, \Gamma(1/2)} \int_{-1}^{1} (1-s^2)^{\nu - 1/2} e^{i\rho s}\, ds \tag{3.2}$$

and

$$I_\nu(\rho) = \frac{(\rho/2)^\nu}{\Gamma(\nu + 1/2)\, \Gamma(1/2)} \int_{-1}^{1} (1-t^2)^{\nu - 1/2} e^{\rho t}\, dt, \tag{3.3}$$

and defining the normalization constant by

$$C_{k,l}^{p,q} = \frac{\Gamma(k + p/2)\, \Gamma(l + q/2)}{\pi \Gamma[k + (p-1)/2]\, \Gamma[l + (q-1)/2]}, \tag{3.4}$$

we find that

$$\mathscr{E}_{k,l}^{p,q}(\rho_1, \rho_2) = 2^{k+l+(p+q)/2-2}\Gamma(k+p/2)\,\Gamma(l+q/2)\,\rho_1^{1-p/2}\rho_2^{1-q/2}$$
$$\times [I_{k+p/2-1}(\rho_1)\,J_{l+q/2-1}(\rho_2) + I_{k+p/2}(\rho_1)\,J_{l+q/2-1}(\rho_2)\,\vec{\omega}_1$$
$$- I_{k+p/2-1}(\rho_1)\,J_{l+q/2}(\rho_2)\,\vec{\omega}_2 - I_{k+p/2}(\rho_1)\,J_{l+q/2}(\rho_2)\,\vec{\omega}_1\vec{\omega}_2]. \tag{3.5}$$

This is our definition of the bi-axial exponential function and it combines two bi-axial exponential functions defined by one of us previously [7]. We have chosen the normalisation constant $C_{k,l}^{p,q}$ as in (3.4) so that

$$\mathscr{E}_{k,l}^{p,q}(\rho_1, \rho_2) \approx \rho_1^k \rho_2^l \tag{3.6}$$

when $\rho_1, \rho_2 \to 0$. We keep this definition of $C_{k,l}^{p,q}$ for the remainder of this work.

The norm of the Clifford number $a \equiv \sum_A a_A e_A$ is defined by

$$|a| = |(a\bar{a})_0|^{1/2}, \tag{3.7}$$

where $(b)_0$ is the scalar part of b and

$$\bar{a} \equiv \sum_A a_A^* \bar{e}_A \tag{3.8}$$

with

$$\bar{e}_A = e_{\alpha_l} \cdots e_{\alpha_1} (-)^l. \tag{3.9}$$

We now obtain an upper bound to $|\mathscr{E}_{k,l}^{p,q}(\rho_1, \rho_2)|$ for all $\rho_1, \rho_2 \geqslant 0$. From the definition (3.7) and the expression (3.5) for the bi-axial exponential function,

$$|\mathscr{E}_{k,l}^{p,q}(\rho_1, \rho_2)|^2$$
$$= 2^{2(k+l)+p+q-4}\rho_1^{2-p}\rho_2^{2-q}\Gamma(k+p/2)^2\,\Gamma(l+q/2)^2$$
$$\times \{I_{k+p/2-1}(\rho_1)^2 + I_{k+p/2}(\rho_1)^2\}\{J_{l+q/2-1}(\rho_2)^2 + J_{l+q/2}(\rho_2)^2\}. \tag{3.10}$$

In our previous work [2] we derived the bound

$$J_{l+q/2-1}(\rho_2)^2 + J_{l+q/2}(\rho_2)^2 \leqslant \frac{\rho_2^{2l+q-2}\Gamma(l+q/2-1/2)}{\pi^{1/2}\Gamma(l+q/2)\,\Gamma(2l+q-1)}. \tag{3.11}$$

We can obtain similar bounds for the modified Bessel functions from relation [8],

$$I_\mu^2(\rho)=\frac{2}{\pi}\int_0^{\pi/2} I_{2\mu}(2\rho\cos\theta)\,d\theta. \tag{3.12}$$

Therefore,

$$\begin{aligned}I_{k+p/2-1}(\rho_1)^2+I_{k+p/2}(\rho_1)^2&=\frac{2(2k+p-1)}{\pi\rho_1}\int_0^{\pi/2} I_{2k+p-1}(2\rho_1\cos\theta)/\cos\theta\,d\theta\\&=\frac{2(2k+p-1)}{\pi^{3/2}\Gamma(2k+p-1/2)\,\rho_1}\int_0^{\pi/2}\int_{-1}^{1}\frac{(\rho_1\cos\theta)^{2k+p-1}}{\cos\theta}\\&\quad\times e^{-2t\rho_1\cos\theta}(1-t^2)^{2k+p-3/2}\,dt\,d\theta,\end{aligned} \tag{3.13}$$

where we have used the representation (3.3) for the modified Bessel function. Bounding $e^{-2t\rho_1\cos\theta}$ by $e^{2\rho_1}$, the integrals may be expressed in terms of beta functions, giving

$$I_{k+p/2-1}(\rho_1)^2+I_{k+p/2}(\rho_1)^2\leqslant\frac{\rho_1^{2k+p-2}\Gamma(k+p/2-1/2)\,e^{2\rho_1}}{\pi^{1/2}\Gamma(k+p/2)\,\Gamma(2k+p-1)}. \tag{3.14}$$

Substituting the bounds (3.11), (3.14) in (3.10) and using the duplication formula for gamma functions,

$$|\mathscr{E}_{k,l}^{p,q}(\rho_1,\rho_2)|\leqslant e^{\rho_1}\rho_1^k\rho_2^l,\qquad \rho_1,\rho_2\geqslant 0, \tag{3.15}$$

which is the required bound for the bi-axial exponential function.

4. Bi-axial Hermite and Gegenbauer Polynomials

Before considering the existence of the generalization of Hermite and Gegenbauer polynomials to the bi-axial case, we consider integral power bi-axial monogenics obtained by substituting $f(Z)=Z^j$, $j=0,1,2,\ldots$, in (2.1). The discussion of non-integer powers involves some modification of the integral transform in (2.1) and is delayed until the next section.

We therefore obtain the definition

$$\begin{aligned}&R_{k,l;j}^{p,q}(\rho_1,\rho_2)\,P_{k,l}(\vec\omega_1,\vec\omega_2)\\&\quad=C_{k,l}^{p,q}\rho_1^k\rho_2^l\int_{-1}^{1}\int_{-1}^{1}(\rho_1 t+i\rho_2 s)^j\,(1-t^2)^{k+(p-3)/2}\,(1-s^2)^{l+(q-3)/2}\\&\quad\times\{1+\vec\omega_1 t+i\vec\omega_2 s+i\vec\omega_1\vec\omega_2 st\}\,dt\,ds\,P_{k,l}(\vec\omega_1,\vec\omega_2).\end{aligned} \tag{4.1}$$

We need the behavior as $\rho_2 \to 0$ of the powers

$$R^{p,q}_{k,l;j}(\rho_1, \rho_2) \approx \frac{\Gamma[(j+1)/2]\, \Gamma(k+p/2)\, \rho_1^{j+k} \rho_2^l}{\Gamma(1/2)\, \Gamma[k+(p+j)/2]}, \qquad j = 0, 2, 4, ..., \tag{4.2}$$

$$\approx \frac{\Gamma(j/2+1)\, \Gamma(k+p/2)\, \vec{x}_1 \rho_1^{j+k-1} \rho_2^l}{\Gamma(1/2)\, \Gamma[k+(p+j+1)/2]}, \qquad j = 1, 3, 5, ..., \tag{4.3}$$

which follows straightforwardly from the representation (4.1).

We now define bi-axial Hermite polynomials $H_{n;k,l;p,q}(\vec{x}_1)$ by requiring that

$$f(\vec{x}) \equiv e^{-\rho_1^2/2} \sum_{n=0}^{\infty} H_{n;k,l;p,q}(\vec{x}_1) \frac{\vec{x}_2^n}{n!} P_{k,l}(\vec{x}_1, \vec{x}_2) \tag{4.4}$$

is monogenic and has the behaviour as $\rho_2 \to 0$ of the form

$$f(\vec{x}) \approx e^{-\rho_1^2/2} P_{k,l}(\vec{x}_1, \vec{x}_2) = \rho_1^k \rho_2^l e^{-\rho_1^2/2} P_{k,l}(\vec{\omega}_1, \vec{\omega}_2). \tag{4.5}$$

An alternative expression for $f(\vec{x})$ is obtained by expanding by RHS of (4.5) in powers of ρ_1 and then replacing these powers by the monogenic power functions $R^{p,q}_{k,l;j}(\rho_1, \rho_2)$. We then find

$$f(\vec{x}) = \sum_{j=0}^{\infty} \frac{(-1/2)^j\, \Gamma(1/2)\, \Gamma(k+p/2+j)}{\Gamma(j+1)\, \Gamma(j+1/2)\, \Gamma(k+p/2)} R^{p,q}_{k,l;2j}(\rho_1, \rho_2)\, P_{k,l}(\vec{\omega}_1, \vec{\omega}_2), \tag{4.6}$$

and it is straightforward to check using (4.2) that $f(\vec{x})$ has the correct limiting behaviour as $\rho_2 \to 0$.

Since the RHS of (4.5) has a unique monogenic extension to domains where $\rho_2 \neq 0$ [9], the expressions (4.4), (4.6) for $f(\vec{x})$ are equivalent. The generalised Hermite polynomials are then obtained by picking out the coefficients of powers of ρ_2 on the RHS of (4.6) using (4.1) for the $R^{p,q}_{k,l;2j}(\rho_1, \rho_2)$. The coefficient of $\rho_2^n P_{k,l}(\vec{x}_1, \vec{x}_2)$ for $n = 2r$, $r = 0, 1, 2, ...$, is then

$$= \frac{\Gamma(l+q/2)(-)^r}{\Gamma(1/2)\, \Gamma[k+(p-1)/2]\, \Gamma[l+(q-1)/2]}$$
$$\times \sum_{j=r}^{\infty} \frac{(-1/2)^j\, \Gamma(k+p/2+j)\, \Gamma(2j+1)\, \rho_1^{2j-2r}}{\Gamma(j+1)\, \Gamma(2r+1)\, \Gamma(2j-2r+1)\, \Gamma(j+1/2)}$$
$$\times \int_{-1}^{1} \int_{-1}^{1} t^{2j-2r} s^{2r} (1-t^2)^{k+(p-3)/2} (1-s^2)^{l+(q-3)/2} (1+i\vec{\omega}_1 \vec{\omega}_2 st)\, dt\, ds. \tag{4.7}$$

The integrals may be expressed as beta functions and the contribution from the $\vec{\omega}_1\vec{\omega}_2$ term is zero. Setting $L=j-r$ and using the duplication formula for gamma functions, it is easy to show that this coefficient

$$=\frac{\Gamma(l+q/2)\,\Gamma(k+p/2+r)}{2^r\Gamma(r+1)\,\Gamma(k+p/2)\,\Gamma(l+q/2+r)}\,\Phi(k+p/2+r;k+p/2;-\rho_1^2/2)$$

$$=\frac{\Gamma(l+q/2)\,\Gamma(k+p/2+r)}{2^r\Gamma(r+1)\,\Gamma(k+p/2)\,\Gamma(l+q/2+r)}\,e^{-\rho_1^2/2}\Phi(-r;k+p/2;\rho_1^2/2), \quad (4.8)$$

where $\Phi(a;b;Z)$ are confluent hypergeometric functions. Comparing this coefficient with that of $\rho_2^n P_{k,l}(\vec{x}_1,\vec{x}_2)$ in (4.4), we finally arrive at the result

$$H_{2r;k,l;p,q}(\vec{x}_1)=\frac{(-2)^r\,(k+p/2)_r\,(1/2)_r}{(l+q/2)_r}\,\Phi(-r;k+p/2;\rho_1^2/2),$$
$$r=0,1,2,\ldots \quad (4.9)$$

Similarly, it may be shown that

$$H_{2r+1;k,l;p,q}(\vec{x}_1)=\frac{(-2)^r\,(k+p/2+1)_r\,(1/2)_{r+1}}{(l+q/2+1)_{r+1}}$$
$$\times\,\vec{x}_1\Phi(-r;k+p/2+1;\rho_1^2/2), \qquad r=0,1,2,\ldots \quad (4.10)$$

It should be noted that the l, q dependence of these polynomials occurs only in the coefficients outside the hypergeometric functions. In fact, these generalised Hermite polynomials are just constant multiples of the $H_{n,p,k}(\vec{x}_1)$ obtained previously in the axial case [2] and more precisely,

$$H_{2r;k,l;p,q}(\vec{x}_1)=\frac{(-)^r\,(1/2)_r}{(l+q/2)_r}\,H_{2r,p,k}(\vec{x}_1), \qquad r=0,1,2,\ldots, \quad (4.11)$$

and

$$H_{2r+1;k,l;p,q}(\vec{x}_1)=\frac{(-)^r\,(1/2)_{r+1}}{(l+q/2)_{r+1}}\,H_{2r+1,p,k}(\vec{x}_1), \qquad r=0,1,2,\ldots \quad (4.12)$$

The generalised bi-axial Gegenbauer polynomials are similarly obtained by requiring

$$f(\vec{x})=(1+\rho_1^2)^{-[\alpha+k+l+(p+q)/2-1]}\sum_{j=0}^{\infty}\left(\frac{1}{1+\rho_1^2}\right)^{j/2}$$
$$\times\,C^{(\alpha)}_{j;p,q;k,l}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right]\vec{x}_2^{\,j}P_{k,l}(\vec{x}_1,\vec{x}_2) \quad (4.13)$$

to be monogenic and have the behaviour as $\rho_2 \to 0$ of the form

$$f(\vec{x}) \approx (1+\rho_1^2)^{-[\alpha+k+l+(p+q)/2-1]} \rho_1^k \rho_2^l P_{k,l}(\vec{\omega}_1, \vec{\omega}_2)$$
$$= \sum_{r=0}^{\infty} \frac{(-)^r (\alpha+k+p/2+q/2-1)_r}{\Gamma(r+1)} \rho_1^{2r+k} \rho_2^l P_{k,l}(\vec{\omega}_1, \vec{\omega}_2). \quad (4.14)$$

Using (4.2) and the fact that as shown in [9] the RHS has a unique monogenic extension to domains with $\rho_2 \neq 0$,

$$f(\vec{x}) = \sum_{r=0}^{\infty} \frac{(-)^r (\alpha+k+l+p/2+q/2-1)_r (k+p/2)_r}{\Gamma(r+1)(1/2)_r}$$
$$\times R_{k,l;2r}^{p,q}(\rho_1, \rho_2) P_{k,l}(\vec{\omega}_1, \vec{\omega}_2). \quad (4.15)$$

Substituting from (4.1) and (3.4),

$$f(\vec{x}) = \sum_{r=0}^{\infty} \frac{(-)^r(\alpha+k+l+p/2+q/2-1)_r \Gamma(k+p/2+r) \Gamma(q+l/2)}{\Gamma(1/2) \Gamma(r+1) \Gamma(r+1/2) \Gamma[k+(p-1)/2] \Gamma[l+(q-1)/2]}$$
$$\times \sum_{n=0}^{2r} \frac{\Gamma(2r+1) \rho_1^{2r-n} (i\rho_2)^n}{\Gamma(n+1) \Gamma(2r-n+1)}$$
$$\times \int_{-1}^{1} \int_{-1}^{1} t^{2r-n} s^n (1-t^2)^{k+(p-3)/2} (1-s^2)^{l+(q-3)/2}$$
$$\times \{1 + \vec{\omega}_1 t + i\vec{\omega}_2 s + i\vec{\omega}_1 \vec{\omega}_2 st\} \, dt \, ds \, P_{k,l}(\vec{x}_1, \vec{x}_2). \quad (4.16)$$

The integrals over the terms proportional to $\vec{\omega}_1$ and to $\vec{\omega}_2$ vanish. The integrals over the other terms give beta functions and thus may be used with the duplication formula for gamma functions to prove that the coefficient of ρ_2^{2j}, $j = 0, 1, 2, \ldots$, on the RHS of (4.16)

$$= \frac{(\alpha+k+l+p/2+q/2-1)_j (k+p/2)_j}{\Gamma(j+1)(l+q/2)_j}$$
$$\times {}_2F_1(\alpha+k+l+p/2+q/2+j-1, k+p/2+j; k+p/2; -\rho_1^2) P_{k,l}(\vec{x}_1, \vec{x}_2), \quad (4.17)$$

where ${}_2F_1(a, b; c; Z)$ is the hypergeometric function. But we have the standard identity

$${}_2F_1(a, b; c; Z) = (1-Z)^{-a} {}_2F_1(a, c-b; c; Z/(Z-1)). \quad (4.18)$$

Using this in (4.17) and comparing with (4.13), it follows that

$$C^{(\alpha)}_{2j;p,q;k,l}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right] = \frac{(-)^j\,(\alpha+k+l+p/2+q/2-1)_j\,(k+p/2)_j}{(l+q/2)_j\,\Gamma(j+1)} \times {}_2F_1(\alpha+k+l+p/2+q/2+j-1,\ -j;\ k+p/2;\ \rho_1^2/(1+\rho_1^2)),\quad j=0,1,2,\ldots \tag{4.19}$$

Similarly,

$$C^{(\alpha)}_{2j+1;p,q;k,l}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right] = \frac{(-)^j\,(\alpha+k+l+p/2+q/2-1)_{j+1}\,(k+1+p/2)_j}{\Gamma(j+1)(l+q/2)_{j+1}} \times \frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\,{}_2F_1\left(\alpha+k+l+p/2+q/2+j,\ -j;\ k+1+p/2;\ \frac{\rho_1^2}{1+\rho_1^2}\right),\quad j=0,1,2,\ldots \tag{4.20}$$

Since the second parameter of the hypergeometric function on the RHS of (4.19) and of (4.20) is a negative integer, it follows that $C^{(\alpha)}_{2j;p,q;k,l}$, $C^{(\alpha)}_{2j+1;p,q;k,l}$ are polynomials of order $2j$ and $2j+1$, respectively, in $\vec{x}_1/(1+\rho_1^2)^{1/2}$.

As in the case of Hermite polynomials, these bi-axial Gegenbauer polynomials are related to those obtained previously in the axial case [2] which we denoted by $C^{(\alpha)}_{n;m,k}[\vec{x}_1/(1+\rho_1^2)^{1/2}]$. In fact,

$$C^{(\alpha)}_{2j;p,q;k,l}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right] = \frac{(-)^j\,(1/2)_j}{(l+q/2)_j}\,C^{(\alpha+q/2+l-1/2)}_{2j;p,k}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right], \tag{4.21}$$

$$C^{(\alpha)}_{2j+1;p,q;k,l}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right] = \frac{(-)^{j+1}\,(1/2)_{j+1}}{(l+q/2)_{j+1}}\,C^{(\alpha+q/2+l-1/2)}_{2j+1;p,k}\left[\frac{\vec{x}_1}{(1+\rho_1^2)^{1/2}}\right], \tag{4.22}$$

for $j=0,1,2,\ldots$ As expected, the bi-axial polynomials reduce to the axial polynomials in the case when $q=1$ and $l=0$ as the latter corresponds to the axial monogenic case.

5. Bi-axial Monogenic Power Functions

We now consider the bi-axial monogenic functions corresponding to $f(Z)=Z^\alpha$ with α non-integer. Unfortunately, if we substitute this into the integral transform (2.1), we see that the integrand has a singularity at $s=t=0$ and is therefore not differentiable at this point. To avoid this problem we consider an alternative definition of the bi-axial monogenic function $f_{k,l}(\vec{x})$ by deforming the contours of integration as was done in the axial case [2]. Instead of integrating along the straight line from $t=-1$ to $t=+1$ we integrate over the semi-circle $\{t=e^{-i\theta}; 0\leqslant\theta\leqslant\pi\}$ and similarly for s. Then we obtain the definition

$$\begin{aligned} f_{k,l}(\vec{x}) = -\int_0^\pi\int_0^\pi & f(\rho_1 e^{-i\theta}+i\rho_2 e^{-i\phi})(1-e^{-2i\theta})^{k+(p-3)/2}\,(1-e^{-2i\phi})^{l+(q-3)/2} \\ & \times\{1+\vec{\omega}_1 e^{-i\theta}+i\vec{\omega}_2 e^{-i\phi}+i\vec{\omega}_1\vec{\omega}_2 e^{-i(\theta+\phi)}\}\,e^{-i(\phi+\theta)}\,d\phi\,d\theta \\ & \times P_{k,l}(\vec{x}_1,\vec{x}_2). \end{aligned} \tag{5.1}$$

It is straightforward to prove as in Section that this is a bi-axial monogenic if $f(Z)$ is holomorphic when $Z=\rho_1 e^{-i\theta}+i\rho_2 e^{-i\phi}$ for all $0\leqslant\theta,\ \phi\leqslant\pi$. Let us consider the case when $f(Z)=Z^\alpha$ and $\rho_1<\rho_2$. Now

$$Z=\rho_1\cos\theta+\rho_2\sin\phi+i(-\rho_1\sin\theta+\rho_2\cos\phi) \tag{5.2}$$

and is real when

$$\cos\phi=\frac{\rho_1}{\rho_2}\sin\theta. \tag{5.3}$$

In that case

$$Z=\pm\rho_1\sqrt{1-\sin^2\theta}+\rho_2\sqrt{1-\frac{\rho_1^2}{\rho_2^2}\sin^2\theta}>0 \qquad \text{when} \quad \rho_2>\rho_1. \tag{5.4}$$

Therefore $f(Z)=Z^\alpha$ is holomorphic for all $0\leqslant\theta,\ \phi\leqslant\pi$, as required.

Substituting this function in (4.1) we have to derive the following integrals:

$$\begin{aligned} I_1=\int_0^\pi\int_0^\pi & (\rho_1 e^{-i\theta}+i\rho_2 e^{\ i\phi})^\alpha\,(1-e^{-2i\theta})^{k+(p-3)/2} \\ & \times(1-e^{-2i\phi})^{l+(q-3)/2}\,e^{-i(\phi+\theta)}\,d\phi\,d\theta \end{aligned} \tag{5.5}$$

$$\begin{aligned} I_2=\int_0^\pi\int_0^\pi & (\rho_1 e^{-i\theta}+i\rho_2 e^{-i\phi})^\alpha\,(1-e^{-2i\theta})^{k+(p-3)/2} \\ & \times(1-e^{-2i\phi})^{l+(q-3)/2}\,e^{-i(\phi+2\theta)}\,d\phi\,d\theta \end{aligned} \tag{5.6}$$

$$I_3 = \int_0^\pi \int_0^\pi (\rho_1 e^{-i\theta} + i\rho_2 e^{-i\phi})^\alpha (1 - e^{-2i\theta})^{k+(p-3)/2} \times (1 - e^{-2i\phi})^{l+(q-3)/2} e^{-i(2\phi+\theta)} \, d\phi \, d\theta \tag{5.7}$$

$$I_4 = \int_0^\pi \int_0^\pi (\rho_1 e^{-i\theta} + i\rho_2 e^{-i\phi})^\alpha (1 - e^{-2i\theta})^{k+(p-3)/2} \times (1 - e^{-2i\phi})^{l+(q-3)/2} e^{-i(2\phi+2\theta)} \, d\phi \, d\theta \tag{5.8}$$

We now obtain expressions for these integrals in terms of hypergeometric functions demonstrating our method for I_1 and just presenting the results for the other integrals.

The binomial expansion may be used for $\rho_1 < \rho_2$ to write

$$I_1 = \rho_2^\alpha \sum_{j=0}^{\infty} \int_0^\pi \int_0^\pi e^{i(\pi/2-\phi)\alpha} \left[-ie^{i(\phi-\theta)} \frac{\rho_1}{\rho_2} \right]^j \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\,\Gamma(\alpha-j+1)} \times (1-e^{-2i\theta})^{k+(p-3)/2} (1-e^{-2i\phi})^{l+(q-3)/2} e^{-i(\phi+\theta)} \, d\phi \, d\theta \tag{5.9}$$

$$= \rho_2^\alpha i^\alpha \sum_{j=0}^{\infty} \frac{\Gamma(\alpha+1)(-i\rho_1/\rho_2)^j}{\Gamma(j+1)\,\Gamma(\alpha-j+1)} \times \left[\int_0^\pi e^{i\phi(-\alpha+j-1)} (1-e^{-2i\phi})^{l+(q-3)/2} \, d\phi \right] \times \left[\int_0^\pi e^{-i\theta(j+1)} (1-e^{-2i\theta})^{k+(p-3)/2} \, d\theta \right]. \tag{5.10}$$

But from standard tables of integrals [10],

$$\int_0^\pi e^{i\phi(-\alpha+j-1)} (1-e^{-2i\phi})^{l+(q-3)/2} \, d\phi$$

$$= \int_0^\pi e^{i\phi(-\alpha+j+1/2-l-q/2)} (2i)^{l+(q-3)/2} \sin^{l+(q-3)/2} \phi \, d\phi$$

$$= \frac{\pi i^{j-1-\alpha} \Gamma[l+(q-1)/2]}{\Gamma[(j+1-\alpha)/2]\, \Gamma[l+(q-j+\alpha)/2]} \tag{5.11}$$

$$= -i^{j-1-\alpha} \frac{\left(\begin{array}{c} \Gamma[l+(q-1)/2]\, \Gamma[1-l+(j-\alpha-q)/2] \\ \times \sin \pi[(j-\alpha-q)/2 - l] \end{array} \right)}{\Gamma[(j+1-\alpha)/2]}, \tag{5.12}$$

where in the last step we have used the reflection formula for gamma functions.

Similarly,

$$\int_0^{\pi} e^{-i\theta(j+1)}(1-e^{-2i\theta})^{k+(p-3)/2}\,d\theta$$

$$=\frac{\Gamma[k+(p-1)/2]\,\Gamma[(j+1)/2]\sin[(j+1)\,\pi/2]}{i^{j+1}\Gamma(j/2+k+p/2)}. \quad (5.13)$$

Substituting in (5.10),

$$I_1=-\rho_2^{\alpha}\,\Gamma(\alpha+1)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]$$
$$\times\sum_{j=0}^{\infty}\frac{\begin{pmatrix} i^{j-2}(-\rho_1/\rho_2)^j\,\Gamma[1-l+(j-\alpha-q)/2] \\ \times\sin[(j-\alpha-2l-q)\,\pi/2]\,\Gamma[(j+1)/2]\sin[(j+1)\,\pi/2]\end{pmatrix}}{\Gamma(j+1)\,\Gamma(\alpha-j+1)\,\Gamma[(j+1-\alpha)/2]\,\Gamma[(j+2k+p)/2]} \quad (5.14)$$

We note that the terms for odd j are zero so that by substituting $j=2r$, $r=0, 1, ...,$

$$I_1=\rho_2^{\alpha}\Gamma(\alpha+1)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]$$
$$\times\sum_{r=0}^{\infty}\frac{\begin{pmatrix}(-)^{r+1}\,\Gamma(r-l+1-\alpha/2-q/2) \\ \times\Gamma(r+1/2)\sin[(\alpha+2l+q)\,\pi/2]\end{pmatrix}}{\Gamma(2r+1)\,\Gamma(\alpha-2r+1)\,\Gamma(r+1/2-\alpha/2)\,\Gamma(r+k+p/2)}(\rho_1/\rho_2)^{2r}. \quad (5.15)$$

Using the reflection and duplication formulae for gamma functions,

$$I_1=\frac{\begin{pmatrix}\rho_2^{\alpha}\Gamma(\alpha+1)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2] \\ \times\sin[(\alpha+2l+q)\,\pi/2]\sin\pi\alpha\end{pmatrix}}{\pi 2^{\alpha+1}}$$
$$\times\sum_{r=0}^{\infty}\frac{\Gamma(r+1-l-q/2-\alpha/2)\,\Gamma(r-\alpha/2)[-\rho_1^2/\rho_2^2]^r}{\Gamma(r+1)\,\Gamma(r+k+p/2)} \quad (5.16)$$
$$=\frac{\begin{pmatrix}\rho_2^{\alpha}\sin\pi\alpha\sin[(\alpha+2l+q)\,\pi/2]\,\Gamma(\alpha+1) \\ \times\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]\end{pmatrix}}{2^{\alpha+1}\pi\Gamma(k+p/2)}$$
$$\times\Gamma(1-l-q/2-\alpha/2)\,\Gamma(-\alpha/2)$$
$$\times {}_2F_1\{1-l-q/2-\alpha/2,\ -\alpha/2;k+p/2;\ -\rho_1^2/\rho_2^2\}. \quad (5.17)$$

A final use of the reflection and duplication formulae for gamma functions leads to the result that

$$I_1 = \frac{-\pi^{1/2}\rho_2^\alpha \Gamma[(\alpha+1)/2]\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]\cos\pi\alpha/2}{\Gamma(l+q/2+\alpha/2)\,\Gamma(k+p/2)} \times {}_2F_1(1-l-q/2-\alpha/2,\ -\alpha/2;k+p/2;\ -\rho_1^2/\rho_2^2). \tag{5.18}$$

Similarly,

$$I_2 = \frac{\pi^{1/2}\rho_2^\alpha \Gamma(\alpha/2+1)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]\sin\pi\alpha/2}{\Gamma(l+q/2+\alpha/2-1/2)\,\Gamma(k+p/2+1)} \times (\rho_1/\rho_2)\,{}_2F_1(3/2-l-q/2-\alpha/2,\ 1/2-\alpha/2;k+p/2+1;\ -\rho_1^2/\rho_1^2). \tag{5.19}$$

$$I_3 = \frac{-i\pi^{1/2}\rho_2^\alpha \Gamma(\alpha/2+1)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]\sin\pi\alpha/2}{\Gamma(l+q/2+\alpha/2+1/2)\,\Gamma(k+p/2)} \times {}_2F_1(1/2-l-q/2-\alpha/2,\ 1/2-\alpha/2;k+p/2;\ -\rho_1^2/\rho_2^2) \tag{5.20}$$

and

$$I_4 = \frac{i\pi^{1/2}\rho_2^\alpha \Gamma(\alpha/2+1)(\alpha/2)\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2]\cos\pi\alpha/2}{\Gamma(l+q/2+\alpha/2)\,\Gamma(k+p/2+1)} \times (\rho_1/\rho_2)\,{}_2F_1(1-l-q/2-\alpha/2,\ 1-\alpha/2;k+p/2+1;\ -\rho_1^2/\rho_2^2). \tag{5.21}$$

Substituting in (5.1) we find that the bi-axial monogenic obtained from $f(Z)=Z^\alpha$ is

$$\begin{aligned} f_{k,l}^\alpha(\vec{x}) \equiv{}& \pi^{1/2}\rho_2^\alpha\,\Gamma[l+(q-1)/2]\,\Gamma[k+(p-1)/2] \\ &\times\Bigg\{\frac{\Gamma[(\alpha+1)/2]\cos\pi\alpha/2}{\Gamma(l+q/2+\alpha/2)\,\Gamma(k+p/2)} \\ &\times {}_2F_1(1-l-q/2-\alpha/2,\ -\alpha/2;k+p/2;\ -\rho_1^2/\rho_2^2) \\ &-\vec{\omega}_1\frac{\Gamma(\alpha/2+1)\sin\pi\alpha/2}{\Gamma(l+q/2+\alpha/2-1/2)\,\Gamma(k+p/2+1)} \\ &\times(\rho_1/\rho_2)\,{}_2F_1(3/2-l-q/2-\alpha/2,\ 1/2-\alpha/2;k+p/2+1;\ -\rho_1^2/\rho_2^2) \\ &-\vec{\omega}_2\frac{\Gamma(\alpha/2+1)\sin\pi\alpha/2}{\Gamma(l+q/2+\alpha/2+1/2)\,\Gamma(k+p/2)} \\ &\times {}_2F_1(1/2-l-q/2-\alpha/2,\ 1/2-\alpha/2;k+p/2;-\rho_1^2/\rho_2^2) \\ &+\vec{\omega}_1\vec{\omega}_2\frac{\Gamma(\alpha/2+1)(\alpha/2)\cos\pi\alpha/2}{\Gamma(l+q/2+\alpha/2)\,\Gamma(k+p/2+1)}(\rho_1/\rho_2) \\ &\times {}_2F_1(1-l-q/2-\alpha/2,\ 1-\alpha/2;k+p/2+1;\ -\rho_1^2/\rho_2^2)\Bigg\}\,P_{k,l}(\vec{x}_1, x_2). \end{aligned} \tag{5.22}$$

Since the conditions (1.12) to (1.15) for $f_{k,l}(\vec{x})$ in (5.1) to be a bi-axial monogenic only couple the scalar with the bivector component and, respectively, the two vector components together, we can change the signs of the vector terms in (5.2) and obtain new bi-axial monogenic power functions

$$g_{k,l}^{(\alpha)}(\vec{x}) \equiv -[I_1 - \vec{\omega}_1 I_2 - i\vec{\omega}_2 I_3 + i\vec{\omega}_1 \vec{\omega}_2 I_4]\, P_{k,l}(\vec{x}_1, \vec{x}_2). \qquad (5.23)$$

These may be combined with the $f_{k,l}^{(\alpha)}(\vec{x})$ to produce bi-axial monogenics which are proportional to ρ_2^α or $\rho_2^\alpha \vec{\omega}_2$ as $\rho_1 \to 0$. Such combinations would correspond to the "inner" and "outer" power functions discussed in the axial case [2].

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