European Journal of Combinatorics 30 (2009) 1540–1563

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Trends and lines of development in scheme theory

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ARTICLE INFO

Article history:
Available online 5 February 2009

ABSTRACT

The concept of an association scheme is one of those mathematical concepts which were utilized as technical tools in various different mathematical areas for a long time before becoming the subject of a theory in their own right. The significance of symmetric schemes, for instance, in the design of (statistical) experiments was recognized as early as the first half of the last century. Coding theory has been associated with commutative schemes for more than three decades, and polynomial schemes have provided the language in which major topics in algebraic graph theory are communicated for about twenty years. The notion of a scheme itself, however – a notion which, if considered in its full generality, generalizes not only the notion of a group but also the notion of a Moore geometry and that of a building in the sense of Jacques Tits – has been considered as the subject of an abstract theory in itself only relatively recently.

It is the purpose of this article to reflect on the lines of development, the *Entwicklungslinien*, along which abstract scheme theory has been developed so far and along which scheme theory might be developed in the future. The emphasis will be not so much on completeness as on an attempt to show exemplarily how naturally and organically the structure theory of association schemes arises from certain aspects in group theory.

1. The origin of schemes in group theory

Let G be a group, let H be a subgroup of G, and set

$$G/H := \{xH \mid x \in G\}.^{1}$$

For each element g in G, we define g^H to be the set of all pairs (yH, zH) with $y \in G$ and $z \in yHgH$.

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[1] Recall that, for each element x in G, xH stands for the set of all products xh with $h \in H$. The set xH is called a (*left*) *coset* of *H in G*.

0195-6698/$ – see front matter
doi:10.1016/j.ejc.2008.11.003

Let e and f be elements in G such that $e^H \cap f^H$ is not empty. Then there exist elements y and z in G such that $(yH, zH) \in e^H$ and $(yH, zH) \in f^H$. It follows that $y^{-1}z \in HeH \cap HfH$. Thus, as the double cosets of H form a partition of G, $HeH = HfH$. From this one obtains $e^H = f^H$. Thus, as $(yH, zH) \in (y^{-1}z)^H$ for any two elements y and z in G, the set

$$G/\!\!/H := \{g^H \mid g \in G\}$$

is a partition of $G/H \times G/H$.

Note also that 1^H is the identity on G/H and that $(g^{-1})^H$ is the inverse relation of g^H, that is the relation which contains (yH, zH) if and only if $(zH, yH) \in g^H$. We shall refer to the above three observations at a later stage of this section.

It is clear that the group G acts transitively on G/H by multiplication from the left hand side with inverses. Note also that, for any four elements a, y, z, and g in G, one has $(yH, zH) \in g^H$ if and only if $(a^{-1}yH, a^{-1}zH) \in g^H$. Thus, the action of G on the cartesian product $G/H \times G/H$ induced by the above-mentioned action of G on G/H preserves the relations on G/H which we collected in $G/\!\!/H$.

We shall see that the orbits of G on the cartesian product $G/H \times G/H$ are exactly the elements of $G/\!\!/H$. In other words, we shall see that, for any five elements v, w, y, z, and g in G satisfying $(vH, wH) \in g^H$ and $(yH, zH) \in g^H$, there exists an element a in G with $a^{-1}vH = yH$ and $a^{-1}wH = zH$.

Since the action of G on G/H is transitive and preserves the relations in $G/\!\!/H$, we may assume that $vH = H$ and $yH = H$. Then we have $w \in HgH$ and $z \in HgH$. It follows that $HwH = HzH$. Thus, there exists an element a in H such that $w \in azH$. From $a \in H$ we obtain $a^{-1}vH = yH$. (Recall that $v \in H$ and $y \in H$.) From $w \in azH$ we obtain $a^{-1}wH = zH$.

As a consequence of this last observation we obtain that the partition $G/\!\!/H$ of $G/H \times G/H$ satisfies the following regularity condition which, and this is the key point here, expresses itself without the action of G on G/H.

Let v, w, y, z, and g be elements in G such that $(vH, wH) \in g^H$ and $(yH, zH) \in g^H$. Then, given elements e and f in G, the number of cosets xH of H in G satisfying $(vH, xH) \in e^H$ and $(xH, wH) \in f^H$ is the same as the number of cosets xH of H in G satisfying $(yH, xH) \in e^H$ and $(xH, zH) \in f^H$.

It is this regularity condition which one puts together with the previously mentioned three observations to define schemes. Here is the definition.

Let X be a set, and let 1 denote the set of all pairs (x, x) with $x \in X$. For each subset r of the cartesian product $X \times X$, we define r^* to be the set of all pairs (y, z) with $(z, y) \in r$. Whenever x stands for an element in X and r for a subset of $X \times X$, we define xr to be the set of all elements y in X such that $(x, y) \in r$.

Let S be a partition of $X \times X$ such that $1 \in S$. Assume that, for each element s in S, $s^* \in S$. The set S is called an *association scheme* or simply a *scheme* on X if, for any three elements p, q, and r in S, there exists a cardinal number a_{pqr} such that, for any two elements y in X and z in yr, $|yp \cap zq^*| = a_{pqr}$. The cardinal numbers a_{pqr} are called the *structure constants* of S.

Referring to this definition the above considerations can now be put together by saying that, for any group G and a given subgroup H of G, the set $G/\!\!/H$ is a scheme on G/H.

2. Schurian schemes and the rise of recognition theorems

Is there a condition which characterizes the schemes arising from a group G and a subgroup of G in the above-described sense within the class of all schemes? To answer this question as generally as possible we define what it means for schemes to be isomorphic.

Let X and X' be sets, let S be a scheme on X, and let S' be a scheme on X'. A bijective map ϕ from X to X' is called an *isomorphism* from S to S' if there exists a bijective map σ from S to S' such that

$$(xs)\phi \subseteq (x\phi)(s\sigma)$$

for any two elements x in X and s in S.[2] The map σ is called the bijection *associated with* ϕ.

2 Note that the definition of an isomorphism does not change if one requires $(xs)\phi = (x\phi)(s\sigma)$ rather than $(xs)\phi \subseteq (x\phi)(s\sigma)$ for any two elements x in X and s in S. Equivalent to our definition is, of course, the requirement that $(y\phi, z\phi) \in s\sigma$ for any three elements y, z in X and s in S with $(y, z) \in s$.

Two schemes S and S' are called *isomorphic* if there exists an isomorphism from S to S'. If two schemes S and S' are isomorphic, one indicates that (as usual) by $S \cong S'$.

A scheme is called *schurian* if it is isomorphic to $G/\!\!/H$ for some group G and some subgroup H of G.

The question which we asked at the beginning of this section can now be restated in the following way. Is there any condition which characterizes the schurian schemes within the class of all schemes?

It seems that there is no genuine, no purely scheme theoretic condition which characterizes the schurian schemes within the class of all schemes. However, there is such a condition in terms of automorphisms of schemes, and we shall now look at this condition. Let us first explain what we mean by an automorphism of a scheme.

An isomorphism ϕ from a scheme to itself is called an *automorphism* if the bijection associated with ϕ is the identity.[3]

It follows immediately from the definition of an automorphism that the set of all automorphisms of a scheme S is a group with respect to composition. This group is called the *automorphism group* of S and will be denoted by $\mathrm{Aut}(S)$.

In the previous section, we saw that the automorphism group of a schurian scheme S on a set X possesses, for any five elements y, y' in X, s in S, z in ys, and z' in $y's$, an element g such that $yg = y'$ and $zg = z'$. Let us now prove that this condition is, in fact, also sufficient for a scheme to be schurian.

Theorem A. *A scheme S on a set X is schurian if and only if, for any five elements y, y' in X, s in S, z in ys, and z' in $y's$, S possesses an automorphism g such that $yg = y'$ and $zg = z'$.*

Proof. Let S be a scheme. We set $G := \mathrm{Aut}(S)$, we fix an element w in X, and we define H to be the set of all elements g in G satisfying $wg = w$. We shall see that $S \cong G/\!\!/H$.

Let x be an element in X, and let g and g' be elements in G such that $xg = w$ and $xg' = w$. Then $wg^{-1}g' = w$. Thus, $g^{-1}g' \in H$, and that means that $gH = g'H$. Thus, setting

$$x\phi := gH$$

for any two elements x in X and g in G with $xg = w$, ϕ is a map from X to G/H.

Let s be an element in S, and let y and z be elements in X such that $z \in ys$. Let e be an element in G such that $y\phi = eH$, and let f be an element in G such that $z\phi = fH$.

Let y' and z' be elements in X such that $z' \in y's$. Let e' be an element in G such that $y'\phi = e'H$, and let f' be an element in G such that $z'\phi = f'H$. We claim that $(e^{-1}f)^H = (e'^{-1}f')^H$.

From $y\phi = eH$ we obtain

$$ye = w.$$

Similarly, we obtain

$$y'e' = w, \qquad zf = w, \qquad z'f' = w$$

from $y'\phi = e'H$, $z\phi = fH$, and $z'\phi = f'H$.

Since $z \in ys$ and $z' \in y's$, G possesses an automorphism g such that $yg = y'$ and $zg = z'$.

From $ye = w$ we obtain $y = we^{-1}$. From $yg = y'$ and $y'e' = w$ we obtain $w = yge'$. Thus, $w = we^{-1}ge'$. Thus, $e^{-1}ge' \in H$. Thus, there exists an element h in H such that $e^{-1}ge' = h$. It follows that $g = ehe'^{-1}$.

From $zf = w$ we obtain $z = wf^{-1}$. From $zg = z'$ and $z'f' = w$ we obtain $w = zgf'$. Thus, $w = wf^{-1}gf'$. Thus, $f^{-1}gf' \in H$.

From $f^{-1}gf' \in H$ and $g = ehe'^{-1}$ we obtain $f^{-1}ehe'^{-1}f' \in H$. Thus, $e'^{-1}f' \in He^{-1}fH$. Thus, $He^{-1}fH = He'^{-1}f'H$. It follows that

$$(e^{-1}f)^H = (e'^{-1}f')^H.$$

Let s be an element in S, and let y and z be elements in X such that $z \in ys$. Let e be an element in G such that $y\phi = eH$, and let f be an element in G such that $z\phi = fH$. Then we define $s\sigma := (e^{-1}f)^H$.

3 Note that the automorphisms of a scheme S on a set X are exactly the permutations α of X which satisfy $(xs)\alpha \subseteq (x\alpha)s$ for any two elements x in X and s in S.

The above reasoning shows that this definition of $s\sigma$ is independent of the choice of e and f. Thus, σ is a map from S to $G/\!\!/H$.

That ϕ is an isomorphism from S to $G/\!\!/H$ (with associated bijection σ) follows right from the definition of ϕ. $\square$

Considering the development which group theory has undergone during the second half of the last century it seems to be a reasonable task to systematically search for scheme theoretic conditions which are sufficient for a scheme to be schurian. Theorem A provides a natural key to such an enterprise.

The search for conditions which are sufficient for a scheme to be schurian leads naturally to the following more specific question. Given a scheme theoretic property σ which is sufficient for a scheme to be schurian, can we specify a group theoretic condition γ such that a group G satisfies γ if and only if it possesses a subgroup H such that $G/\!\!/H$ satisfies σ?

A theorem which, in this sense, associates a group theoretic condition with a given scheme theoretic condition has been called a recognition theorem in [22]. This is because the initially given scheme theoretic condition σ recognizes the group theoretic condition γ. One may also say that σ identifies or characterizes the groups which satisfy γ. One obtains a characterization of the groups satisfying γ in terms which cannot be expressed solely in group theoretic terms.

It is the purpose of Part A of this article to review some of the currently existing recognition theorems.

3. Thin schemes and the rise of structure theorems

A scheme is called *thin* if it is isomorphic to $G/\!\!/\{1\}$ for some group G. Recalling the definition of schurian schemes one sees that thin schemes are schurian.

In Section 2, we mentioned that no scheme theoretic condition is known which characterizes the schurian schemes within the class of all schemes. For thin schemes the situation is different. Setting $n_s := a_{ss^*1}$ for each scheme element s and calling this cardinal number the *valency* of s, one has the following.

Theorem $\mathbf{B_1}$. *A scheme is thin if and only if all of its elements have valency* 1.

The proof of this theorem is straightforward and follows the lines of the proof of Theorem A. The same is true for the following theorem which shows that our notion of a scheme isomorphism generalizes that of a group isomorphism.

Theorem $\mathbf{B_2}$. *Let G and G' be groups, let H be the identity subgroup of G, and let H' be the identity subgroup of G'. Then $G \cong G'$ if and only if $G/\!\!/H \cong G'/\!\!/H'$.*

Theorem B_2 allows us to view the class of all groups as a distinguished class of schemes, namely as the class of the thin schemes. It is tempting to consider this observation as a justification for far-reaching and ambitious conjectures. One would like to know to what extent basic group theoretic definitions and results can be generalized to scheme theory in such a way that the thin versions of the scheme theoretic generalizations correspond to the group theoretic originals that one starts with.

In fact, scheme theory allows quite a few steps in this direction, and it is the purpose of Part B of this article to present a collection of structure theorems of schemes which generalize group theoretic structure theorems.

4. Preliminaries

In this section, the letter X stands for a set, the letter S for a scheme on X.

For each nonempty subset R of S, we define n_R to be the sum of the cardinalities n_r with $r \in R$. The cardinality n_R is called the *valency* of R.

Note that $n_S = |X|$. Moreover, for each element s in S, one has $|s| = n_s n_S$. If S has finite valency, the latter observation yields $n_{s^*} = n_s$ for each element s in S. Occasionally, we shall refer to this equation without further mention.

Here are the most fundamental equations which structure constants of schemes have to satisfy.[4]

Lemma 4.1. *Let p and q be elements of S, and assume that n_p and n_q are finite. Then the following hold.*

(i) *We have*
$$\sum_{s\in S} a_{psq} = n_p.$$
(ii) *We have*
$$\sum_{s\in S} a_{pqs}n_s = n_p n_q.$$
(iii) *For each element s in S, we have $a_{psq}n_q = a_{qs^*p}n_p$.*

For any two nonempty subsets P and Q of S, we define PQ to be the set of all elements s in S such that there exist elements p in P and q in Q with $a_{pqs} \neq 0$. The set PQ is called the *complex product* of P and Q.

It is easy to see that complex multiplication is associative and generalizes complex multiplication in group theory.

For each nonempty subset R of S, we define R^* to be the set of all elements r^* with $r \in R$.

Note that $(PQ)^* = Q^*P^*$ for any two nonempty subsets P and Q of S. Note also that, for any two nonempty subsets P and Q of S, $1 \in P^*Q$ if and only if $P \cap Q$ is not empty.

The following lemma provides a link between complex products and valencies.

Lemma 4.2. *Let P and Q be nonempty subsets of S, and assume that n_P and n_Q are finite. Then the following hold.*

(i) *We have $n_Q \leq n_{PQ}$.*
(ii) *We have $n_Q = n_{PQ}$ if and only if $Q = P^*PQ$.*

For any two elements p and q of S, we write pq instead of $\{p\}\{q\}$.

Lemma 4.3. *Let p and q be elements of S, and assume that n_p and n_q are finite. Then $|p^*q|$ is less than or equal to the greatest common divisor of n_p and n_q.*

A nonempty subset R of S is called *closed* if $R^*R \subseteq R$. Closed subsets generalize subgroups.

Similarly to subgroups, closed subsets contain 1 as an element. One also verifies easily that intersections of closed subsets of S are closed and that the valency of a closed subset of S divides n_S if n_S is finite. This latter observation is the scheme theoretic generalization of Lagrange's Theorem for finite groups.

The proof of the following lemma can be translated word by word from the corresponding proof in group theory.

Lemma 4.4. *Let T and U be closed subsets of S. Then TU is closed if and only if $TU = UT$.*

Given an element s of S and a nonempty subset R of S we write Rs instead of $R\{s\}$ and sR instead of $\{s\}R$.

Let T and U be closed subsets of S, and assume that $T \subseteq U$. The closed subset T is called *normal* in U if $Tu = uT$ for each element u in U. (It is easy to see that $Tu \subseteq uT$ is equivalent to $Tu = uT$ if U has finite valency.)

A closed subset T of S is called *simple* if $\{1\}$ and T are the only normal closed subsets of T.

For each closed subset T of S, $N_S(T)$ stands for the *normalizer* of T *in* S, that is the set of all elements s in S which satisfy $Ts = sT$. Thus, like in group theory, a closed subset T of S is normal in S if and only if $N_S(T) = S$.

[4] For a proof of Lemma 4.1 we refer the reader to [22, Lemma 1.1.3]. The proofs of the remaining lemmata in this section are given in [22, Lemma 1.4.4], [22, Lemma 1.5.2], [22, Lemma 2.1.1], [22, Lemma 2.2.1], [22, Lemma 2.3.3], [23, (1.7)(i)], and [22, Theorem 4.1.3(iii)], respectively.

Note that $TU = UT$ for any two closed subsets T and U of S with $U \subseteq N_S(T)$. Thus, by Lemma 4.4, TU is closed for any two such closed subsets of S.

The following scheme theoretic version of Dedekind's 'Modularity Laws' will be useful in Section 9.

Lemma 4.5. *Let P and Q be nonempty subsets of S, and let T be a closed subset of S. Then we have the following.*

(i) *If $P \subseteq T$, $T \cap PQ = P(T \cap Q)$.*
(ii) *If $Q \subseteq T$, $T \cap PQ = (T \cap P)Q$.*

Here is a link between closed subsets and valencies.

Lemma 4.6. *Let s be an element in S, and let T and U be closed subsets of S. Assume that s, T, and U have finite valency. Then n_{TsU} divides $n_T n_s n_U$.*

For each subset R of S, we define $\langle R\rangle$ to be the intersection of all closed subsets T of S satisfying $R \subseteq T$. The set $\langle R\rangle$ is called the *span* of R in S, and we say that R *spans* or *generates* $\langle R\rangle$.

Since intersections of closed subsets are closed, spans of subsets of S are closed subsets of S.

We mentioned earlier that complex multiplication is associative. Thus, given a nonempty subset R of S and a positive integer n, we may inductively define R^n.

The following characterization of spans is fundamental and appears in one form or another in every algebraic theory.

Lemma 4.7. *Let R be a subset of S. Then $\langle R\rangle$ is equal to the union of the sets $(R^* \cup R)^n$ where n is a non-negative integer.*

In contrast to group theory, schemes of finite valency allow us to define a quotient structure for any closed subset, not only for normal closed subsets. Let us look at the definition of quotient schemes.

Assume n_S to be finite, and let T be a closed subset of S. For each element x in X, we define xT to be the union of all sets xt with $t \in T$. We define

$$X/T := \{xT \mid x \in X\}.$$

For each element s in S, we define s^T to be the set of all pairs (yT, zT) with $y \in X$ and $z \in yTsT$. It is not difficult to see that

$$S/\!\!/T := \{s^T \mid s \in S\}$$

is a scheme on X/T; cf. [22, Theorem 4.1.3(i)]. This scheme is called the *quotient scheme* of S over T.

As for the valencies of the elements in $S/\!\!/T$ we have the following.

Lemma 4.8. *Assume that S has finite valency, let T be a closed subset of S, and let s be an element in S. Then we have $n_{s^T} n_T = n_{TsT}$.*

Assume that S has finite valency, let T be a closed subset of S, and let s be an element in S. Then, by Lemma 4.8, $n_{s^T} n_T = n_{TsT}$. From Lemma 4.6 we also know that n_{TsT} divides $n_T n_s n_T$. Thus, n_{s^T} divides $n_s n_T$. We shall refer to this observation in Sections 9 and 10.

A. Recognition Theorems

There are several different ways to express what it means for a scheme to be thin. The closer one stays to one or another of these conditions, the better the chances are of finding a condition which leads to schurian schemes.

In this part of our article, we shall look at three different conditions which can be considered to be close to the condition of being thin.

Theorem B_1 says that the thin schemes are exactly the schemes in which all elements have valency 1. This observation suggests investigating schemes in which all valencies are still relatively small. Mitsugu Hirasaka and Mikhail Muzychuk looked at schemes of finite valency all elements of which have valency at most 2; cf. [10,12,13]. In Section 5, we shall focus on a specific class of these schemes, a class which turns out to consist of schurian schemes.

Given a scheme S one defines $O_\vartheta(S)$ to be the set of all elements in S which have valency 1. The set $O_\vartheta(S)$ is called the *thin radical* of S. Theorem B_1 says that a scheme S is thin if and only if $O_\vartheta(S) = S$.

Let S be a scheme of finite valency. One defines $O^\vartheta(S)$ to be the span of the union of all subsets s^*s with $s \in S$. The closed subset $O^\vartheta(S)$ of S is called the *thin residue* of S.[5] From Lemma 4.1(ii) one obtains easily that, for each element s in S, $n_s = 1$ is equivalent to $s^*s = \{1\}$. Thus, according to Theorem B_1 a scheme S of finite valency is thin if and only if $O^\vartheta(S) = \{1\}$.

We so have seen that both equations, $O_\vartheta(S) = S$ and $O^\vartheta(S) = \{1\}$, express the fact that a scheme S of finite valency is thin. A natural relaxation of the fact that S is thin is, therefore, the condition $O^\vartheta(S) \subseteq O_\vartheta(S)$. We shall look more closely at this condition in Section 6.

The third condition which we shall discuss in this part of our article deals with schemes in which 'many' structure constants are equal to 1. The condition deals with schemes generated by involutions. In contrast to the first two conditions, this condition is not restricted to schemes of finite valency. It brings buildings into the game and will be considered in Section 7.

5. Schemes and Glauberman involutions

In this section, all schemes are assumed to have finite valency. We shall look at schemes (of finite valency) all elements of which have valency at most 2. Since our focus is on recognition theorems, we do not follow the above-mentioned path of Hirasaka and Muzychuk. We start, however, with a lemma which is the key also to their investigation and which is implicit in [12, Lemma 3.1].

Lemma 5.1. *Let S be a scheme, and let s be an element in S with $n_s = 2$. Then $s^*s \setminus \{1\}$ possesses a symmetric element r with $n_r \leq 2$ and $\{1, r\} = s^*s$.*

Proof. Assuming $n_s = 2$ we must have $2 \leq |s^*s|$. On the other hand, $n_s = 2$ also yields $|s^*s| \leq 2$; cf. Lemma 4.3. Thus, $|s^*s| = 2$.

Since $1 \in s^*s$, we obtain from $|s^*s| = 2$ that $s^*s \setminus \{1\}$ possesses an element r such that $s^*s = \{1, r\}$.

From $\{1, r\} = s^*s$ and $(s^*s)^* = s^*s$ we obtain $r^* = r$.

Applying Lemma 4.1(ii) to s^* and s in place of p and q, we obtain from $s^*s = \{1, r\}$ that $a_{s^*s1} + a_{s^*sr}n_r = n_{s^*}n_s$. Thus, as $a_{s^*s1} = n_{s^*} = 2$, $n_r \leq 2$. □

From Lemma 5.1 one obtains $n_{s^*s} \in \{2, 3\}$ for each scheme element s of valency 2. The elements s satisfying $n_{s^*s} = 2$ are the ones which prevent schemes (in which all elements have valency 1 or 2) from being schurian. In fact, there exists a famous non-schurian scheme of valency 28 which has four elements of valency 1 and twelve elements of valency 2. In the following, we shall denote this scheme by $HM_{176}(28)$.

All elements s of valency 2 of $HM_{176}(28)$ satisfy $n_{s^*s} = 2$. We shall see in the next section that $HM_{176}(28)$ is also responsible for the necessity of additional conditions which one needs to impose in order to obtain schurity from the condition $O^\vartheta(S) \subseteq O_\vartheta(S)$.

The following result is [16, (4.1)]. Its proof consists of the concrete construction of automorphisms, and that will enable us to apply Theorem A. It refers to Lemma 5.1 and depends at various instances decisively on the hypothesis $n_{s^*s} \neq 2$.

Proposition 5.2. *Let S be a scheme, and assume that, for each element s in S, $n_s \leq 2$ and $n_{s^*s} \neq 2$. Then S is schurian.*

Proposition 5.2 establishes a sufficient condition for a scheme S to be schurian. According to our remarks in Section 2 this raises the question of what the corresponding recognition theorem says. In order to answer this question we determine the one-point stabilizer of the automorphism group of S; cf. [16, (4.2)] and [16, (4.3)].

5 The closed subset $O^\vartheta(S)$ owes its name to the fact that it is the uniquely determined smallest closed subset T of S such that $S/\!\!/T$ is thin.

Lemma 5.3. *Let X be a set, let S be a scheme on X, and assume that $n_s \leq 2$ for each element s in S. Set $G := \mathrm{Aut}(S)$, fix an element x in X, and define H to be the set of all elements g in G satisfying $xg = x$. Then the following hold.*

(i) *If S is not thin, $2 \leq |H|$.*
(ii) *If $O^{\vartheta}(S)$ does not have valency 2, $|H| \leq 2$.*

Let S be a non-thin scheme such that $n_s \leq 2$ and $n_{s^*s} \neq 2$ for each element s in S. Then, by Lemma 5.3, $|H| = 2$. Let us see what $G/\!\!/H$ looks like if G is a group and H a subgroup of order 2 of G.

Lemma 5.4. *Let G be a group, and let H be a subgroup of order 2 of G. Then the following hold.*

(i) *For each element g in G, we have $n_{gH} \leq 2$.*
(ii) *For each element g in G, we have $n_{gH} = 2$ if and only if $g \in G \setminus C_G(H)$.*
(iii) *Let g be an element in G, and assume that $n_{gH} = 2$. Then $n_{(g^H)^* g^H} \neq 2$ if and only if $HH^g \neq H^gH$.*

The first statement of Lemma 5.4 follows immediately from Lemma 4.8. Its second statement is straightforward, and its third statement is an application of Lemma 5.1.

The group theoretic condition given in Lemma 5.4(iii) is well known and arises in a famous context in group theory.

Let G be a group, and let l be an involution of G. If G has finite order, the local condition that $ll^g \neq l^g l$ for each element g in $G \setminus C_G(l)$ can be expressed globally, referring to the uniquely determined maximal normal subgroup $O(G)$ of odd order of G. This fact, namely that condition (b) of the following theorem is a consequence of condition (a), is George Glauberman's Z^*-Theorem; cf. [3, Theorem 1]. Its proof is considered to be a highlight of modular representation theory of finite groups.

Theorem 5.5. *Let G be a finite group, and let H be a subgroup of order 2 of G. Then the following conditions are equivalent.*

(a) *For each element g in $G \setminus C_G(H)$, $HH^g \neq H^gH$.*
(b) *We have $G = [O(G), H]C_G(H)$.*

Let G be a simple group satisfying condition (b) in Theorem 5.5. Then $H = G$ or $[O(G), H] = G$. In the second case, $O(G) = G$, and that means that G has odd order, contradicting the fact that H has even order. Thus, G must have order 2 if it satisfies condition (b) of Theorem 5.5.

Theorem 5.5 allows us to state Proposition 5.2 in a more precise way. The proof of the following theorem was given first in [16, (5.1)]. We include it here since it shows by example how scheme theory and group theory work together.

Theorem 5.6. *Let S be a non-thin scheme. Assume that, for each element s in S, $n_s \leq 2$ and $n_{s^*s} \neq 2$. Then there exists a finite group G and a subgroup H of G such that $|H| = 2$, $G = [O(G), H]C_G(H)$, and $S \cong G/\!\!/H$.*

Proof. Set $G := \mathrm{Aut}(S)$, fix an element x in X, and define H to be the set of all elements g in G satisfying $xg = x$. Then, by Lemma 5.3, $|H| = 2$. Thus, there exists an element h in $G \setminus \{1\}$ such that $\{1, h\} = H$.

Let g be an element in $G \setminus C_G(h)$. According to Theorem 5.5 we shall be done if we succeed in showing that $hh^g \neq h^g h$.

Let us denote by s the uniquely determined element in S satisfying $xg \in xs$. Then

$$xgh \in xsh \subseteq xhs = xs,$$

and so $\{xg, xgh\} \subseteq xs$.

Assume that $xgh = xg$. Then $xghg^{-1} = x$. Thus, $ghg^{-1} \in H$. Thus, as $\{1, h\} = H$, $ghg^{-1} = h$. It follows that $g \in C_G(h)$, contradicting the choice of g. Thus, we must have $xgh \neq xg$.

From $\{xg, xgh\} \subseteq xs$ and $xgh \neq xg$ we obtain $\{xg, xgh\} = xs$. (Recall that $n_s \leq 2$.) In particular,

$$xgs^* \cup xghs^* = xss^*.$$

Since $xg \in xs$, $x \in xsg^{-1} \subseteq xg^{-1}s$. Thus, $xg^{-1} \in xs^*$, so

$$xh^g = xg^{-1}hg \in xs^*hg \subseteq xhgs^* = xgs^*.$$

On the other hand, we obtain from $xg \in xs$ also that $x \in xgs^*$. Thus, as $x \neq xh^g$,

$$\{x, xh^g\} = xgs^*.$$

From this we obtain

$$\{x, xh^g h\} = xghs^*.$$

Thus, as $xgs^* \cup xghs^* = xss^*$, $\{x, xh^g, xh^g h\} = xss^*$. Thus, as we are assuming that $n_{ss^*} \neq 2$, we conclude that $xh^g \neq xh^g h$. In particular, $hh^g \neq h^g h$. □

Let S be a scheme of finite valency, and assume that $O_\vartheta(O^\vartheta(S))$ has odd valency. Then we have $n_{ss^*} \neq 2$ for each element s in S. Thus, the conclusion of Theorem 5.6 remains valid if we assume that all elements of S have valency at most 2 and the valency of $O^\vartheta(S)$ is odd.

Here is the converse of Theorem 5.6. It is an immediate consequence of Lemma 5.4 and Theorem 5.5. Together with Theorem 5.6 it is a recognition theorem. Its proof is straightforward and was given in [16, (5.2)].

Theorem 5.7. *Let G be a finite group, and let H be a subgroup of G such that $|H| = 2$ and $G = [O(G), H]C_G(H)$. Then $G/\!\!/H$ has finite valency and, for each element g in G, one has $n_{g^H} \leq 2$ and $n_{(g^H)^* g^H} \neq 2$.*

As a consequence of Theorem 5.6 one obtains the following.

Corollary 5.8. *Let S be a scheme of finite valency, and assume that, for each element s in S, $n_s \leq 2$ and $n_{s^*s} \neq 2$. Then $S = O^\vartheta(S)O_\vartheta(S)$, and $O^\vartheta(S)$ has odd valency.*

There are more results in this direction. Remarkable among them is a theorem of Hirasaka and Muzychuk which says that schemes of valency $4p$, p a prime different from 7, are schurian if all of their elements have valency at most 2; cf. [12, Theorem 5.2]. The hypothesis $p \neq 7$ is, of course, needed because of $HM_{176}(28)$.

6. Schemes and the generalized Fitting subgroup

In the previous section, we saw that having elements of valency 1 or 2 only is not sufficient for a scheme to be schurian. The scheme $HM_{176}(28)$ is not schurian although all of its elements have valency 1 or 2. It is the same scheme which shows that the condition $O^\vartheta(S) \subseteq O_\vartheta(S)$ is not sufficient for a scheme S to be schurian. In fact, the thin residue of $HM_{176}(28)$ is equal to its thin radical and is an elementary abelian group of order 4.

All schemes in this section are assumed to have finite valency. We shall look at conditions which guarantee that schemes S (of finite valency) satisfying $O^\vartheta(S) \subseteq O_\vartheta(S)$ are schurian.

Let s be an element of a scheme S. From Lemma 4.3 we know that $|st| = 1$ for each element t in $O_\vartheta(S)$. We define

$$T_s := \{t \in O_\vartheta(S) \mid st = \{s\}\}.$$

Since $spq = sq = \{s\}$ for any two elements p and q in T_s, T_s is closed. Moreover, one has $T_s = \{1\}$ for each element s in $O_\vartheta(S)$.

It turns out that the closed subsets T_s with $s \in S$ rule over the structure of schemes S that satisfy $O^\vartheta(S) \subseteq O_\vartheta(S)$. In fact, assuming $O^\vartheta(S) \subseteq O_\vartheta(S)$ one obtains $T_s = s^*s$ for each element s in S. Moreover, one obtains that, for each element s in S, T_s is a normal closed subset of $O^\vartheta(S)$ and has valency (order) n_s.[6] (All this is, for instance, contained in [22, Lemma 6.7.1].) This indicates that the structure of S depends heavily on the (group theoretic) structure of $O^\vartheta(S)$.

As a preliminary result in this direction one has the following theorem; cf. [15, Theorem B].

[6] Recall that a closed subset T of S is called normal in S if $Ts = sT$ for each element s in S.

Theorem 6.1. *Let S be a scheme with $O^{\vartheta}(S) \subseteq O_{\vartheta}(S)$, and assume that the set $\{s^*s \mid s \in S\}$ is linearly ordered with respect to set theoretic inclusion. Then S is schurian.*

In the following, we try to get away from the restrictive hypothesis of Theorem 6.1. We shall deal with schemes S in which $O^{\vartheta}(S)$ is the direct product of two thin simple closed subsets (two simple groups) which we call C and D.[7] The scheme $HM_{176}(28)$ forces us also to assume that C and D are not isomorphic. We assume that they have different order.

Note that $O^{\vartheta}(S)$ has exactly four normal closed subsets, namely $\{1\}$, C, D, and $O^{\vartheta}(S)$. Recall also that, for each element s in S, T_s is a normal closed subset of $O^{\vartheta}(S)$. Thus, we must have

$$T_s = C, \qquad T_s = D, \quad \text{or} \quad T_s = O^{\vartheta}(S)$$

for each element s in $S \setminus O_{\vartheta}(S)$.

Define U to be the set of all elements s in S with $T_s = \{1\}$ or $T_s = C$. Similarly, let V denote the set of all elements s in S with $T_s = \{1\}$ or $T_s = D$.

It is not too difficult to show that U and V are closed. Also, one has $U \subseteq N_S(D)$ and $V \subseteq N_S(C)$.

Referring to the above notion we can show the following.

Lemma 6.2. *Let y be an element in X, let s be an element in V, and let z and z' be elements in ys. Then S possesses an automorphism ϕ such that $x\phi = x$ for each element x in $X \setminus zU$ and $z\phi = z'$.*

With the help of Lemma 6.2 one can go one step further.

Lemma 6.3. *Assume that $O_{\vartheta}(S) \neq U$ and that $O_{\vartheta}(S) \neq V$. Let y be an element in X, let s be an element in $S \setminus (U \cup V)$, and let z and z' be elements in ys. Then S possesses an automorphism ϕ such that $y\phi = y$ and $z\phi = z'$.*

Lemma 6.2 and Lemma 6.3 tell us that, for any two elements x in X and s in S, the stabilizer of x in the automorphism group of S acts transitively on xs. This is half of schurity. For the other half we are lucky. That a scheme S satisfying $O^{\vartheta}(S) \subseteq O_{\vartheta}(S)$ has a transitive automorphism group if $O^{\vartheta}(S)$ is a product of two simple groups of different order follows immediately from a more general result of Hirasaka; cf. [11, Theorem 1.2]. Thus, we have the following.

Proposition 6.4. *Let S be a scheme in which $O^{\vartheta}(S)$ is the direct product of two thin simple closed subsets of different order. Then S is schurian.*

Proposition 6.4 is similar to Proposition 5.2. Both results provide a sufficient condition for a scheme of finite valency to be schurian. Again we want to know what the schurian schemes satisfying the hypothesis of Proposition 6.4 look like. In order to see this, we have to translate the conditions of Proposition 6.4 into group theory.

Let G be a finite group, and let H be a subgroup of G. It is easy to see that

$$O_{\vartheta}(G /\!\!/ H) = N_G(H) /\!\!/ H$$

and

$$O^{\vartheta}(G /\!\!/ H) = \langle H^g \mid g \in G \rangle /\!\!/ H.$$

Thus, one has $O^{\vartheta}(G /\!\!/ H) \subseteq O_{\vartheta}(G /\!\!/ H)$ if and only if $\langle H^g \mid g \in G \rangle \subseteq N_G(H)$.

Referring to this observation, it is not difficult to prove the following theorem.

Theorem 6.5. *Let S be a scheme in which $O^{\vartheta}(S)$ is the direct product of two thin simple closed subsets of different order. Then there exist a finite group G, normal subgroups M_1 and M_2 of G, and maximal normal subgroups H_1 of M_1 and H_2 of M_2 satisfying the following conditions.*

(i) *If M_1 is commutative, M_1 is elementary abelian and H_1 contains no normal subgroup of G different from $\{1\}$. The same is true for M_2.*

7 Recall that S is called simple if $\{1\}$ and T are the only normal closed subsets of S.

(ii) *If* M_1 *is not commutative,* M_1 *is a minimal normal subgroup of* G*. The same is true for* M_2*.*
(iii) *We have* $H_1 \ntrianglelefteq G$, $H_2 \ntrianglelefteq G$*, and* $S \cong G /\!\!/ H_1H_2$*.*

The following theorem, together with Theorem 6.5, is one of our recognition theorems.

Theorem 6.6. *Let* G *be a finite group, let* M_1 *and* M_2 *be normal subgroups of* G *such that* $M_1 \cap M_2 = \{1\}$*, let* H_1 *be a maximal normal subgroup of* M_1*, let* H_2 *be a maximal normal subgroup of* M_2*, and assume that* $H_1 \ntrianglelefteq G$ *and* $H_2 \ntrianglelefteq G$*. Then* $O^{\vartheta}(G /\!\!/ H_1H_2) \cong M_1/H_1 \times M_2/H_2$*.*

Theorem 6.6 says, in particular, that finite groups with two different components (quasisimple subnormal subgroups) give rise to schemes satisfying the hypothesis of Proposition 6.4. It would be interesting to see whether, generally, the generalized Fitting subgroup of a finite group plays a major role in the investigation of schemes with thin thin residue.

It seems to be an interesting question which finite groups G guarantee that schemes S with $O^{\vartheta}(S) \cong G$ are schurian. The scheme $HM_{176}(28)$ shows that the elementary abelian group of order 4 does not have this property. From Theorem 6.1 one obtains that simple groups do have this property. Direct products of two simple non-abelian groups of different order do have this property, too; cf. Proposition 6.4.

7. Involutions, the exchange condition, and buildings

In this section, we present a recognition theorem which is not restricted to schemes of finite valency. It deals with involutions. Involutions in scheme theory generalize the group theoretic notion of an involution.

A scheme element s is called an *involution* if $|\langle\{s\}\rangle| = 2$. Note that involutions are necessarily symmetric.

Let us fix a nonempty set of involutions of a scheme S and call it L.

Since involutions are symmetric, we obtain from Lemma 4.7 that $\langle L\rangle$ is the union of the sets L^n with n a non-negative integer. Thus, for each element s in $\langle L\rangle$, one obtains a non-negative integer n with $s \in L^n$. The smallest such integer is called the *L-length* of s and will be denoted by $\ell_L(s)$. If there is no danger of ambiguity (as is the case at the moment), we shall speak simply of the length of an element in $\langle L\rangle$ rather than of the L-length and write ℓ instead of ℓ_L.

Let p and q be elements in $\langle L\rangle$. It follows right from the definition of ℓ that

$$\ell(r) \le \ell(p) + \ell(q)$$

for each element r in pq. For each element q in $\langle L\rangle$, we define $S_1(q)$ to be the set of all elements p in $\langle L\rangle$ such that pq possesses an element r with $\ell(r) = \ell(p) + \ell(q)$.

Here are the two main definitions.

(i) The set L is called *constrained* if $|pq| = 1$ for any two elements q in $\langle L\rangle$ and p in $S_1(q)$.
(ii) We say that L satisfies the *exchange condition* if, for any three elements h, k in L and s in $S_1(k)$, $h \in S_1(s)$ implies $hs = sk$ or $hs \subseteq S_1(k)$.

A constrained set of involutions is called a *Coxeter set* if it satisfies the exchange condition. An association scheme is called *Coxeter scheme* (of *rank* n) if it is the span of a Coxeter set (of cardinality n).

The definition of a Coxeter scheme has two interesting features. Firstly, thin Coxeter schemes are the same thing as Coxeter groups. Secondly, Coxeter schemes are the same thing as buildings. Indeed, we have the following.

Theorem 7.1. *There is a natural and well-understood bijective map between the class of all buildings and the class of all Coxeter schemes.*

To give a rough idea about the bijective map in this theorem we mention that this bijective map relates buildings of type A_2 (projective planes) to Coxeter schemes defined by Coxeter sets $\{h, k\}$ satisfying $hkh = khk$.

The fact that thin Coxeter schemes and Coxeter groups are the same thing raises the question of which of the elementary facts about Coxeter groups can be generalized to Coxeter schemes.

There are surprisingly many features of Coxeter groups which carry over to Coxeter schemes. For instance, subsets of Coxeter sets are Coxeter sets. One also has $K = L \cap \langle K \rangle$ for each subset K of a Coxeter set L. More is said in the following lemma.

Lemma 7.2. *Let L be a Coxeter set of a scheme S, and let K be a nonempty subset of L. Define $S_1(K)$ to be the intersection of the sets $S_1(k)$ with $k \in K$. Then we have $S_1(\langle K \rangle) = S_1(K)$, $\langle L \setminus K \rangle \subseteq S_1(\langle K \rangle)$, and $S_1(K)\langle K \rangle = \langle L \rangle$.*

Proofs of the statements of Lemma 7.2 are given in [24, Lemma 3.5, Lemma 3.6, and Lemma 3.7].

Given a Coxeter set L and an element q in $\langle L \rangle$ we define $S_{-1}(q)$ to be the set of all elements r in $\langle L \rangle$ such that there exists an element p in $\langle L \rangle$ with $r \in pq$ and $\ell(r) = \ell(p) + \ell(q)$.

Dually to the first equation of Lemma 7.2 we obtain the following lemma; cf. [24, Lemma 3.4].

Lemma 7.3. *Let L be a Coxeter set of a scheme S, and let K be a nonempty subset of L. Define $S_{-1}(K)$ to be the intersection of the sets $S_{-1}(k)$ with $k \in K$. Then we have $S_{-1}(\langle K \rangle) = S_{-1}(K)$.*

It is probably not an exaggeration to say that the first equation of Lemma 7.2 and the equation in Lemma 7.3 are the most useful and clarifying results in the basic theory of Coxeter sets.

While the proofs of the above two lemmata are straightforward generalizations of the corresponding proofs for Coxeter groups, the proof of the following proposition is quite involved.

Proposition 7.4. *Let S be a finite Coxeter scheme of rank at least 3, and assume that $O_\vartheta(S) = \{1\}$. Then S is schurian.*[8]

Coxeter schemes of rank 2 are not necessarily schurian. A Coxeter scheme of finite valency and of type A_2 is schurian if and only if it corresponds to a desarguesian projective plane; cf. [17].

The proof of Proposition 7.4 was first given in [24]. Together with Theorem 7.1 it provides an alternate proof of Tits' reduction theorems for buildings of spherical type; cf. [20, Theorem 4.1.2] and [21, Proposition 11.13]. It aims for an application of Theorem A. In fact, in order to prove Proposition 7.4 one constructs, for any five elements y, y' in X, s in S, z in ys, and z' in $y's$, an automorphism g of S such that $yg = y'$ and $zg = z'$. The automorphism g is constructed by extending the map ϕ from $\{y, z\}$ to X which sends y to y' and z to z' step by step to an automorphism of S.

In order to explain the individual steps in which ϕ is extended it is useful to introduce the notion of a faithful map.

Let W be a subset of X. A map χ from W to X is called *faithful* if, for any three elements y, z in W and s in S, $z \in ys$ implies $z\chi \in y\chi s$.

Note that faithful maps are injective and that a surjective faithful map from X to X is an automorphism of S. Note also that the above-defined map ϕ is a faithful map from $\{y, z\}$ to X.

The extension of the faithful map ϕ to a faithful map from X to X comes now in three steps. We define V to be the union of the sets $\langle M \rangle$ with $M \subseteq L$ and $|M| \leq 2$.

(i) Given elements x, y, and z in X each faithful map from $\{y\} \cup zV$ extends to a faithful map from $\{x, y\} \cup zV$ to X.

(ii) Let χ be a faithful map from $yV \cup \{z\}$ to X. Then χ extends to a faithful map from $yV \cup zV$ to X.

These first two steps do not require S to be finite. But finiteness will be used for the third step in which we need elements of maximal length. In the same way as one shows that finite Coxeter groups possess a uniquely determined element of maximal length one proves this fact for finite Coxeter schemes. We call this element m.

(iii) Let y be an element in X, let z be an element in ym. Then each faithful map χ from $yV \cup \{z\}$ to X extends to an automorphism of S.

[8] We call a scheme *finite* if it has finitely many elements. Each scheme of finite valency is finite, but the converse does not hold.

The above three steps in our proof of Proposition 7.4 are modeled after Tits' procedure in his treatment of buildings of spherical type in [20].

Proposition 7.4 is similar to Propositions 6.4 and 5.2. Similar to these two propositions it provides a sufficient condition for S to be schurian. Again, we would like to know the group theoretic condition which is characterized by the schemes considered in Proposition 7.4.

This time the answer refers to Tits systems. Let us explain what one means by a Tits system of a group.

Let G be a group, let H be a subgroup of G, and let J be a subset of G such that $G = \langle H \cup J \rangle$.

Assume that $H \cap \langle J \rangle$ is normal in $\langle J \rangle$. Assume that, for each element j in J, $j^2 \in H$ and

$$H \neq HjHjH.$$

Assume, finally, that

$$HgHjH \subseteq HgjH \cup HgH$$

for any two elements $j \in J$ and g in $\langle J \rangle$. Then (H, J) is called a *Tits system* for G.

We can now state Proposition 7.4 in a more precise way.

Theorem 7.5. *Let S be a finite Coxeter scheme of rank at least 3, and assume that $O_\vartheta(S) = \{1\}$. Then there exists a group G with a Tits system (H, J) such that $S \cong G/\!\!/H$.*

Theorem 7.5 is a consequence of [22, Theorem 12.3.4]. Its converse is the following theorem, a result that says that Tits systems give rise to Coxeter sets. It is a consequence of [22, Theorem12.3.5]. Together with Theorem 7.5, Theorem 7.6 is a recognition theorem.

Theorem 7.6. *Let G be a group which possesses a Tits system (H, J). Then $G/\!\!/H$ is a scheme with $O_\vartheta(G/\!\!/H) = \{H\}$, and $J/\!\!/H$ is a Coxeter set which spans $G/\!\!/H$.*

In Theorem 7.6 one does not automatically obtain that $G/\!\!/H$ is finite and that $J/\!\!/H$ has at least three elements.

B. Structure Theorems

In this second part of the article, we shall discuss five themes from group theory which contribute to a conceptional understanding of the structure of association schemes.

In Section 8, we shall present the Homomorphism Theorem, the two Isomorphism Theorems, and the Jordan–Hölder Theorem for schemes of finite valency. We follow the lines of [18].

Section 9 deals with simplicity and primitivity of schemes. Section 10 presents the generalized Sylow Theorems as they have been proven in [14]. In Section 11, we present an advanced result on involutions, and in the last section, we glimpse at representation theory of schemes of finite valency.

8. Subnormal closed subsets

All schemes in this section are assumed to have finite valency.

Let X and X' be sets, let S be a scheme on X, and let S' be a scheme on X'. A map ϕ from X to X' is called a *morphism* from S to S' if there exists a map σ from S to S' such that $(xs)\phi \subseteq (x\phi)(s\sigma)$ for any two elements x in X and s in S. The map σ is called the map *associated with* ϕ.

A morphism ϕ from S to S' will be called a *homomorphism* if, for any three elements y, z in X and s in S with $z\phi \in (y\phi)(s\sigma)$, there exist elements v in X and w in vs such that $v\phi = y\phi$ and $w\phi = z\phi$.

Note that a bijective morphism is an isomorphism and that isomorphisms are homomorphisms.

Given a homomorphism ϕ from S we define the *kernel* of ϕ to be the set of all elements s in S satisfying $s\phi = 1\phi$. The kernel of a homomorphism ϕ is denoted by $\ker(\phi)$.

It follows right from the definition of the kernel that kernels are closed. Here is the Homomorphism Theorem for schemes.

Theorem 8.1. *Let X be a set, and let S be a scheme on X. Let ϕ be a homomorphism from S to a scheme S' with associated map σ, and set $T := \ker(\phi)$. For each element x in X, set $(xT)\psi := x\phi$. For each element s in S, set $(s^T)\tau := s\sigma$. Then ψ is an injective homomorphism from $S/\!\!/T$ to S', and τ is the map associated with ψ.*

We shall now come to the Isomorphism Theorems for schemes.

Theorem 8.2. *Let T and U be closed subsets of a scheme S, and assume that $T \subseteq U$. Then $(S/\!\!/T)/\!\!/(U/\!\!/T) \cong S/\!\!/U$.*

Recall that, for each closed subset T of a scheme S, $N_S(T)$ is our notation for the set of all elements s in S which satisfy $Ts \subseteq sT$.

The following lemma is a straightforward generalization of a standard group theoretic result.

Lemma 8.3. *Let T and U be closed subsets of a scheme S, and assume that $T \subseteq N_S(U)$. Then $T \cap U$ is normal in T and U is normal in TU.*

Let X be a set, let S be a scheme on X, let x be an element in X, and let T be a closed subset of S. It is obvious that, for each element t in T, $(t_{xT})^* = (t^*)_{xT}$. Note also that T_{xT} is a scheme on xT.

Let x be an element in X, and let T be a closed subset of S. We call T_x the *subscheme* of S *defined by* xT.

Theorem 8.4. *Let X be a set, let S be a scheme on X, let x be an element in X, and let T and U be closed subsets of S such that $T \subseteq N_S(U)$. Then we have $(T/\!\!/T \cap U)_x \cong (TU/\!\!/U)_x$.*

Let X be a set, let S be a scheme on X, and let $\mathcal{T}$ be a set of closed subsets of S such that $\{1\} \in \mathcal{T}$ and $S \in \mathcal{T}$. Let us assume that, for any two elements U and V in $\mathcal{T}$, $U \subseteq V$ or $V \subseteq U$.

For each element T in $\mathcal{T} \setminus \{S\}$, we define $T^{\mathcal{T}}$ to be the intersection of all elements U of $\mathcal{T} \setminus \{T\}$ which contain T as a subset. (Since S is assumed to have finite valency, we have $T^{\mathcal{T}} \in \mathcal{T}$.) The set $\mathcal{T}$ is called a *subnormal series* of S if, for each element T in $\mathcal{T}$, T is normal in $T^{\mathcal{T}}$. A maximal subnormal series of S is called a *composition series* of S.

Two composition series $\mathcal{T}$ and $\mathcal{U}$ of S are called *isomorphic* if there exists a bijective map η from $\mathcal{T} \setminus \{S\}$ to $\mathcal{U} \setminus \{S\}$ such that, for any two elements x in X and T in $\mathcal{T} \setminus \{S\}$,

$$(T^{\mathcal{T}}/\!\!/T)_x \cong (T^{\eta\mathcal{U}}/\!\!/T^{\eta})_x.$$

The following theorem generalizes a famous group theoretic theorem of Otto Hölder to scheme theory.

Theorem 8.5. *Any two composition series of a scheme of finite valency are isomorphic.*

Theorem 8.5 suggests investigating schemes S in which $\{1\}$ and S are the only normal closed subsets. Recall that such schemes were called simple.

9. Primitivity and simplicity

In this section, the letter X stands for a finite set, the letter S for a scheme on X.

Let T and U be closed subsets of S, and assume that $T \subseteq U$. Recall that T is called normal in U if, for each element u in U, $Tu = uT$.

Lemma 9.1. *Let T and U be closed subsets of S. Assume that T is normal in S. Then TU is closed and $TU/\!\!/U$ is normal in $S/\!\!/U$.*

Proof. Since T is assumed to be normal in S, TU is closed; cf. Lemma 4.4. Moreover, as T is assumed to be normal in S, we have $TUsU = UsTU$ for each element s in S. Thus, by [22, Lemma 4.1.5], $TU/\!\!/U$ is normal in $S/\!\!/U$. □

Theorem 9.2. *Let T, U, and V be closed subsets of S. Assume that TU is closed, that $U \subseteq V \subseteq TU$, and that $T \cap V /\!\!/ T \cap U$ is normal in $T /\!\!/ T \cap U$. Then $V /\!\!/ U$ is normal in $TU /\!\!/ U$.*

Proof. We are assuming that TU is closed. Thus, by Lemma 4.4, $TU = UT$. Thus, as $V \subseteq TU$, $V \subseteq UT$.

We are assuming that $U \subseteq V$. Thus, by Lemma 4.5(i), $UT \cap V = U(T \cap V)$. Thus, as $V \subseteq UT$,

$$V = U(T \cap V).$$

From Lemma 4.5(ii) we know that $TU \cap V = (T \cap V)U$. Thus, as $V \subseteq TU$,

$$V = (T \cap V)U.$$

We are assuming that $T \cap V /\!\!/ T \cap U$ is normal in $T /\!\!/ T \cap U$. According to [22, Lemma 4.1.5] this means that, for each element t in T,

$$(T \cap V)t(T \cap U) \subseteq (T \cap U)t(T \cap V).$$

Thus, for each element t in T,

$$VtU = U(T \cap V)t(T \cap U)U \subseteq U(T \cap U)t(T \cap V)U = UtV.$$

We shall now see that, for each element s in TU, $VsU \subseteq UsV$ (not only for elements s in T).

Let s be an element in TU. Then there exist elements t in T and u in U such that $s \in tu$. Thus, by [22, Lemma 1.3.3(i)], $t \in su^*$. From $s \in tu$, $t \in su^*$, and $u^* \in U \subseteq V$ we obtain

$$VsU \subseteq VtuU = VtU \subseteq UtV \subseteq Usu^*V = UsV.$$

Since s has been chosen arbitrarily in TU, this proves that $V /\!\!/ U$ is normal in $TU /\!\!/ U$; cf. [22, Lemma 4.1.5]. □

A closed subset T of S is called a *Dedekind* set if each closed subset of T is normal in T.

It follows right from the definition of Dedekind sets that closed subsets of Dedekind sets are Dedekind sets. The following lemma says that quotients of Dedekind set are Dedekind sets.

Lemma 9.3. *Let T and U be closed subsets of S such that $T \subseteq U$. Assume that U is a Dedekind set. Then $U /\!\!/ T$ is a Dedekind set.*

Proof. Let V be a closed subset of U such that $T \subseteq V$. Then, as U is assumed to be a Dedekind set, V is normal in U. Thus, by Lemma 9.1, $V /\!\!/ T$ is normal in $U /\!\!/ T$. □

Let T be a closed subset of S, and assume that $T \neq \{1\}$. The set T is called *primitive* if $\{1\}$ and T are the only closed subsets of T. (Recall that T is called simple if $\{1\}$ and T are the only normal closed subsets of T.)

It follows right from these definitions that primitive closed subsets are simple. Of course, for Dedekind sets the converse holds, too. We are interested in other circumstances under which the converse holds.

Lemma 9.4. *Let T and U be closed subsets such that TU is closed. Assume that $T /\!\!/ T \cap U$ is a Dedekind set. Then the following hold.*

(i) *The set $TU /\!\!/ U$ is a Dedekind set.*
(ii) *If $TU /\!\!/ U$ is simple, $TU /\!\!/ U$ is primitive.*

Proof. (i) This follows immediately from Theorem 9.2.

(ii) Assuming $TU /\!\!/ U$ to be simple we obtain from (i) that $TU /\!\!/ U$ is primitive. □

Theorem 9.5. *Let T be a closed subset of S. Assume that S possesses a normal closed subset U such that $U \not\subseteq T$ and $U /\!\!/ T \cap U$ is a Dedekind set. Then, if $S /\!\!/ T$ is simple, $S /\!\!/ T$ is primitive.*

Proof. Since U is assumed to be normal in S, TU is closed and $TU/\!\!/T$ is normal in $S/\!\!/T$; cf. Lemma 9.1. Thus, as $S/\!\!/T$ is assumed to be simple, we must have $TU/\!\!/T = \{1^T\}$ or $TU/\!\!/T = S/\!\!/T$.

Since we are assuming that $U \not\subseteq T$, we cannot have $TU/\!\!/T = \{1^T\}$. Thus, $TU/\!\!/T = S/\!\!/T$, and this implies $TU = S$.

We are assuming that $S/\!\!/T$ is simple. Thus, as $TU = S$, $TU/\!\!/T$ is simple. Thus, by Lemma 9.4(ii), $TU/\!\!/T$ is primitive. Thus, as $TU = S$, $S/\!\!/T$ is primitive. □

If group theoretic theorems do not right away generalize to schemes, one may wish to generalize them first to schurian schemes. Sergei Evdokimov and Ilia Ponomarenko, to whom most of the remaining results of this section are due, did that with the Odd-Order Theorem, the theorem of Walter Feit and John Thompson which says that finite groups of odd order are solvable.

The scheme S of finite valency is said to be of *odd order* if, for each element s in S, $|s|$ is odd.

The following lemma provides a useful characterization of schemes of odd order.

Lemma 9.6. *A scheme is of odd order if and only if the identity is its only symmetric element.*

Proof. Let X be a finite set, let S be a scheme on X, and assume first that S is of odd order. Let s be a symmetric element of S. Then we have $(z, y) \in s$ for any two elements y and z in X with $(y, z) \in s$. Thus, as $|s|$ is assumed to be odd, there exist elements y and z in X with $(y, z) = (z, y) \in s$. It follows that $1 \cap s$ is not empty. Thus, $s = 1$.

Let us now assume that 1 is the only symmetric element of S. Then S possesses a subset R such that $\{R^*, R\}$ is a partition of $S \setminus \{1\}$. Thus, for each element s in S,

$$n_s = \sum_{r\in S} a_{srs} = a_{s1s} + 2\sum_{r\in R} a_{srs} = 1 + 2\sum_{r\in R} a_{srs}.$$

(The first equation follows from Lemma 4.1(i), the second equation from Lemma 4.1(iii).) □

The following two lemmata will not be needed in the remainder of this section. They shows how being of odd order is inherited.

Lemma 9.7. *Let T be a closed subset of S. Then S is of odd order if and only if T and $S/\!\!/T$ are of odd order.*

Proof. Assume first that S is of odd order. Then, by definition, $|s|$ is odd for each element s in S. In particular, $|t|$ is odd for each element t in T, so T is of odd order.

In order to show that $S/\!\!/T$ is of odd order, we fix an element in S and call it s. Then, by Lemma 4.8 and Lemma 4.6, n_{s^T} divides $n_s n_T$. Thus, $|s^T| = n_{s^T} n_{S/\!\!/T}$ divides $n_s n_T n_{S/\!\!/T} = n_s n_S = |s|$, and we are done.

We now assume that T and $S/\!\!/T$ are of odd order, and we fix an element s in S.

If $s \in S \setminus T$, $s^T \neq 1^T$. Thus, as $S/\!\!/T$ is assumed to be of odd order, $(s^T)^* \neq s^T$; cf. Lemma 9.6. Thus, as $(s^T)^* = (s^*)^T$, $s^* \neq s$.

If $s \in T \setminus \{1\}$, one obtains $s^* \neq s$ from the hypothesis that T is of odd order. Thus, 1 is the only symmetric element in S, so, by Lemma 9.6, S is of odd order. □

Lemma 9.8. *Let T and U be closed subsets of S, and assume that $S/\!\!/T$ and $S/\!\!/U$ are of odd order. Then $S/\!\!/(T \cap U)$ is of odd order.*

Proof. Let T and U be closed subsets of S such that $S/\!\!/T$ and $S/\!\!/U$ are of odd order. Then none of the elements in $S/\!\!/T \setminus \{1^T\}$ or in $S/\!\!/U \setminus \{1^U\}$ is symmetric; cf. Lemma 9.6. It follows that $Ts^*T \cap TsT = \emptyset$ for each element s in $S \setminus T$ and $Us^*U \cap UsU = \emptyset$ for each element s in $S \setminus U$.

Let s be an element in $S \setminus (T \cap U)$. Then $s \in S \setminus T$ or $s \in S \setminus U$. Assume, without loss of generalization, that $s \in S \setminus T$. Then

$$(T \cap U)s^*(T \cap U) \cap (T \cap U)s(T \cap U) \subseteq Ts^*T \cap TsT = \emptyset.$$

Thus, none of the elements of $S/\!\!/(T \cap U) \setminus \{1^{T\cap U}\}$ is symmetric. Thus, by Lemma 9.6, $S/(T \cap U)$ is of odd order. □

Recall that the set of all automorphisms of S is a group with respect to composition and denoted by $\mathrm{Aut}(S)$.

Lemma 9.9. *If S is of odd order,* $\mathrm{Aut}(S)$ *has odd order.*

Proof. Assume, by way of contradiction, that $\mathrm{Aut}(S)$ possesses an element g of order 2. Then there exists an element x in X such that $xg \neq x$.

Let us denote by s the uniquely determined element in S which satisfies $xg \in xs$. Then $xg^2 \in xsg = xgs$. However, as g has order 2, $xg^2 = x$. Thus, $x \in xgs$. Thus, as $xg \in xs$, s is symmetric. Thus, by Lemma 9.6, $s = 1$. Thus, $xg = x$, contradiction. □

Lemma 9.10. *Assume S to be schurian and simple. Assume that S has a commutative group of automorphisms acting transitively on X. Then S is primitive.*

Proof. Since S is assumed to be schurian, there exists a group G and a subgroup H of G such that $S \cong G/\!\!/H$. Thus, as S is assumed to be simple, $G/\!\!/H$ is simple.

By hypothesis, G possesses a commutative subgroup A acting transitively on X. Since A acts transitively on X, $AH = G$. Thus, by Lemma 9.4(ii), $G/\!\!/H$ must be primitive. Thus, as $S \cong G/\!\!/H$, S is primitive. □

Lemma 9.11. *Assume S to be schurian and simple. Assume that* $\mathrm{Aut}(S)$ *has a commutative normal subgroup A different from* $\{1\}$*. Then S is primitive.*

Proof. We set $G := \mathrm{Aut}(S)$. Since S is assumed to be schurian, G possesses a closed subset H such that $S \cong G/\!\!/H$.

By hypothesis, G has a commutative normal subgroup A with $A \neq \{1\}$. Since $A \neq \{1\}$, A is not a subset of H. Thus, the lemma follows from Theorem 9.5. □

Scheme theoretically, the Feit–Thompson Theorem says that thin simple schemes of odd order are primitive. Referring to this theorem we can now say a little bit more.

Theorem 9.12. *Schurian simple schemes of odd order are primitive.*

Proof. Let S be a schurian scheme of odd order, and set $G := \mathrm{Aut}(S)$. Then G has odd order; cf. Lemma 9.9. Thus, by [2], G is solvable. Thus, G has a commutative normal subgroup A different from $\{1\}$. Thus, by Lemma 9.11, S is primitive. □

It seems to be unknown whether or not non-schurian simple schemes of odd order are generally primitive. No imprimitive simple scheme of odd order is known. It also seems to be an open question whether primitive schemes of odd order are commutative.

10. Sylow theory

All schemes in this section are assumed to have finite valency.

Let S be a scheme, and let p be a prime number. An element s in S is called *p-valenced* if n_s is a power of p. A nonempty subset of S is called *p-valenced* if each of its elements is p-valenced.

Recall that $O_\vartheta(S)$ is our notation for the thin radical of S, that is the set of all elements in S which have valency 1. One obviously has $1 \in O_\vartheta(S)$.

It is easy to see that the following lemma generalizes the fact that finite p-groups have nontrivial centers.

Lemma 10.1. *Let p be a prime number, and let T be a closed p-valenced subset of a scheme S. Assume that p divides n_T. Then $O_\vartheta(T) \neq \{1\}$.*

Proof. By definition, n_T is the sum of the integers n_t with $t \in T$. Since T is assumed to be p-valenced, n_t is a power of p for each element t in T. Thus, as we are assuming that p divides n_T, p divides $n_{O_\vartheta(T)}$. □

Again, let p be a prime number. A nonempty p-valenced subset R of a scheme S is called a *p-subset* of S if n_R is a power of p. A closed p-subset T of a scheme S is called a *Sylow* p-subset of S if p does not divide $n_{S/\!\!/T}$.

One cannot expect that (like in group theory) each scheme of finite valency possesses p-Sylow subsets. In fact, for each integer n with $2 \leq n$, there exists a scheme of valency n which has only two elements, the identity and the non-identity. To find an appropriate condition which guarantees the existence of Sylow subsets was, therefore, a certain challenge in the development of the structure theory of schemes of finite valency.

The situation changed when Hirasaka, whose work on schemes of finite valency had already reflected specific features of the arithmetic of the valencies of schemes, observed that, since thin schemes are p-valenced for any prime number p, a 'p-Sylow theorem' for p-valenced schemes would be a genuine generalization of Sylow's group theoretic theorems [19]. His suggestion of searching for Sylow p-subsets only in p-valenced schemes led to the Sylow theorems for association schemes as they later were established in [14].

Theorem 10.2. *Each p-valenced scheme possesses at least one Sylow p-subset.*

Proof. Let S be a minimal counterexample. Then p divides n_S. Thus, by Lemma 10.1, p divides $n_{O_\vartheta(S)}$. Thus, by Cauchy's Lemma, $O_\vartheta(S)$ possesses a closed subset T of valency p. From Lemma 4.6 and Lemma 4.8 we now obtain that $S/\!\!/T$ is p-valenced. So, by induction, $S/\!\!/T$ possesses a Sylow p-subset $U/\!\!/T$. It follows that U is a Sylow p-subset of S, contradiction. □

The key for our next theorem on Sylow subsets is the following analogue of the conjugation property of Sylow subgroups. We include a proof also of this result since it is one of the most convincing applications of Lemma 4.2.

Proposition 10.3. *Let S be a p-valenced scheme, let T be a closed p-subset of S, and let U be a Sylow p-subset of S. Then there exists an element s in S such that $s^*Ts \subseteq U$.*

Proof. Let R be a subset of S which contains exactly one element of each double coset of T and U in S. Then, as the double cosets of T and U in S form a partition of S, we have

$$n_S = \sum_{r \in R} n_{TrU}.$$

Now recall that n_T and n_U are assumed to be powers of p. Moreover, as S is assumed to be p-valenced, n_r is a power of p for each element r in R. Thus, for each element r in R, n_{TrU} is a power of p; cf. Lemma 4.6.

Now recall that $n_U \leq n_{TrU}$ for each element r in R; cf. Lemma 4.2(i). Thus, as n_U is the highest power of p dividing n_S, R possesses an element s such that $n_U = n_{TsU}$. From $n_U = n_{TsU}$ we obtain $s^*Ts \subseteq U$; cf. Lemma 4.2(ii). □

For each p-valenced scheme S, we define $\mathrm{Syl}_p(S)$ to be the set of all Sylow p-subsets of S.

Theorem 10.4. *Let S be a p-valenced scheme, and let T be a Sylow p-subset of S. Then* $\mathrm{Syl}_p(S) = \{s^*Ts \mid s \in S,\ ss^* \subseteq T\}$.

In Theorem 10.2, we saw that each p-valenced scheme possesses at least one Sylow p-subset. Generalizing this theorem we can now say more about the number of Sylow p-subsets of p-valenced schemes.

Theorem 10.5. *The number of Sylow p-subsets of a p-valenced scheme is congruent to* 1 *modulo p.*

At this point it might be worth mentioning that the theory of table algebras allows a Sylow theory which is similar to the one which we presented in this section, a Sylow theory which is, of course, more general than the one for schemes; cf. [1].

11. Conjugate constrained sets of involutions

Let S be a p-valenced scheme, and let T and U be Sylow p-subsets of S. Then, by Proposition 10.3, there exists an element s in S such that $s^*Ts = U$. Like in group theory, one might say that the Sylow p-subsets are 'conjugate'.

In this section, we shall deal with conjugation of Coxeter sets. Let S be a scheme, and let L be a set of involutions of S. Assume that L is a Coxeter set, and that $\langle L\rangle$ is finite. Then $\langle L\rangle$ possesses a uniquely determined element of maximal length. (We mentioned this in Section 7. A proof of this fact was given in [24, Lemma 6.2].) Let us call this element m_L.

From [24, Lemma 6.1, Lemma 2.1] one obtains, for each element r in $\langle L\rangle$, a uniquely determined element $r^{(L)}$ in $\langle L\rangle$ such that

$$m_L \in r^{(L)}r \quad \text{and} \quad \ell_L(m_L) = \ell_L(r^{(L)}) + \ell_L(r).$$

From $m_L \in 1^{(L)}1$ we now obtain $1^{(L)} = m_L$. Thus, we have $1^{(L)} \in r^{(L)}r$ and $\ell_L(1^{(L)}) = \ell_L(r^{(L)}) + \ell_L(r)$ for each element r in $\langle L\rangle$. Thus, as L is assumed to be constrained, we obtain

$$r^{(L)}r = \{1^{(L)}\}$$

for each element r in $\langle L\rangle$.

It turns out that these equations, together with the observation that $^{(L)}$ is injective (a fact which was proven in [22, Lemma 12.1.1(i)]), take care of many of the structural results in the theory of finite Coxeter schemes. We, therefore, isolate these two facts from the initial setup in finite Coxeter schemes and turn them into the starting point of the following somewhat more abstract considerations.

Let S be a scheme, and let K be a constrained set of involutions. Assume that there exists an injective map $^{\rho}$ from $\langle K\rangle$ to S such that, for each element q in $\langle K\rangle$,

$$q^{\rho}q = \{1^{\rho}\}.$$

We set $m := 1^{\rho}$.

The difference between the maps $^{(L)}$ and $^{\rho}$ is that domain and codomain of $^{(L)}$ are equal, whereas $^{\rho}$ does not necessarily send its elements back to its domain. A satisfactory picture of the image of $^{\rho}$ is given in the following theorem, the proof of which is not straightforward.

Theorem 11.1. *We have* $\langle K\rangle^{\rho} = m\langle K\rangle$.

Since the cosets of a closed subset of S form a partition of S, Theorem 11.1 implies that the image of $^{\rho}$ is either equal to its domain or disjoint from its domain.

We shall now deal with the set on which S is a scheme, and we shall denote this set by X.

Let y be an element in X, and let z be an element in ym. It is not too difficult to see that, for each element r in $\langle K\rangle$, $yr^{*\rho} \cap zr$ contains exactly one element.

We define C_{yz} to be the union of the sets $yr^{*\rho} \cap zr$ with $r \in \langle K\rangle$.

Since $z \in ym$ and $m = 1^{\rho}$, $z \in y1^{\rho}$. Thus, $z \in C_{yz}$.

Like the proof of Theorem 11.1, the proof of the following theorem is not straightforward.

Theorem 11.2. *If* K *is a Coxeter set,* C_{yz} *is an apartment of* $\langle K\rangle$.

Apartments have been introduced by Tits as one of the indispensable tools in the theory of buildings. Scheme theory allows to generalize this notation in the following way. Let T be a closed subset of a scheme S on X. A subset W of X is called *apartment* of T if $|W \cap wt| = 1$ for any two elements $w \in W$ and $t \in T$.

We now assume that S possesses a second constrained set of involutions. We call this set H and assume that there exists an injective map also from $\langle H\rangle$ to S. This map will be called λ, and we assume that

$$pp^{\lambda} = \{1^{\lambda}\}$$

for each element p in $\langle H\rangle$. Finally, we assume that $1^{\lambda} = m$ and that

$$\langle H\rangle m = m\langle K\rangle.$$

From a group theoretic point of view, this last equation suggests considering $\langle H\rangle$ and $\langle K\rangle$ to be 'conjugate'.

From Theorem 11.1 (together with [22, Lemma 1.3.2(iii)]) one obtains $\langle H\rangle^{\lambda} = \langle H\rangle m$. This yields

$$\langle H\rangle^{\lambda} = \langle K\rangle^{\rho}.$$

Thus, as $^{\rho}$ is assumed to be injective, we obtain the following.

Lemma 11.3. *For each element p in $\langle H\rangle$, there exists exactly one element q in $\langle K\rangle$ such that $p^{\lambda} = q^{\rho}$.*

Similarly one obtains, of course, for each element q in $\langle K\rangle$, a uniquely determined element p in $\langle H\rangle$ with $p^{\lambda} = q^{\rho}$.

If H and K satisfy the exchange condition, there is a long list of natural consequences of our setup.[9] The following two lemmata might give an impression.

Lemma 11.4. *Let p be an element in $\langle H\rangle$, and let q denote the uniquely determined element in $\langle K\rangle$ which satisfies $p^{\lambda} = q^{\rho}$. Then we have $p^{*\lambda} = q^{*\rho}$. Moreover, if p is thin, so is q.*

Lemma 11.5. *Let p be an element in $\langle H\rangle$, and let q denote the uniquely determined element in $\langle K\rangle$ which satisfies $p^{\lambda} = q^{\rho}$. Assume that H does not contain thin elements. Then we have $\ell_H(p) = \ell_K(q)$. Moreover, if $p \in H$, then $q \in K$ and $n_p = n_q$.*

We now fix elements y in X and z in ym and define apartments as we did earlier. We define A_{yz} to be the union of the sets $yp \cap zp^{\lambda *}$ with $p \in \langle H\rangle$. By B_{yz} we mean the union of the sets $yq^{*\rho} \cap zq$ with $q \in \langle K\rangle$.

Let v be an element in A_{yz}, let w be an element in B_{yz}. Let p denote the uniquely determined element of $\langle H\rangle$ satisfying $v \in yp$, and let q denote the uniquely determined element of $\langle K\rangle$ satisfying $w \in zq$. Then one obtains from Lemma 11.3 and Lemma 11.5 that $p^{\lambda} = q^{\rho}$ if and only if $w \in vm$.

Theorem 11.6. *For each element v in A_{yz}, there exists exactly one element w in B_{yz} such that $w \in vm$.*

From Theorem 11.6 one obtains that, for each element w in B_{yz}, there exists exactly one element v in A_{yz} such that $w \in vm$. Thus, the relation m establishes a bijective map between A_{yz} and B_{yz}.

12. Representations of schemes of finite valency

The extent to which the arithmetic of the structure constants of association schemes rules over the structure of association schemes is visible not only in the Sylow Theorems for schemes; it is even more apparent in the representation theory of schemes of finite valency.

Representation theory of association schemes is the oldest part of scheme theory and deals with schemes of finite valency. It obtained its first substantial contributions from Donald Higman's investigations on coherent configurations; cf. [9]. Many papers have been published on the representation theory of specific classes of association schemes. In particular the literature on eigenvalues of commutative and, even more specifically, of symmetric association schemes is overwhelming.

In this final section, we shall not make any attempt to survey representation theory of schemes. The intention is again to just highlight a few analogies to group theory. The latest achievements in representation theory of association schemes (of finite valency) are discussed in a wider framework in Akihide Hanaki's contribution to this volume; cf. [4].

Let X be a finite set, and let C be a field. For each element x in X, we fix an element c_x in C. We write

$$\sum_{x \in X} c_x x$$

to denote the map from X to C which sends each element x in X to c_x.

[9] Recall that H and K are assumed to be constrained. Thus, H and K are Coxeter sets if they satisfy the exchange condition.

The set CX of all maps from X to C is a vector space over C with respect to componentwise addition and componentwise multiplication with elements of C.

Each element x in X can be identified with the map from X to C which maps x to 1 and each element different from x to 0. Thus, X can be viewed as a subset of CX. In fact, X is a basis of the vector space CX.

Let S be a scheme on X, and let s be an element in S. Since X is a basis of CX, the endomorphism ring $\mathrm{End}_C(CX)$ of CX possesses a uniquely defined element σ_s such that

$$x\sigma_s = \sum_{y \in xs} y$$

for each element x in X.

For each nonempty subset R of S, we define CR to be the set of all finite sums of products $c\sigma_r$ with $c \in C$ and $r \in R$.

Note that, for each nonempty subset R of S, CR is a vector space over C with respect to componentwise addition and componentwise multiplication with elements of C. The set $\{\sigma_r \mid r \in R\}$ is a basis of CR.

It follows right from the regularity condition for schemes that

$$\sigma_p\sigma_q = \sum_{s \in S} a_{pqs}\sigma_s$$

for any two elements p and q in S. Thus, CS is a subring of $\mathrm{End}_C(CX)$.

Since $1 \in S$, $\sigma_1 \in CS$. Thus, CS is a ring with 1. It is called the *adjacency algebra* of S *over* C or the *scheme ring* of S *over* C.[10] The field C is called the *base field* of CS.

Since CS is a subring of $\mathrm{End}_C(CX)$, CX is a CS-module. This module is called the *standard module* of CS.

Since CS is a ring with 1, the elements of C can be identified with the multiples of σ_1. In particular, C can be viewed as a subfield of $Z(CS)$, the center of CS. This enables us to define a character for each CS-module which is finitely generated over C.

Recall that the standard module CX of CS is finitely generated over C. The character of CS afforded by the standard module is called the *standard character* of CS and denoted by χ_{CX}.

The following lemma gives some information about the standard character.

Lemma 12.1. *The following statements hold.*

(i) *We have* $\chi_{CX}(\sigma_1) = n_S$.
(ii) *For each element* s *in* $S \setminus \{1\}$, *we have* $\chi_{CX}(\sigma_s) = 0$.
(iii) *For any two elements* p *and* q *in* S, $\chi_{CX}(\sigma_{p^*}\sigma_q) = \delta_{pq}|p^*|$.
(iv) *For each element* s *in* S, *let* c_s *be an element in* C. *Set*

$$\sigma := \sum_{s \in S} c_s\sigma_s.$$

Then, for each element s *in* S, $\chi_{CX}(\sigma_{s^*}\sigma) = c_s|s^*|$.

The standard module possesses an irreducible submodule which induces a character of CS all values of which can be computed explicitly. In order to introduce this module we (temporarily) set

$$j := \sum_{x \in X} x.$$

Note that, for each element s in S,

$$j\sigma_s = \sum_{x \in X} \sum_{y \in xs} y = n_{s^*} \sum_{x \in X} x = n_{s^*} j.$$

[10] In investigations on commutative association schemes scheme rings are usually called Bose–Mesner algebra. Note also that scheme rings of thin schemes are nothing but group rings.

Thus, Cj is a submodule of the CS-module CX.[11] It is called the *principal module* of CS. The character afforded by the principal module is called the *principal character* of CS. We denote it by 1_{CS}.

The above equation tells us that, for each element s in S,

$$1_{CS}(\sigma_s) = n_s.$$

(Recall that we have $n_{s^*} = n_s$ for each element s in S, since S is assumed to have finite valency.)

The key for all computations with characters is the following structure theorem for scheme rings.

Theorem 12.2. *Assume that, for each element s in S, the characteristic of C does not divide $|s|$. Then CS is semisimple.*

If CS is semisimple, we may apply the well-known theorem of Emil Artin and Joseph Wedderburn on completely reducible rings. Thus, there exists exactly one maximal homogeneous submodule H_χ of the CS-module CS such that $\chi = \psi_{H_\chi}$. We set

$$\epsilon_\chi := 1_{H_\chi}.$$

Let us denote by $\mathrm{Irr}(CS)$ the set of all irreducible characters of CS. Then there exists, for each irreducible character χ of CS, a non-negative integer m_χ such that

$$\chi_{CX} = \sum_{\chi\in\mathrm{Irr}(CS)} m_\chi \chi.$$

The integers m_χ are called the *multiplicities* of χ.

Lemma 12.3. *Assume that, for each element s in S, the characteristic of C does not divide $|s|$. Let χ be an irreducible character of CS. Then we have*

$$\epsilon_\chi = \frac{m_\chi}{n_S} \sum_{s\in S} \frac{\chi(\sigma_{s^*})}{n_{s^*}} \sigma_s.$$

Lemma 12.3 is the key in the proof of the following theorem. The equations in this theorem are usually called the *orthogonality relations* for schemes of finite valency.

Theorem 12.4. *Assume that, for each element s in S, the characteristic of C does not divide $|s|$. Then we have*

$$\frac{1}{n_S} \sum_{s\in S} \frac{1}{n_{s^*}} \phi(\sigma_{s^*})\psi(\sigma_s) = \delta_{\phi\psi} \frac{\phi(\sigma_1)}{m_\phi}$$

for any two irreducible characters ϕ and ψ of CS.

The orthogonality relations are the key for quite a few results in group theory. This is due to the fact that they bring algebraic integers on the left hand side of the equations together with rational numbers on the right hand side. Since the ring of the integers is integrally closed, this can lead to interesting divisibility conditions. As is well known, this is the case in Burnside's proof of the solvability of groups of order $p^\alpha q^\beta$, but it is also the case in the proof of the theorem of Feit and Graham Higman on finite polygons or, as we would say, on Coxeter schemes of rank 2 and finite valency.

In the proof of this latter theorem one first computes completely the irreducible characters of CS like one can completely compute the irreducible characters of a dihedral group. Independently from this one knows the multiplicities.

So far we have assumed that the characteristic of C does not divide any of the integers $|s|$ with $s \in S$. Like in group theory the theory changes considerably if one omits this hypothesis. It is the merit mainly of Akihide Hanaki to have seriously looked at the *modular* representation theory of schemes of finite valency. All his considerations are based on the following observation.

11 Recall that Cj denotes the set of all elements cj with $c \in C$.

Proposition 12.5. *Let R be an integral domain of characteristic* 0*, and let p be a prime number. Assume that p is not a unit in R and that n_S is a power of p. Then* 1 *is the only idempotent element of RS.*

From this he obtained the following; cf. [5, Theorem 3.4].

Theorem 12.6. *Let R be a complete discrete valuation ring of characteristic* 0*, and let p be a prime number. Assume that p is not a unit in R and that n_S is a power of p. Then RS is local.*

As an application he, jointly with Katsuhiro Uno, obtained the following structural result for schemes of prime valency.

Lemma 12.7. *Let R be a complete discrete valuation ring of characteristic* 0*, and let p be a prime number. Assume that p is not a unit in R and that $n_S = p$. Let F denote the field of fractions of R. Then any two irreducible characters of FS different from 1_{FS} are algebraically conjugate.*

On the other hand, from the powerful fact that the *Frame number*

$$n_S^{|S|} \frac{\prod_{s\in S} n_s}{\prod_{\chi\in \mathrm{Irr}(CS)} m_\chi^{\chi(1)^2}}$$

is an integer, one obtains the following.

Proposition 12.8. *If all nontrivial irreducible characters of a scheme S have the same multiplicity, then all elements in $S \setminus \{1\}$ have the same valency and S is commutative.*

Since algebraically conjugate characters have the same multiplicity, the last two results yield the following; cf. [8].

Theorem 12.9. *Let S be a scheme such that n_S is a prime number. Then S is commutative.*

In two forthcoming papers, one of them jointly with Hirasaka and Uno, Hanaki has investigated schemes whose valency is the square of a prime number. The best result so far is the following; cf. [6,7].

Theorem 12.10. *Let S be a scheme such that n_S is the square of a prime number. Assume that $O_\vartheta(S) \neq \{1\}$ or that $O^\vartheta(S) \neq S$. Then S is commutative.*

It seems that no noncommutative scheme of prime square valency is known.

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