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Some subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function

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ABSTRACT

In the present paper, we introduce and investigate each of the following new subclasses:

$$\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi),\qquad \mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi)\quad\text{and}\quad \mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$$

as well as

$$\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi),\qquad \mathfrak{G}_p^{q,s}(\alpha;\alpha_1;\phi)\quad\text{and}\quad \mathfrak{H}_p^{q,s}(\alpha;\alpha_1;\phi)$$

of meromorphically p-valent functions, which is defined by means of a certain meromorphically p-modified version of the familiar Dziok–Srivastava linear operator involving the generalized hypergeometric function. Such results as inclusion relationships, integral representations and convolution properties for these function classes are proved. The results presented here provide extensions of those given in some earlier works. Several other new results are also obtained.

1. Introduction, definitions and preliminaries

Let Σ_p denote the class of functions of the form:

$$f(z)=z^{-p}+\sum_{n=1}^{\infty}a_{n-p}z^{n-p}\quad(p\in\mathbb{N}:=\{1,2,3,\ldots\}),\tag{1.1}$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{U}^*:=\{z:z\in\mathbb{C}\quad\text{and}\quad 0<|z|<1\}=:\mathbb{U}\setminus\{0\}.$$

For simplicity, we write

$$\Sigma_1=:\Sigma.$$

Let $f,g\in\Sigma_p$, where $f(z)$ is given by (1.1) and $g(z)$ is defined by

$$g(z)=z^{-p}+\sum_{n=1}^{\infty}b_{n-p}z^{n-p}.$$

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Then the Hadamard product (or convolution) $f * g$ is defined by

$$(f * g)(z) := z^{-p} + \sum_{n=1}^{\infty} a_{n-p} b_{n-p} z^{n-p} =: (g * f)(z).$$

For parameters

$$\alpha_j \in \mathbb{C} \quad (j = 1, \ldots, q) \quad \text{and} \quad \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^- \quad (\mathbb{Z}_0^- := \{0, -1, -2, \ldots\}; j = 1, \ldots, s),$$

the generalized hypergeometric function

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$$

is defined by the following infinite series:

$${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$
$$(q \leqq s + 1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Corresponding to the function

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

defined by

$$h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) = z^{-p} \cdot {}_q F_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z),$$

we consider a linear operator (which is essentially a *meromorphically* p-modified version of the familiar Dziok–Srivastava linear operator [1,2])

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_, \ldots, \beta_s) : \Sigma_p \longrightarrow \Sigma_p$$

defined by the following Hadamard product:

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) := h_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z) * f(z)$$
$$(q \leqq s + 1; q, s \in \mathbb{N}_0; z \in \mathbb{U}^*). \tag{1.2}$$

If $f \in \Sigma_p$ is given by (1.1), then we have

$$H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s)f(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} a_{n-p} \frac{z^{n-p}}{n!} \quad (n \in \mathbb{N}; z \in \mathbb{U}^*).$$

In order to make the notation simple, we write

$$H_p^{q,s}(\alpha_1) := H_p(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s) \quad (q \leqq s + 1; q, s \in \mathbb{N}_0).$$

It is easily verified from the definition (1.2) that

$$z \left(H_p^{q,s}(\alpha_1)f\right)'(z) = \alpha_1 H_p^{q,s}(\alpha_1 + 1)f(z) - (\alpha_1 + p)H_p^{q,s}(\alpha_1)f(z) \quad (f \in \Sigma_p). \tag{1.3}$$

In recent years, Liu and Srivastava [3], Raina and Srivastava [4], and Aouf [5] obtained many interesting results involving the linear operator $H_p^{q,s}(\alpha_1)$. In particular, for

$$q = 2, \qquad s = 1, \qquad \alpha_1 = a, \qquad \beta_1 = c, \quad \text{and} \quad \alpha_2 = 1,$$

we obtain the following linear operator:

$$\mathcal{L}_p(a, c)f(z) := H_p(\alpha_1, 1; \beta_1)f(z) \quad (z \in \mathbb{U}^*),$$

which was introduced and investigated earlier by Liu and Srivastava [6], and was *further* studied in a subsequent investigation by Srivastava et al. [7]. It should also be remarked that the linear operator $H_p^{q,s}(\alpha_1)$ is a generalization of other linear operators considered in many earlier investigations (see, for example, [8–16]).

Let $\mathcal{P}$ denote the class of functions of the form:

$$\mathfrak{p}(z) = 1 + \sum_{n=1}^{\infty} \mathfrak{p}_n z^n,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the following condition:

$$\Re(\mathfrak{p}(z)) > 0 \quad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function $\omega(z)$, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Throughout this paper, we assume that

$$p, k \in \mathbb{N}, \qquad q, s \in \mathbb{N}_0, \qquad \varepsilon_k = \exp\left(\frac{2\pi \mathrm{i}}{k}\right),$$

$$f_{p,k}^{q,s}(\alpha_1; z) = \frac{1}{k}\sum_{j=0}^{k-1} \varepsilon_k^{jp}\left(H_p^{q,s}(\alpha_1)f\right)(\varepsilon_k^j z) = z^{-p} + \cdots \quad (f \in \Sigma_p), \tag{1.4}$$

$$g_p^{q,s}(\alpha_1; z) = \frac{1}{2}\left[H_p^{q,s}(\alpha_1)f(z) + \overline{H_p^{q,s}(\alpha_1)f(\overline{z})}\right] = z^{-p} + \cdots \quad (f \in \Sigma_p), \tag{1.5}$$

and

$$h_p^{q,s}(\alpha_1; z) = \frac{1}{2}\left[H_p^{q,s}(\alpha_1)f(z) - \overline{H_p^{q,s}(\alpha_1)f(-\overline{z})}\right] = z^{-p} + \cdots \quad (f \in \Sigma_p). \tag{1.6}$$

Clearly, for $k = 1$, we have

$$f_{p,1}^{q,s}(\alpha_1; z) = H_p^{q,s}(\alpha_1)f(z).$$

Making use of the integral operator $H_p^{q,s}(\alpha_1)$ and the above-mentioned principle of subordination between analytic functions, we now introduce and investigate the following subclasses of the class Σ_p of meromorphically p-valent functions.

Definition 1. A function $f \in \Sigma_p$ is said to be in the class $\mathcal{F}_{p,k}^{q,s}(\alpha; \alpha_1; \phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)\right]} \prec \phi(z) \quad (z \in \mathbb{U}), \tag{1.7}$$

for some α ($\alpha \geqq 0$), where $\phi \in \mathcal{P}$, $f_{p,k}^{q,s}(\alpha_1; z)$ is defined by (1.4) and

$$f_{p,k}^{q,s}(\alpha_1+1; z) \neq 0 \quad (z \in \mathbb{U}^*).$$

For simplicity, we write

$$\mathcal{F}_{p,k}^{q,s}(0; \alpha_1; \phi) =: \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi).$$

Remark 1. In a recent paper, Zou and Wu [17] introduced and investigated a subclass $MS_s^*(\alpha)$ of Σ consisting of functions which are meromorphically α-starlike with respect to symmetric points and satisfy the following inequality:

$$\Re\left(-\frac{z[(1+\alpha)f'(z) + \alpha\left(zf'(z)\right)']}{(1+\alpha)T_s f(z) + \alpha z\left(T_s f(z)\right)'}\right) > 0 \quad (z \in \mathbb{U}),$$

where

$$T_s f(z) = \frac{1}{2}\left[f(z) - f(-z)\right].$$

It is easy to see by setting

$$p=1,\qquad k=2,\qquad q=2,\qquad s=1,\qquad \alpha_1=\alpha_2=\beta_1=1,\quad \text{and}\quad \phi(z)=\frac{1+z}{1-z}$$

that the class $\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ reduces to the known function class $MS_s^*(\alpha)$. More recently, Srivastava et al. [7] studied a subclass $\Sigma_{p,k}(a,c;\phi)$ of Σ_p consisting of functions which are satisfy the following subordination condition:

$$-\frac{z\left(\mathcal{L}_p(a,c)f\right)'(z)}{pf_{p,k}(a,c;z)}\prec\phi(z)\quad(z\in\mathbb{U}),$$

where $\phi\in\mathcal{P}$ and

$$f_{p,k}(a,c;z)=\frac{1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\left(\mathcal{L}_p(a,c)f\right)(\varepsilon_k^j z)\neq0\quad(z\in\mathbb{U}^*).$$

It is also easy to see by setting

$$q=2,\qquad s=1,\qquad \alpha_1=a,\qquad \beta_1=c,\qquad \alpha_2=1,\quad\text{and}\qquad \alpha=0$$

that the class $\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ reduces to the aforementioned known function class $\Sigma_{p,k}(a,c;\phi)$.

Definition 2. A function $f\in\Sigma_p$ is said to be in the class $\mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)g_p^{q,s}(\alpha_1;z)+\alpha g_p^{q,s}(\alpha_1+1;z)\right]}\prec\phi(z)\quad(z\in\mathbb{U};\alpha\geqq0),$$

where $\phi\in\mathcal{P}$, $g_p^{q,s}(\alpha_1;z)$ is defined by (1.5), and

$$g_p^{q,s}(\alpha_1+1;z)\neq0\quad(z\in\mathbb{U}^*).$$

For simplicity, we write

$$\mathcal{G}_p^{q,s}(0;\alpha_1;\phi)=:\mathcal{G}_p^{q,s}(\alpha_1;\phi).$$

Definition 3. A function $f\in\Sigma_p$ is said to be in the class $\mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)h_p^{q,s}(\alpha_1;z)+\alpha h_p^{q,s}(\alpha_1+1;z)\right]}\prec\phi(z)\quad(z\in\mathbb{U};\alpha\geqq0),$$

where $\phi\in\mathcal{P}$, $h_p^{q,s}(\alpha_1;z)$ is defined by (1.6), and

$$h_p^{q,s}(\alpha_1+1;z)\neq0\quad(z\in\mathbb{U}^*).$$

For simplicity, we write

$$\mathcal{H}_p^{q,s}(0;\alpha_1;\phi)=:\mathcal{H}_p^{q,s}(\alpha_1;\phi).$$

Remark 2. In a recent paper, Zou and Wu [18] introduced and investigated a subclass $MS_{sc}^*(\alpha)$ of Σ consisting of functions which are meromorphically α-starlike with respect to symmetric conjugate points and satisfy the following inequality:

$$\Re\left(-\frac{z[(1+\alpha)f'(z)+\alpha\left(zf'(z)\right)']}{(1+\alpha)T_{sc}f(z)+\alpha z\left(T_{sc}f(z)\right)'}\right)>0\quad(z\in\mathbb{U}),$$

where

$$T_{sc}f(z)=\frac{1}{2}\left(f(z)-\overline{f(-\bar{z})}\right).$$

It is easy to see by setting

$$p=1,\qquad q=2,\qquad s=1,\qquad \alpha_1=\alpha_2=\beta_1=1,\quad\text{and}\quad \phi(z)=\frac{1+z}{1-z}$$

that the class $\mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$ reduces to the above-mentioned function class $MS_{sc}^*(\alpha)$.

Definition 4. A function $f \in \Sigma_p$ is said to be in the class $\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)+\alpha\mathfrak{f}_{p,k}^{q,s}(\alpha_1+1;z)\right]}\prec\phi(z)\quad(z\in\mathbb{U})$$
$$\left(\alpha\geqq 0;\ \mathfrak{f}\in\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)\right),$$

where $\phi\in\mathcal{P}$, $\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)$ is defined as in (1.4), and

$$\mathfrak{f}_{p,k}^{q,s}(\alpha_1+1;z)\neq 0\quad(z\in\mathbb{U}^*).$$

For simplicity, we write

$$\mathfrak{F}_{p,k}^{q,s}(0;\alpha_1;\phi)=:\mathfrak{F}_{p,k}^{q,s}(\alpha_1;\phi).$$

Remark 3. If we set

$$q=2,\qquad s=1,\qquad \alpha_1=a,\qquad \beta_1=c,\qquad \alpha_2=1,\quad\text{and}\quad\alpha=0$$

in the class $\mathfrak{F}_{p,k}^{l,m}(\lambda;\alpha_1;\phi)$, then it reduces to the class $\mathcal{K}_{p,k}(a,c;\phi)$, which was also introduced and studied recently by Srivastava et al. [7].

Definition 5. A function $f \in \Sigma_p$ is said to be in the class $\mathfrak{G}_p^{q,s}(\alpha;\alpha_1;\phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)\mathfrak{g}_p^{q,s}(\alpha_1;z)+\alpha\mathfrak{g}_p^{q,s}(\alpha_1+1;z)\right]}\prec\phi(z)\quad(z\in\mathbb{U})$$
$$\left(\alpha\geqq 0;\ \mathfrak{g}\in\mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi)\right),$$

where $\phi\in\mathcal{P}$, $\mathfrak{g}_p^{q,s}(\alpha_1;z)$ is defined as in (1.5), and

$$\mathfrak{g}_p^{q,s}(\alpha_1+1;z)\neq 0\quad(z\in\mathbb{U}^*).$$

For simplicity, we write

$$\mathfrak{G}_p^{q,s}(0;\alpha_1;\phi)=:\mathfrak{G}_p^{q,s}(\alpha_1;\phi).$$

Definition 6. A function $f \in \Sigma_p$ is said to be in the class $\mathfrak{H}_p^{q,s}(\alpha;\alpha_1;\phi)$ if it satisfies the following subordination condition:

$$-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)\mathfrak{h}_p^{q,s}(\alpha_1;z)+\alpha\mathfrak{h}_p^{q,s}(\alpha_1+1;z)\right]}\prec\phi(z)\quad(z\in\mathbb{U})$$
$$\left(\alpha\geqq 0;\ \mathfrak{h}\in\mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)\right),$$

where $\phi\in\mathcal{P}$, $\mathfrak{h}_p^{q,s}(\alpha_1;z)$ is defined as in (1.6), and

$$\mathfrak{h}_p^{q,s}(\alpha_1+1;z)\neq 0\quad(z\in\mathbb{U}^*).$$

For simplicity, we write

$$\mathfrak{H}_p^{q,s}(0;\alpha_1;\phi)=:\mathfrak{H}_p^{q,s}(\alpha_1;\phi).$$

In order to establish our main results, we shall also make use of each of the following lemmas.

Lemma 1 *(See [19,20]). Let $\beta,\gamma\in\mathbb{C}$. Suppose also that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0)=1\quad\text{and}\quad\Re\left(\beta\phi(z)+\gamma\right)>0\quad(z\in\mathbb{U}).$$

If $\mathfrak{p}(z)$ is analytic in $\mathbb{U}$ with $\mathfrak{p}(0)=1$, then the following subordination:

$$\mathfrak{p}(z)+\frac{z\mathfrak{p}'(z)}{\beta\mathfrak{p}(z)+\gamma}\prec\phi(z)\quad(z\in\mathbb{U})$$

implies that

$$\mathfrak{p}(z)\prec\phi(z)\quad(z\in\mathbb{U}).$$

Lemma 2 (*See [21]*). *Let $\beta, \gamma \in \mathbb{C}$. Suppose that $\phi(z)$ is convex and univalent in $\mathbb{U}$ with*

$$\phi(0) = 1 \quad \text{and} \quad \Re(\beta\phi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$

Also let

$$\mathfrak{q}(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

If $\mathfrak{p}(z) \in \mathcal{P}$ and satisfies the following subordination:

$$\mathfrak{p}(z) + \frac{z\mathfrak{p}'(z)}{\beta\mathfrak{q}(z) + \gamma} \prec \phi(z) \quad (z \in \mathbb{U}),$$

then

$$\mathfrak{p}(z) \prec \phi(z) \quad (z \in \mathbb{U}).$$

Lemma 3. *Let $f \in \mathcal{F}_{p,k}^{q,s}(\alpha; \alpha_1; \phi)$. Then*

$$-\frac{z\left[(1+\alpha)\left(f_{p,k}^{q,s}(\alpha_1)f\right)'(z) + \alpha\left(f_{p,k}^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)\right]} \prec \phi(z) \quad (z \in \mathbb{U}). \tag{1.8}$$

Furthermore, if $\phi \in \mathcal{P}$ with

$$\Re\left(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}),$$

then

$$-\frac{z\left(f_{p,k}^{q,s}(\alpha_1; z)\right)'}{pf_{p,k}^{q,s}(\alpha_1; z)} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Proof. Making use of (1.4), we have

$$\begin{aligned} f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) &= \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{np}\left(H_p^{q,s}(\alpha_1)f\right)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp} \cdot \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{(n+j)p}\left(H_p^{q,s}(\alpha_1)f\right)(\varepsilon_k^{n+j}z) \\ &= \varepsilon_k^{-jp} f_{p,k}^{q,s}(\alpha_1; z) \quad (j \in \{0, 1, \ldots, k-1\}) \end{aligned} \tag{1.9}$$

and

$$\left(f_{p,k}^{q,s}(\alpha_1; z)\right)' = \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{j(p+1)}\left(H_p^{q,s}(\alpha_1)f\right)(\varepsilon_k^j z). \tag{1.10}$$

Replacing α_1 by $\alpha_1 + 1$ in (1.9) and (1.10), respectively, we can get

$$f_{p,k}^{q,s}(\alpha_1+1; \varepsilon_k^j z) = \varepsilon_k^{-jp} f_{p,k}^{q,s}(\alpha_1+1; z) \quad (j \in \{0, 1, \ldots, k-1\}) \tag{1.11}$$

and

$$\left(f_{p,k}^{q,s}(\alpha_1+1; z)\right)' = \frac{1}{k}\sum_{n=0}^{k-1} \varepsilon_k^{j(p+1)}\left(H_p^{q,s}(\alpha_1+1)f\right)(\varepsilon_k^j z). \tag{1.12}$$

From (1.9) to (1.12), we can get

$$\begin{aligned} &-\frac{z\left[(1+\alpha)\left(f_{p,k}^{q,s}(\alpha_1)f\right)'(z) + \alpha\left(f_{p,k}^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)\right]} \\ &\quad = -\frac{1}{k}\sum_{j=0}^{k-1} \frac{\varepsilon_k^{j(p+1)} z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(\varepsilon_k^j z) + \alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1; z) + \alpha f_{p,k}^{q,s}(\alpha_1+1; z)\right]} \end{aligned}$$

$$= -\frac{1}{k}\sum_{j=0}^{k-1}\frac{\varepsilon_k^j z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(\varepsilon_k^j z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;\varepsilon_k^j z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;\varepsilon_k^j z)\right]}\quad (z\in\mathbb{U}). \tag{1.13}$$

Moreover, since $f\in\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$, it follows that

$$-\frac{\varepsilon_k^j z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(\varepsilon_k^j z)+\alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(\varepsilon_k^j z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;\varepsilon_k^j z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;\varepsilon_k^j z)\right]}\prec\phi(z) \tag{1.14}$$

$(z\in\mathbb{U}; j\in\{0,1,\ldots,k-1\})$.

By noting that $\phi(z)$ is convex and univalent in $\mathbb{U}$, we conclude from (1.13) and (1.14) that the assertion (1.8) of Lemma 3 holds true.

Next, making use of the relationships (1.3) and (1.4), we have

$$\begin{aligned} z\left(f_{p,k}^{q,s}(\alpha_1;z)\right)'+(\alpha_1+p)f_{p,k}^{q,s}(\alpha_1;z) &= \frac{\alpha_1}{k}\sum_{j=0}^{k-1}\varepsilon_k^{jp}\left(H_p^{q,s}(\alpha_1+1)f\right)(\varepsilon_k^j z)\\ &= \alpha_1 f_{p,k}^{q,s}(\alpha_1+1;z)\quad (f\in\Sigma_p). \end{aligned} \tag{1.15}$$

Let $f\in\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ and suppose that

$$\psi(z) = -\frac{z\left(f_{p,k}^{q,s}(\alpha_1;z)\right)'}{pf_{p,k}^{q,s}(\alpha_1;z)}\quad (z\in\mathbb{U}). \tag{1.16}$$

Then $\psi(z)$ is analytic in $\mathbb{U}$ and $\psi(0)=1$. It follows from (1.15) and (1.16) that

$$\alpha_1+p-p\psi(z) = \alpha_1\frac{f_{p,k}^{q,s}(\alpha_1+1;z)}{f_{p,k}^{q,s}(\alpha_1;z)}. \tag{1.17}$$

From (1.16) and (1.17), we can get

$$z\left(f_{p,k}^{q,s}(\alpha_1+1;z)\right)' = -\frac{p}{\alpha_1}\left(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z)\right)f_{p,k}^{q,s}(\alpha_1;z)\quad (z\in\mathbb{U}^*). \tag{1.18}$$

It now follows from (1.8) and (1.16)–(1.18) that

$$\begin{aligned} &-\frac{z\left[(1+\alpha)\left(f_{p,k}^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(f_{p,k}^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\alpha f_{p,k}^{q,s}(\alpha_1+1;z)\right]}\\ &= \frac{p(1+\alpha)\psi(z)f_{p,k}^{q,s}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p\left(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z)\right)f_{p,k}^{q,s}(\alpha_1;z)}{p(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z)+\frac{\alpha}{\alpha_1}p[\alpha_1+p-p\psi(z)]f_{p,k}^{q,s}(\alpha_1;z)}\\ &= \frac{(1+\alpha)\psi(z)+\frac{\alpha}{\alpha_1}\left(z\psi'(z)+[\alpha_1+p-p\psi(z)]\psi(z)\right)}{(1+\alpha)+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}\\ &= \frac{\frac{\alpha}{\alpha_1}z\psi'(z)+\left((1+\alpha)+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]\right)\psi(z)}{(1+\alpha)+\frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}\\ &= \psi(z)+\frac{z\psi'(z)}{\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\psi(z)}\prec\phi(z)\quad (z\in\mathbb{U}). \end{aligned} \tag{1.19}$$

Thus, since

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right)>0\quad (\alpha>0; z\in\mathbb{U}),$$

by means of (1.19) and Lemma 1, we find that

$$\psi(z) = -\frac{z\left(f_{p,k}^{q,s}(\alpha_1;z)\right)'}{pf_{p,k}^{q,s}(\alpha_1;z)}\prec\phi(z)\quad (z\in\mathbb{U}).$$

This completes the proof of Lemma 3. □

By similarly applying the method of proof of Lemma 3, we can easily get the following results for the classes $\mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi)$ and $\mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$.

Lemma 4. *Let $f \in \mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi)$. Then*

$$-\frac{z\left[(1+\alpha)\left(g_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(g_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)g_p^{q,s}(\alpha_1;z)+\alpha g_p^{q,s}(\alpha_1+1;z)\right]} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Furthermore, if $\phi \in \mathcal{P}$ with

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}),$$

then

$$-\frac{z\left(g_p^{q,s}(\alpha_1;z)\right)'}{p g_p^{q,s}(\alpha_1;z)} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Lemma 5. *Let $f \in \mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$. Then*

$$-\frac{z\left[(1+\alpha)\left(h_p^{q,s}(\alpha_1)f\right)'(z)+\alpha\left(h_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)h_p^{q,s}(\alpha_1;z)+\alpha h_p^{q,s}(\alpha_1+1;z)\right]} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Furthermore, if $\phi \in \mathcal{P}$ with

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}),$$

then

$$-\frac{z\left(h_p^{q,s}(\alpha_1;z)\right)'}{p h_p^{q,s}(\alpha_1;z)} \prec \phi(z) \quad (z \in \mathbb{U}).$$

In the present paper, we aim at proving such results as inclusion relationships, integral representations and convolution properties for each of the following function classes which we have introduced here:

$$\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi), \qquad \mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi) \quad \text{and} \quad \mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$$

as well as

$$\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi), \qquad \mathfrak{G}_p^{q,s}(\alpha;\alpha_1;\phi) \quad \text{and} \quad \mathfrak{H}_p^{q,s}(\alpha;\alpha_1;\phi).$$

The results presented in this paper would provide extensions of those given in a number of earlier works. Several other new results are also obtained as corollaries and consequences of our main results.

2. A set of inclusion relationships

We first provide some inclusion relationships for the following function classes:

$$\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi), \qquad \mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi) \quad \text{and} \quad \mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)$$

as well as

$$\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi), \qquad \mathfrak{G}_p^{q,s}(\alpha;\alpha_1;\phi) \quad \text{and} \quad \mathfrak{H}_p^{q,s}(\alpha;\alpha_1;\phi),$$

each of which was defined in the preceding section.

Theorem 1. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}).$$

Then

$$\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi) \subset \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi).$$

Proof. Let $f \in \mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ and suppose that

$$q(z) = -\frac{z\left(H_p^{q,s}(\alpha_1;z)f\right)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \quad (z \in \mathbb{U}). \tag{2.1}$$

Then $q(z)$ is analytic in $\mathbb{U}$ and $q(0) = 1$. It follows from (1.3) and (2.1) that

$$q(z)f_{p,k}^{q,s}(\alpha_1;z) = -\frac{\alpha_1}{p}H_p^{q,s}(\alpha_1+1)f(z) + \frac{\alpha_1+p}{p}H_p^{q,s}(\alpha_1)f(z). \tag{2.2}$$

Differentiating both sides of (2.2) with respect to z and using (2.1), we have

$$zq'(z) + \left(\alpha_1 + p + \frac{z\left(f_{p,k}^{q,s}(\alpha_1;z)\right)'}{f_{p,k}^{q,s}(\alpha_1;z)}\right)q(z) = -\frac{\alpha_1}{p}\;\frac{z\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)}{f_{p,k}^{q,s}(\alpha_1;z)}. \tag{2.3}$$

It now follows from (1.7), (1.16), (1.17), (2.1) and (2.3) that

$$\begin{aligned}
&-\frac{z\left[(1+\alpha)\left(H_p^{q,s}(\alpha_1)f\right)'(z) + \alpha\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)\right]}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \alpha f_{p,k}^{q,s}(\alpha_1+1;z)\right]}\\
&= \frac{p(1+\alpha)q(z)f_{p,k}^{q,s}(\alpha_1;z) + \frac{\alpha}{\alpha_1}p\left[zq'(z) + [\alpha_1+p-p\psi(z)]q(z)\right]f_{p,k}^{q,s}(\alpha_1;z)}{p\left[(1+\alpha)f_{p,k}^{q,s}(\alpha_1;z) + \frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]f_{p,k}^{q,s}(\alpha_1;z)\right]}\\
&= \frac{(1+\alpha)q(z) + \frac{\alpha}{\alpha_1}\left[zq'(z) + [\alpha_1+p-p\psi(z)]q(z)\right]}{(1+\alpha) + \frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}\\
&= \frac{\frac{\alpha}{\alpha_1}zq'(z) + \left[(1+\alpha) + \frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]\right]q(z)}{(1+\alpha) + \frac{\alpha}{\alpha_1}[\alpha_1+p-p\psi(z)]}\\
&= q(z) + \frac{zq'(z)}{\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\psi(z)} \prec \phi(z) \quad (z \in \mathbb{U}).
\end{aligned} \tag{2.4}$$

Moreover, since

$$\Re\left(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}),$$

by Lemma 3, we have

$$\psi(z) = -\frac{z\left(f_{p,k}^{q,s}(\alpha_1;z)\right)'}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z) \quad (z \in \mathbb{U}).$$

Thus, by (2.4) and Lemma 2, we find that

$$q(z) \prec \phi(z) \quad (z \in \mathbb{U}),$$

that is, that $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi)$. This implies that

$$\mathcal{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi) \subset \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi).$$

The proof of Theorem 1 is evidently completed. □

In view of Lemmas 4 and 5, and by similarly applying the method of proof of Theorem 1, we can easily get the following inclusion relationships.

Corollary 1. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha} + 2\alpha_1 + p - p\phi(z)\right) > 0 \quad (\alpha > 0; z \in \mathbb{U}).$$

Then

$$\mathcal{G}_p^{q,s}(\alpha;\alpha_1;\phi) \subset \mathcal{G}_p^{q,s}(\alpha_1;\phi).$$

Corollary 2. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right)>0 \quad (\alpha>0; z\in\mathbb{U}).$$

Then

$$\mathcal{H}_p^{q,s}(\alpha;\alpha_1;\phi)\subset\mathcal{H}_p^{q,s}(\alpha_1;\phi).$$

Theorem 2. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right)>0 \quad (\alpha>0; z\in\mathbb{U}).$$

Then

$$\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)\subset\mathfrak{F}_{p,k}^{q,s}(\alpha_1;\phi).$$

Proof. Let $f\in\mathfrak{F}_{p,k}^{q,s}(\alpha;\alpha_1;\phi)$ and suppose that

$$p(z)=-\frac{z\left(H_p^{q,s}(\alpha_1;z)f\right)'(z)}{p\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)} \quad (z\in\mathbb{U}). \tag{2.5}$$

Then $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. It follows from (1.3) and (2.5) that

$$p(z)\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)=-\frac{\alpha_1}{p}H_p^{q,s}(\alpha_1+1)f(z)+\frac{\alpha_1+p}{p}H_p^{q,s}(\alpha_1)f(z). \tag{2.6}$$

Differentiating both sides of (2.6) with respect to z and using (2.5), we have

$$zp'(z)+\left(\alpha_1+p+\frac{z\left(\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)\right)'}{\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)}\right)p(z)=-\frac{\alpha_1}{p}\frac{z\left(H_p^{q,s}(\alpha_1+1)f\right)'(z)}{\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)}.$$

Furthermore, we suppose that

$$\varphi(z)=-\frac{z\left(\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)\right)'}{p\mathfrak{f}_{p,k}^{q,s}(\alpha_1;z)} \quad (z\in\mathbb{U}).$$

The remainder of the proof of Theorem 2 is similar to that of Theorem 1. We, therefore, choose to omit the analogous details involved. We thus find that

$$p(z)\prec\phi(z) \quad (z\in\mathbb{U}),$$

which implies that $f\in\mathfrak{F}_{p,k}^{q,s}(\alpha_1;\phi)$. The proof of Theorem 2 is thus completed. □

In view of Lemmas 4 and 5, and by similarly applying the method of proof of Theorem 2, we can easily get the following inclusion relationships.

Corollary 3. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right)>0 \quad (\alpha>0; z\in\mathbb{U}).$$

Then

$$\mathfrak{G}_p^{q,s}(\alpha;\alpha_1;\phi)\subset\mathfrak{G}_p^{q,s}(\alpha_1;\phi).$$

Corollary 4. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\frac{\alpha_1}{\alpha}+2\alpha_1+p-p\phi(z)\right)>0 \quad (\alpha>0; z\in\mathbb{U}).$$

Then

$$\mathfrak{H}_p^{q,s}(\alpha;\alpha_1;\phi)\subset\mathfrak{H}_p^{q,s}(\alpha_1;\phi).$$

In view of Lemmas 3 to 5, and by similarly applying the methods of proofs of Theorems 1 and 2 obtained by Srivastava et al. [7], we can also easily get the following inclusion relationships.

Corollary 5. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathcal{F}_{p,k}^{q,s}(\alpha_1 + 1; \phi) \subset \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi).$$

Corollary 6. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathcal{G}_{p}^{q,s}(\alpha_1 + 1; \phi) \subset \mathcal{G}_{p}^{q,s}(\alpha_1; \phi).$$

Corollary 7. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathcal{H}_{p}^{q,s}(\alpha_1 + 1; \phi) \subset \mathcal{H}_{p}^{q,s}(\alpha_1; \phi).$$

Corollary 8. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathfrak{F}_{p,k}^{q,s}(\alpha_1 + 1; \phi) \subset \mathfrak{F}_{p,k}^{q,s}(\alpha_1; \phi).$$

Corollary 9. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathfrak{G}_{p}^{q,s}(\alpha_1 + 1; \phi) \subset \mathfrak{G}_{p}^{q,s}(\alpha_1; \phi).$$

Corollary 10. *Let $\phi \in \mathcal{P}$ with*

$$\Re\left(\alpha_1 + p - p\phi(z)\right) > 0 \quad (z \in \mathbb{U}).$$

Then

$$\mathfrak{H}_{p}^{q,s}(\alpha_1 + 1; \phi) \subset \mathfrak{H}_{p}^{q,s}(\alpha_1; \phi).$$

3. Integral representations

In this section, we prove a number of integral representations associated with the function classes

$$\mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi), \quad \mathcal{G}_{p}^{q,s}(\alpha_1; \phi) \quad \text{and} \quad \mathcal{H}_{p}^{q,s}(\alpha_1; \phi).$$

Theorem 3. *Let $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. Then*

$$f_{p,k}^{q,s}(\alpha_1; z) = z^{-p} \cdot \exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^j \xi)\right) - 1}{\xi}\mathrm{d}\xi\right), \tag{3.1}$$

where $f_{p,k}^{q,s}(\alpha_1; z)$ is defined by (1.4) and $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. We observe that the condition (1.7) (with $\alpha = 0$) can be written as follows:

$$-\frac{z\left(H_p^{q,s}(\alpha_1)f\right)'(z)}{pf_{p,k}^{q,s}(\alpha_1; z)} = \phi\left(\omega(z)\right) \quad (z \in \mathbb{U}), \tag{3.2}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Replacing z by $\varepsilon_k^j z$ ($j = 0, 1, \ldots, k-1$) in the Eq. (3.2), we find that (3.2) also holds true, that is, that

$$-\frac{\varepsilon_k^j z\left(H_p^{q,s}(\alpha_1)f\right)'(\varepsilon_k^j z)}{pf_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z)} = \phi\left(\omega(\varepsilon_k^j z)\right) \quad (z \in \mathbb{U}). \tag{3.3}$$

We note that

$$f_{p,k}^{q,s}(\alpha_1; \varepsilon_k^j z) = \varepsilon_k^{-jp} f_{p,k}^{q,s}(\alpha_1; z) \quad (z \in \mathbb{U}).$$

Thus, by letting $j = 0, 1, \ldots, k-1$ in (3.3), successively, and summing the resulting equations, we get

$$-\frac{z\left(f_{p,k}^{q,s}(\alpha_1; z)\right)'}{pf_{p,k}^{q,s}(\alpha_1; z)} = \frac{1}{k}\sum_{j=0}^{k-1} \phi\left(\omega(\varepsilon_k^j z)\right) \quad (z \in \mathbb{U}). \tag{3.4}$$

We next find from (3.4) that

$$\frac{\left(f_{p,k}^{q,s}(\alpha_1; z)\right)'}{f_{p,k}^{q,s}(\alpha_1; z)} + \frac{p}{z} = -\frac{p}{k}\sum_{j=0}^{k-1} \frac{\phi\left(\omega(\varepsilon_k^j z)\right) - 1}{z} \quad (z \in \mathbb{U}^*), \tag{3.5}$$

which, upon integration, yields

$$\log\left(z^p f_{p,k}^{q,s}(\alpha_1; z)\right) = -\frac{p}{k}\sum_{j=0}^{k-1} \int_0^z \frac{\phi\left(\omega(\varepsilon_k^j \xi)\right) - 1}{\xi} \mathrm{d}\xi. \tag{3.6}$$

The assertion (3.1) of Theorem 3 can now easily be derived from (3.6). □

Theorem 4. *Let $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. Then*

$$H_p^{q,s}(\alpha_1)f(z) = -p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right) \cdot \exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^\zeta \frac{\phi\left(\omega(\varepsilon_k^j \xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta, \tag{3.7}$$

where $\omega(z)$ is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. Suppose that $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. Then, in light of (3.1) and (3.2), we have

$$\begin{aligned}\left(H_p^{q,s}(\alpha_1)f\right)'(z) &= -\frac{pf_{p,k}^{q,s}(\alpha_1; z)}{z} \cdot \phi\left(\omega(z)\right) \\ &= -pz^{-p-1}\phi(\omega(z)) \cdot \exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^z \frac{\phi\left(\omega(\varepsilon_k^j \xi)\right) - 1}{\xi}\mathrm{d}\xi\right),\end{aligned} \tag{3.8}$$

which, upon integration, leads us easily to the assertion (3.7) of Theorem 4. □

In view of Lemma 3, we can get another integral representation for the function class $\mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$.

Theorem 5. *Let $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. Then*

$$H_p^{q,s}(\alpha_1)f(z) = -p\int_0^z \zeta^{-p-1}\phi\left(\omega_2(\zeta)\right) \cdot \exp\left(-p\int_0^\zeta \frac{\phi\left(\omega_1(\xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta, \tag{3.9}$$

where the functions $\omega_j(z)$ $(j = 1, 2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_j(0) = 0 \quad \textit{and} \quad \left|\omega_j(z)\right| < 1 \quad (z \in \mathbb{U}; j = 1, 2).$$

Proof. Suppose that $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1; \phi)$. We then find from (1.8) (with $\alpha = 0$) that

$$-\frac{z\left(f_{p,k}^{q,s}(\alpha_1; z)\right)'}{pf_{p,k}^{q,s}(\alpha_1; z)} = \phi\left(\omega_1(z)\right) \quad (z \in \mathbb{U}), \tag{3.10}$$

where $\omega_1(z)$ is analytic in $\mathbb{U}$ and $\omega_1(0) = 0$. Thus, by similarly applying the method of proof of Theorem 3, we find that

$$f_{p,k}^{q,s}(\alpha_1; z) = z^{-p} \cdot \exp\left(-p\int_0^z \frac{\phi\left(\omega_1(\xi)\right) - 1}{\xi}\mathrm{d}\xi\right). \tag{3.11}$$

It now follows from (3.2) and (3.11) that

$$\begin{aligned}\left(H_p^{q,s}(\alpha_1)f\right)'(z) &= -\frac{pf_{p,k}^{q,s}(\alpha_1; z)}{z} \cdot \phi\left(\omega_2(z)\right) \\ &= -pz^{-p-1}\phi\left(\omega_2(z)\right) \cdot \exp\left(-p\int_0^z \frac{\phi\left(\omega_1(\xi)\right) - 1}{\xi}\mathrm{d}\xi\right),\end{aligned} \tag{3.12}$$

where the functions $\omega_j(z)$ $(j = 1, 2)$ are analytic in $\mathbb{U}$ with

$$\omega_j(0) = 0 \quad \text{and} \quad \left|\omega_j(z)\right| < 1 \quad (z \in \mathbb{U}; j = 1, 2).$$

Upon integrating both sides of (3.12), we readily arrive at the assertion (3.9) of Theorem 5. □

Remark 4. The result of Theorem 5 also holds true for the classes $\mathcal{G}_p^{q,s}(\alpha_1; \phi)$ and $\mathcal{H}_p^{q,s}(\alpha_1; \phi)$. Here we choose to omit the details involved.

In view of Lemmas 4 and 5, and by similarly applying the methods of proofs of Theorems 3 and 4, we can easily get the following results for the function classes $\mathcal{G}_p^{q,s}(\alpha_1; \phi)$ and $\mathcal{H}_p^{q,s}(\alpha_1; \phi)$.

Corollary 11. *Let* $f \in \mathcal{G}_p^{q,s}(\alpha_1; \phi)$. *Then*

$$g_p^{q,s}(\alpha_1; z) = z^{-p} \cdot \exp\left(-\frac{p}{2}\int_0^z \frac{\phi\left(\omega(\xi)\right) + \overline{\phi\left(\omega(\overline{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right),$$

where $g_p^{q,s}(\alpha_1; z)$ *is defined by* (1.5) *and* $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Corollary 12. *Let* $f \in \mathcal{G}_p^{q,s}(\alpha_1; \phi)$. *Then*

$$H_p^{q,s}(\alpha_1)f(z) = -p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right) \cdot \exp\left(-\frac{p}{2}\int_0^\zeta \frac{\phi\left(\omega(\xi)\right) + \overline{\phi\left(\omega(\overline{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta,$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Corollary 13. *Let* $f \in \mathcal{H}_p^{q,s}(\alpha_1; \phi)$. *Then*

$$h_p^{q,s}(\alpha_1; z) = z^{-p} \cdot \exp\left(-\frac{p}{2}\int_0^z \frac{\phi\left(\omega(\xi)\right) - \overline{\phi\left(\omega(-\overline{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right),$$

where $h_p^{q,s}(\alpha_1; z)$ *is defined by* (1.6) *and* $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Corollary 14. *Let* $f \in \mathcal{H}_p^{q,s}(\alpha_1;\phi)$. *Then*

$$H_p^{q,s}(\alpha_1)f(z) = -p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right)\cdot \exp\left(-\frac{p}{2}\int_0^\zeta \frac{\phi\left(\omega(\xi)\right) - \overline{\phi\left(\omega(-\overline{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta,$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

4. Convolution properties

In this section, we derive several convolution properties for the function classes

$$\mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi), \qquad \mathcal{G}_p^{q,s}(\alpha_1;\phi) \quad \text{and} \quad \mathcal{H}_p^{q,s}(\alpha_1;\phi).$$

Theorem 6. *Let* $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi)$. *Then*

$$f(z) = \left[-p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right)\cdot \exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^\zeta \frac{\phi\left(\omega(\varepsilon_k^j\xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta\right] * \left(\sum_{n=0}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_s)_n}{(\alpha_1)_n\cdots(\alpha_q)_n}z^{n-p}\right), \tag{4.1}$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Proof. From (1.2) and (3.7), we know that

$$\begin{aligned} &-p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right)\cdot \exp\left(-\frac{p}{k}\sum_{j=0}^{k-1}\int_0^\zeta \frac{\phi\left(\omega(\varepsilon_k^j\xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta \\ &\quad = \left[z^{-p}\cdot {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)\right] * f(z). \end{aligned} \tag{4.2}$$

Thus, from (4.2), we can easily get the assertion (4.1) of Theorem 6. □

Theorem 7. *Let* $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi)$. *Then*

$$f(z) = \left[-p\int_0^z \zeta^{-p-1}\phi\left(\omega_2(\zeta)\right)\cdot \exp\left(-p\int_0^\zeta \frac{\phi\left(\omega_1(\xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta\right] * \left(\sum_{n=0}^{\infty}\frac{n!(\beta_1)_n\cdots(\beta_s)_n}{(\alpha_1)_n\cdots(\alpha_q)_n}z^{n-p}\right), \tag{4.3}$$

where the functions $\omega_j(z)$ $(j = 1, 2)$ *are analytic in* $\mathbb{U}$ *with*

$$\omega_j(0) = 0 \quad \textit{and} \quad \left|\omega_j(z)\right| < 1 \quad (z \in \mathbb{U}; j = 1, 2).$$

Proof. From (1.2) and (3.9), we know that

$$-p\int_0^z \zeta^{-p-1}\phi\left(\omega_2(\zeta)\right)\cdot \exp\left(-p\int_0^\zeta \frac{\phi\left(\omega_1(\xi)\right) - 1}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta = \left[z^{-p}\cdot {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)\right] * f(z). \tag{4.4}$$

Thus, from (4.4), we easily arrive at (4.3). □

Remark 5. In view of Remark 4, we know that the convolution property of Theorem 7 also holds true for the function classes $\mathcal{G}_p^{q,s}(\alpha_1;\phi)$ and $\mathcal{H}_p^{q,s}(\alpha_1;\phi)$. Here we choose to omit the details involved.

Theorem 8. *Let*

$$f \in \Sigma_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi)$ *if and only if*

$$f * \left[\left(-pz^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}\, \frac{n-p}{n!}\, z^{n-p}\right) \right.$$
$$\left. + \ p\phi(\mathrm{e}^{\mathrm{i}\theta}) \left(z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}\, \frac{1}{n!}\, z^{n-p}\right) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{1}{z^p(1-\varepsilon^{\upsilon} z)}\right)\right] \neq 0 \tag{4.5}$$
$$(z \in \mathbb{U}^*;\ 0 \leqq \theta < 2\pi).$$

Proof. Suppose that $f \in \mathcal{F}_{p,k}^{q,s}(\alpha_1;\phi)$. Since the following subordination condition:

$$-\frac{z\left(H_p^{q,s}(\alpha_1)f\right)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \prec \phi(z) \quad (z \in \mathbb{U})$$

is equivalent to

$$-\frac{z\left(H_p^{q,s}(\alpha_1)f\right)'(z)}{pf_{p,k}^{q,s}(\alpha_1;z)} \neq \phi(\mathrm{e}^{\mathrm{i}\theta}) \quad (z \in \mathbb{U};\ 0 \leqq \theta < 2\pi), \tag{4.6}$$

it is easy to see that the condition (4.6) can be written as follows:

$$z\left(H_p^{q,s}(\alpha_1)f\right)'(z) + pf_{p,k}^{q,s}(\alpha_1;z)\phi(\mathrm{e}^{\mathrm{i}\theta}) \neq 0 \quad (z \in \mathbb{U}^*;\ 0 \leqq \theta < 2\pi). \tag{4.7}$$

On the other hand, we find from (1.2) that

$$z\left(H_p^{q,s}(\alpha_1)f\right)'(z) = \left(-pz^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}\, \frac{n-p}{n!}\, z^{n-p}\right) * f(z). \tag{4.8}$$

Moreover, from the definition of $f_{p,k}^{q,s}(\alpha_1;z)$, we have

$$f_{p,k}^{q,s}(\alpha_1;z) = H_p^{q,s}(\alpha_1)f(z) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{1}{z^p(1-\varepsilon^{\upsilon} z)}\right)$$
$$= \left(z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n}\, \frac{1}{n!}\, z^{n-p}\right) * \left(\frac{1}{k}\sum_{\upsilon=0}^{k-1} \frac{1}{z^p(1-\varepsilon^{\upsilon} z)}\right) * f(z). \tag{4.9}$$

Upon substituting from (4.8) and (4.9) into (4.7), we easily arrive at the convolution property (4.5) asserted by Theorem 8. □

In view of Corollaries 12 and 14, and by similarly applying the method of proof of Theorem 6, we can easily get the following results for the function classes $\mathcal{G}_p^{q,s}(\alpha_1;\phi)$ and $\mathcal{H}_p^{q,s}(\alpha_1;\phi)$.

Corollary 15. *Let* $f \in \mathcal{G}_p^{q,s}(\alpha_1;\phi)$. *Then*

$$f(z) = \left[-p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right) \cdot \exp\left(-\frac{p}{2}\int_0^{\zeta} \frac{\phi\left(\omega(\xi)\right) + \overline{\phi\left(\omega(\bar{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta\right] * \left(\sum_{n=0}^{\infty} \frac{n!(\beta_1)_n \cdots (\beta_s)_n}{(\alpha_1)_n \cdots (\alpha_q)_n}\, z^{n-p}\right),$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

Corollary 16. *Let* $f \in \mathcal{H}_p^{q,s}(\alpha_1;\phi)$. *Then*

$$f(z) = \left[-p\int_0^z \zeta^{-p-1}\phi\left(\omega(\zeta)\right) \cdot \exp\left(-\frac{p}{2}\int_0^{\zeta} \frac{\phi\left(\omega(\xi)\right) - \overline{\phi\left(\omega(-\bar{\xi})\right)} - 2}{\xi}\mathrm{d}\xi\right)\mathrm{d}\zeta\right]$$
$$* \left(\sum_{n=0}^{\infty} \frac{n!(\beta_1)_n \cdots (\beta_s)_n}{(\alpha_1)_n \cdots (\alpha_q)_n}\, z^{n-p}\right),$$

where $\omega(z)$ *is analytic in* $\mathbb{U}$ *with*

$$\omega(0) = 0 \quad \textit{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}).$$

By similarly applying the method of proof of Theorem 8, we can easily get the following convolution properties for the function classes $\mathcal{G}_p^{q,s}(\alpha_1;\phi)$ and $\mathcal{H}_p^{q,s}(\alpha_1;\phi)$.

Corollary 17. *Let*

$$f \in \Sigma_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{G}_p^{q,s}(\alpha_1;\phi)$ *if and only if*

$$f * \left[\left(-pz^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \, \frac{n-p}{n!} \, z^{n-p}\right) + \frac{p\phi(\mathrm{e}^{i\theta})}{2} h\right] + \frac{p\phi(\mathrm{e}^{i\theta})}{2} \overline{(h*f)(\overline{z})} \neq 0 \quad (z \in \mathbb{U}^*;\ 0 \leqq \theta < 2\pi),$$

where $h(z)$ *is given by*

$$h(z) = z^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \, \frac{1}{n!} \, z^{n-p}. \tag{4.10}$$

Corollary 18. *Let*

$$f \in \Sigma_p \quad \textit{and} \quad \phi \in \mathcal{P}.$$

Then $f \in \mathcal{H}_p^{q,s}(\alpha_1;\phi)$ *if and only if*

$$f * \left[\left(-pz^{-p} + \sum_{n=1}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \, \frac{n-p}{n!} \, z^{n-p}\right) + \frac{p\phi(\mathrm{e}^{i\theta})}{2} h\right] - \frac{p\phi(\mathrm{e}^{i\theta})}{2} \overline{(h*f)(-\overline{z})} \neq 0 \quad (z \in \mathbb{U}^*;\ 0 \leqq \theta < 2\pi),$$

where $h(z)$ *is given by* (4.10).

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