Note

Ears of triangulations and Catalan numbers∗

F. Hurtado, M. Noy∗

Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya, Pau Gargallo 5,
08028 Barcelona, Spain

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Abstract

It is known that a convex polygon of n sides admits Cn−2 triangulations, where Cn is
a Catalan number. We classify these triangulations (considered as outerplanar graphs) accord-
ing to their dual trees, and prove the following formula for the number of triangulations of
a convex n-gon whose dual tree has exactly k leaves:

\[ \frac{n^{2n-2k}}{k^2} \frac{C_{n-2}}{2k-4} \cdot \]

The proof is bijective and provides a recursive formula for the Catalan numbers similar to, but
different from, a classical identity of Touchard. An averaging argument allows one to deduce
Touchard’s formula from ours.

Keywords: Catalan numbers; Triangulation; Polygon; Outerplanar graph; Dual graph

1. Introduction

There are many combinatorial objects counted by the Catalan numbers. Some of
them have been widely studied, specially binary trees and lattice paths (see [10,
Chapter 3]), while others have received less attention. One of the latter is the case for
the triangulations of a convex polygon, a problem that goes back to Euler [2, 9]. If we
let \( t_n \) be the number of ways of triangulating a convex polygon of n sides by means of
nonintersecting diagonals, it can be shown that the following recurrence holds:

\[ t_n = t_2 t_{n-2} + \cdots + t_k t_{n-k} + \cdots + t_2 t_{n-2}. \]

This, together with the convention \( t_2 = 1 \), gives

\[ t_n = C_{n-2} = \frac{1}{n-1} \binom{2n-4}{n-2}. \]

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∗ Corresponding author. E-mail: noy@ma2.upc.es.
where \( C_n \) denotes a Catalan number. For example, a quadrangle admits \( C_2 = 2 \) triangulations and a hexagon admits as many as \( C_4 = 14 \).

If the polygon is not convex, an exact formula can be obtained for families of polygons with a high degree of 'regularity' [3]. Also, in [4] tight bounds for the number of triangulations of a general simple polygon were obtained as functions of the number of concave vertices.

A triangulation of a polygon, whether convex or not, is a planar graph (actually, a maximal outerplanar graph) and one can consider the dual graph associated with it. We will always exclude the unbounded face; in this way the dual graph becomes a tree, since without using additional internal points no cycle can be produced (see Fig. 1). The leaves of the tree are called ears of the triangulation, or of the polygon; there are at least two of them, and this fact provides the basis for several inductive proofs in the combinatorial geometry of polygons [6].

We consider the problem of counting how many triangulations has a convex polygon with a prescribed number of ears. The answer is contained in Theorem 1 and is found by considering the homeomorphy classes of the dual trees, whose number turns out to be again a Catalan number. As a consequence we obtain in Theorem 2 a recursive formula for the Catalan numbers, very similar in appearance to a classical identity of Touchard. By computing the mean number of ears of an arbitrary triangulation we deduce Touchard's identity from ours.

### 2. Counting triangulations by the ears

The dual tree \( U \) of a triangulation has the property that the degree of any vertex is at most three, since a triangle is adjacent to at most three other triangles. When rooted, it becomes a binary tree in which every vertex has either 0, 1 or 2 sons and there is a distinction between a right and a left son. Leaves correspond to ears and vertices of degree 3 to triangles neither of whose sides is an edge of the polygon. We will be concerned with the homeomorphy class of \( U \), that is, the tree \( U' \) obtained by removing all the degree 2 vertices. Then \( U' \) is a full binary tree having the same number \( k \) of leaves as \( U \) and \( k - 1 \) internal vertices (excluding the root). We recall for future reference that the number of full binary trees with \( n \) internal vertices is equal to \( C_n \) [10].

**Theorem 1.** The number of triangulations of a convex polygon with \( n \) sides having exactly \( k \) ears is equal to

\[
e(n,k) = \frac{n}{k} 2^{n-2k} \binom{n-4}{2k-4} C_{k-2}, \quad 2 \leq k \leq \lfloor n/2 \rfloor.
\]

**Proof.** Select a vertex \( v \) and consider triangulations having \( v \) as an ear. We now consider the dual tree \( U \) rooted at the ear \( v \) together with its reduction \( U' \) (see Fig. 1). In this way \( U \) and \( U' \) become planted binary trees.
If $k$ is the number of ears then, as explained above, $U'$ has $k - 1$ leaves and $k - 2$ internal vertices, and its order is $2k - 2$, so that there are $C_{k-2}$ possible choices for $U'$. Choose a fixed $U'$ and consider the possible dual trees $U$ giving rise to the selected $U'$. The order of $U$ will be $n - 2$, the number of triangles in any triangulation of a polygon with $n$ sides. Hence we must insert $n - 2 - (2k - 2) = n - 2k$ vertices of degree 2 and they can be inserted anywhere in the edges of $U'$. In order to show how the terms $2^{n-2k}$ and $\binom{n-4}{k-2}$ enter into the formula we make the following two observations.

(a) There are $2^m$ ways in which $m$ vertices of degree 2 can be chosen, any of which can have either a left son or a right son. Geometrically it corresponds to the selection of $m$ diagonals and each of them can be selected in two ways, according to whether they produce a left or right son in the dual tree. (Fig. 2 shows one such selection). Moreover, the selection of chains in different edges of $U'$ are independent and we are left with a total of $2^{n-2k}$ choices.

(b) We can linearly order the tree $U$ (say by preorder) and then select the position of the $2k - 2$ vertices not of degree 2 among a total of $n - 2$ positions. But rooting the tree at the ear $v$ fixes two of these vertices, namely the root and its unique son, hence we must select $2k - 4$ objects among $n - 4$.

Finally we have $n$ choices for $v$ and each triangulation has been counted $k$ times, once for every ear, hence the factor \( \frac{n}{k} \). The theorem follows. \(\square\)

3. Formulas for the Catalan numbers

**Theorem 2.** The Catalan numbers $C_n$ satisfy the following recurrence:

$$C_{n+1} = (n + 3) \sum_{k=0}^{\left\lfloor (n-1)/2 \right\rfloor} \frac{1}{k + 2} 2^{n-2k-1} \binom{n-1}{2k} C_k.$$  \hspace{1cm} (2)
Proof. This is a consequence of Theorem 1. In fact, summing $e(n + 3, s)$ for $s$ one gets

$$C_{n+1} = t_{n+3} = \sum_{s=2}^{[n+3/2]} e(n + 3, s) = \sum_{s=2}^{[n+3/2]} \frac{n + 3}{s} 2^{n+3-2s} \binom{n-1}{2s-4} C_{s-2},$$

and the change of index $k = s - 2$ gives the desired result. \(\square\)

This identity has a quite interesting feature: the Catalan number on the left hand counts triangulations while those in the right hand count binary trees. Also, the identity is very similar to Touchard's formula (see [11]), which is

$$C_{n+1} = \sum_{k=0}^{[n/2]} 2^{n-2k} \binom{n}{2k} C_k. \quad (3)$$

Formulas (2) and (3) provide different decompositions of $C_n$ as shown in the following table.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$C_n$</th>
<th>Ears</th>
<th>Touchard</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>5</td>
<td>5</td>
<td>4 + 1</td>
</tr>
<tr>
<td>4</td>
<td>14</td>
<td>12 + 2</td>
<td>8 + 6</td>
</tr>
<tr>
<td>5</td>
<td>42</td>
<td>28 + 14</td>
<td>16 + 24 + 2</td>
</tr>
<tr>
<td>6</td>
<td>132</td>
<td>64 + 64 + 4</td>
<td>32 + 80 + 20</td>
</tr>
<tr>
<td>7</td>
<td>429</td>
<td>144 + 240 + 45</td>
<td>64 + 240 + 120 + 5</td>
</tr>
</tbody>
</table>

We finally proceed to deduce Touchard’s formula from (2). To this end we prove the following lemma.
**Lemma.** The mean number of ears in a triangulation of a convex n-gon is equal to \( nt_{n-1}/t_n = nC_{n-3}/C_{n-2} \).

**Proof.** Let us count in two ways the pairs \((T, v)\), where \( T \) is a triangulation of the convex \( n \)-gon and \( v \) is an ear of \( T \). If we denote by \( e(T) \) the number of ears of \( T \) then

\[
\sum_T e(T) = nt_{n-1},
\]

since there are \( t_{n-1} \) triangulations having a fixed ear. The result follows dividing by \( t_n \), the size of the population. \( \Box \)

Now the mean number of ears is obviously equal to

\[
\frac{1}{t_n} \sum_k e(n, k)k.
\]

Combining this with the lemma and Theorem 1 we arrive at

\[
t_{n-1} = \sum_k \left( \frac{n - 4}{2k - 4} \right) 2^{n-2k} C_{k-2}
\]

and changing variables as before one gets (3). Different combinatorial proofs of (3) can be found in [7] using noncrossing partitions or in [1] using lattice paths.

**Remark.** A consequence of the previous lemma is that, since the Catalan numbers satisfy the first order recurrence \((n - 1)t_n = 2(2n - 5)t_{n-1} \), the mean number of ears of a triangulation is

\[
\frac{n(n - 1)}{2(2n - 5)}.
\]

In other words, about one fourth of the vertices are ears. A routine calculation shows that the distribution \( e(n, k) \) for fixed \( n \) is unimodal with a peak at \( \lfloor \frac{n^2 - n + 4}{4n - 2} \rfloor \). Thus, asymptotically, the mean and the mode of the distribution are both at \( n/4 \).

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**References**