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Oscillation Theorems for *n*th-Order Functional Differential Equations

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Two oscillation theorems for even-order equations $x^{(n)}(t)+p(t)f(x(t),x(h(t)))=0$ are established. These results are the extensions of those reported by Hamedani for the second-order equation $x''(t)+p(t)f(x(t),x(h(t)))=0$.

INTRODUCTION

The purpose of this paper is to study the oscillatory behavior of the differential equation

$$x^{(n)}(t)+p(t)f(x(t),x(h(t)))=0, \qquad t\geqslant t_0, \tag{1}$$

where n is even, $n\geqslant 2$. We shall restrict our attention to those solutions of (1) which exist on some ray $[T,\infty)$, where $T\geqslant t_0$, and which are nontrivial in any neighborhood of infinity. Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise, the solution is said to be *nonoscillatory*. An equation is said to be *oscillatory* if all of its solutions are oscillatory.

For the sake of completeness, we shall first state a few conditions and recall various oscillation results concerning the equation

$$x''(t)+p(t)f(x(t),x(h(t)))=0. \tag{2}$$

Conditions:

(C_1) $p\in C[t_0,\infty)$, $p(t)\geqslant 0$, $t\geqslant t_0$;

(C_2) $h\in C[t_0,\infty)$, $g(t)\leqslant h(t)$, and $0<k\leqslant g'(t)\leqslant 1$;

(C_3) $f\in C(R\times R)$, $R=(-\infty,\infty)$, and $f(x,y)$ has the sign of x and y when they have the same sign;

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(C_4) there exists $M>0$ such that, uniformly for $x \geqslant M$,

$$\liminf_{|y| \to \infty} \left| \frac{f(x, y)}{y} \right| \geqslant c > 0;$$

(C_5) $\limsup_{t \to \infty} t \int_t^\infty p(s)\, ds = \infty$;

(C_6) $\limsup_{t \to \infty} t^{1-m}(1/m!) \int_{t_0}^t (t-u)^{m-1} p(u)\, du = \infty$, for some $m>2$;

(C_7) $h \in C[t_0, \infty)$, $h(t) \leqslant t$, for $t \geqslant t_0$, $\lim_{t \to \infty} h(t) = \infty$;

(C_8) $\limsup_{t \to \infty} t \int_t^\infty p(s)(h(s)/s)\, ds > c^{-1}$, where c is as in (C_4);

(C_9) $\limsup_{t \to \infty} t \int_{\gamma(t)}^\infty p(s)\, ds > c^{-1}$, where c is as in (C_4) and $\gamma(t) = \sup\{s \geqslant t_0 \mid h(s) \leqslant t\}$ for $t \geqslant t_0$;

(C_{10}) $\int_{t_0}^\infty p(s)(h(s)/s)\, ds < \infty$;

(C_{11}) there exists a positive integer K such that $\alpha_0(t) = \delta c \int_t^\infty p(s) (h(s)/s) ds$ and $\alpha_m(t) = \int_t^\infty \alpha_{m-1}^2(s) ds + \alpha_0(t)$ are defined for $m = 1, 2, ..., K-1$, but $\lim_{t \to \infty} \int_{t_0}^t \alpha_{K-1}^2(s)\, ds = \infty$, where c is as in (C_4) and δ is a constant, $0 < \delta < 1$;

(C_{12}) $\alpha_m(t)$ is defined for $m = 1, 2, ...,$ such that $\lim_{m \to \infty} \alpha_m(t) = \infty$ pointwise for all large t.

Remarks 1. (i) Travis [6] proved that under conditions (C_1)–(C_5) all solutions of (2) are oscillatory.

(ii) In [8], Yeh showed that under conditions (C_1)–(C_4) and (C_6) all solutions of (2) are oscillatory, extending the result in (i).

(iii) Recently, Hamedani [4] reported that under conditions (C_1), (C_3), (C_4), (C_7), and either (C_8) or (C_9) all solutions of (2) are oscillatory.

(iv) Hamedani [4] also showed that under (C_1), (C_3), (C_4), (C_7), (C_{10}), and either (C_{11}) or (C_{12}) all solutions of (2) are oscillatory, extending the results of Yan [7] for the linear case $x''(t) + p(t)\, x(h(t)) = 0$.

In a recent paper, Grace and Lalli [2] extended Yeh's result (ii) to Eq. (1) without further restrictions on the functions involved. In this paper we shall extend the results given in (iii) and (iv) to Eq. (1) without further restrictions on the functions involved.

RESULTS

The following lemmas are basic for our discussions. The first two are taken from [3], the third from [1], and the fourth from [5]. They are given here for the sake of completeness.

LEMMA 1. *Let x be a positive, n times differentiable function on $[t_0, \infty)$. If $x^{(n)}(t)$ is of constant sign and not identically zero in any interval $[t_1, \infty)$, then there exists $t_x \geqslant t_0$ and an integer j, $0 \leqslant j \leqslant n$ with $n+j$ even for $x^{(n)} \geqslant 0$ or $n+j$ odd for $x^{(n)} \leqslant 0$, such that $j > 0$ implies $x^{(k)}(t) > 0$ for $t \geqslant t_x$ $(k = 0, 1, ..., j-1)$ and $j \leqslant n-1$ implies that $(-1)^{j+k} x^{(k)}(t) > 0$ for $t \geqslant t_x$ $(k = j, j+1, ..., n-1)$.*

LEMMA 2. *If the function x is as in Lemma 1 and $x^{(n-1)}(t)\, x^{(n)}(t) \leqslant 0$ for $t \geqslant t_x$, then for every λ, $0 < \lambda < 1$, there exists an $M_1 > 0$ such that*

$$x(\lambda t) \geqslant M_1 t^{n-1} |x^{(n-1)}(t)|$$

for all large t. In addition, if $\lim_{t \to \infty} x(t) \neq 0$, then there exists $M_2 > 0$ such that

$$x(t) \geqslant M_2 t^{n-1} |x^{(n-1)}(t)|$$

for all large t.

LEMMA 3. *Let (C_7) hold and let $x(t) \in C^2[T, \infty)$, $x(t) > 0$, $x'(t) > 0$, and $x''(t) < 0$ for $t \geqslant T$. Then for each $k_1 \in (0, 1)$ there exists a $T_{k_1} \geqslant T$ such that*

$$x(h(t)) \geqslant k_1 \frac{h(t)}{t} x(t), \qquad t \geqslant T_{k_1}.$$

LEMMA 4. *Let $x(t) \in C^2[T, \infty)$ with $x(t) > 0$, $x'(t) > 0$, and $x''(t) \leqslant 0$ for $t \geqslant T$. Then for each $k_2 \in (0, 1)$ there is a $T_{k_2} \geqslant T$ such that*

$$x(t) \geqslant k_2 t x'(t), \qquad \text{for} \quad t \geqslant T_{k_2}.$$

Observe that in Lemma 4, the choice of $k_2 \in (0, 1)$ is arbitrary, while in Lemma 2, for $n = 2$, the M_2 "exists."

THEOREM 1. *Under the conditions stated in* (iii), *Eq.* (1) *is oscillatory.*

Proof. W.l.o.g., in (C_4), we may assume $\liminf_{|y| \to \infty} |f(x, y)/y| > c$ uniformly in $x \geqslant M$. In view of (C_8) or (C_9) it suffices to consider only unbounded solutions.

Let $x(t)$ be a positive unbounded nonoscillatory solution of Eq. (1) on an interval $[t_1, \infty)$, $t_1 \geqslant t_0$. Let $t_2 \geqslant t_1$ be chosen so that

$$h(t) \geqslant t_1 \qquad \text{for all} \quad t \geqslant t_2.$$

Then $x(h(t)) > 0$ for all $t \geqslant t_2$. From (1) and (C_3), it follows that $x^{(n)}(t) < 0$ for $t \geqslant t_2$. By Lemma 1, there exist an odd integer j, $1 \leqslant j \leqslant n-1$, and a $t_3 \geqslant t_2$ such that $x^{(k)}(t) > 0$ for $t \geqslant t_3$ $(k = 0, 1, ..., j-1)$ and $(-1)^{j+k} x^{(k)}(t) > 0$ for $t \geqslant t_3$ $(k = j, j+1, ..., n-1)$.

Integrating Eq. (1) from t to ∞ $(t \geqslant t_3)$, we have

$$x^{(n-1)}(t) \geqslant \int_t^\infty p(s) f(x(s), x(h(s)))\, ds. \tag{3}$$

Now suppose that (C_8) holds. If $j = 1$, then $x''(t) < 0$ for $t \geqslant t_3$. By Lemma 3, for each $k_1 \in (0, 1)$, there exists $t_4 \geqslant t_3$ such that

$$x(h(t)) \geqslant k_1 \frac{h(t)}{t} x(t), \qquad \text{for} \quad t \geqslant t_4. \tag{4}$$

Let $k_2 \in (0, 1)$. If $n = 2$, then by Lemma 4, there exists a $t_5 \geqslant t_4$ such that

$$x(t) \geqslant k_2 t x^{(n-1)}(t), \qquad \text{for} \quad t \geqslant t_5.$$

On the other hand, if $n > 2$, by applying the second half of Lemma 2 and observing that $M_2 t^{n-1} \geqslant k_2 t$ for all t sufficiently large, we again obtain

$$x(t) \geqslant k_2 t x^{(n-1)}(t), \qquad \text{for} \quad t \geqslant t_5,$$

for some $t_5 \geqslant t_4$. Combining this with (3) and (4) yields

$$\begin{aligned} x(t) &\geqslant k_2 t \int_t^\infty p(s) f(x(s), x(h(s)))\, ds \\ &\geqslant k_1 k_2 t \int_t^\infty p(s) \frac{h(s)}{s} \left(\frac{f(x(s), x(h(s)))}{x(h(s))} \right) x(s)\, ds \\ &\geqslant k_1 k_2 x(t) \inf_{s \geqslant t} \left(\frac{f(x(s), x(h(s)))}{x(h(s))} \right) t \int_t^\infty p(s) \frac{h(s)}{s}\, ds, \end{aligned}$$

for $t \geqslant t_5$. From this inequality we see that

$$c^{-1} \geqslant k_1 k_2 \limsup_{t \to \infty} t \int_t^\infty p(s) \frac{h(s)}{s}\, ds. \tag{5}$$

From (5) and (C_8), it follows that

$$c^{-1} < a = \limsup_{t \to \infty} t \int_t^\infty p(s) \frac{h(s)}{s}\, ds < \infty. \tag{6}$$

From (5) and (6), observing that $k_1, k_2 \in (0, 1)$ are arbitrary, we obtain the contradiction $c^{-1} < a \leqslant c^{-1}$.

If $3 \leqslant j < n-1$, then by Lemma 2, there exists t_6 such that

$$x^{(j-1)}(t) \geqslant M_2 t^{n-j} x^{(n-1)}(t) \qquad \text{for} \quad t \geqslant t_6. \tag{7}$$

By Taylor's formula and Lemma 1, we have

$$\begin{aligned} x(t) &= x(T_0) + x'(T_0)(t-T_0) + \cdots + \frac{x^{(j-1)}(T_0)}{(j-1)!}(t-T_0)^{j-1} \\ &\quad + \frac{x^{(j)}(t_1^*)}{j!}(t-T_0)^j \\ &\geqslant \frac{x^{(j-1)}(T_0)}{(j-1)!}(t-T_0)^{j-1}, \end{aligned} \tag{8}$$

and

$$\begin{aligned} x^{(j-1)}(t) &= x^{(j-1)}(T_0) + x^{(j)}(T_0)(t-T_0) + \frac{x^{(j+1)}(t_2^*)}{2!}(t-T_0)^2 \\ &\leqslant x^{(j-1)}(T_0) + x^{(j)}(T_0)(t-T_0), \end{aligned} \tag{9}$$

where $T_0 = \max\{t_6, t_x\} \leqslant t_1^*, t_2^* \leqslant t$.

From (8) and (9), it follows that there exist $M_3 > 0$ and $t_7 \geqslant T_0$ such that

$$x(t) \geqslant M_3 t^{j-2} x^{(j-1)}(t), \qquad \text{for} \quad t \geqslant t_7. \tag{10}$$

Since $x^{(j-1)}(t) > 0$, $x^{(j)}(t) > 0$, and $x^{(j+1)}(t) < 0$, by Lemma 3, for $k_1 \in (0, 1)$, there exists $t_8 \geqslant t_7$ such that

$$x^{(j-1)}(h(t)) \geqslant k_1 \frac{h(t)}{t} x^{(j-1)}(t) \qquad \text{for} \quad t \geqslant t_8. \tag{11}$$

Now from (3), (7), (11), and (C_3), for $t \geqslant t_8$, we have

$$\begin{aligned} x^{(j-1)}(t) &\geqslant M_2 t^{n-j} \int_t^\infty p(s)\, f(x(s), x(h(s)))\, ds \\ &\geqslant k_1 M_2 t^{n-j} \int_t^\infty p(s) \frac{h(s)}{s} \left(\frac{f(x(s), x(h(s)))}{x^{(j-1)}(h(s))} \right) x^{(j-1)}(s)\, ds \\ &\geqslant k_1 M_2 t^{n-j} x^{(j-1)}(t) \int_t^\infty p(s) \frac{h(s)}{s} \left(\frac{f(x(s), x(h(s)))}{x^{(j-1)}(h(s))} \right) ds. \end{aligned} \tag{12}$$

Using (10) in (12) we obtain

$$1 \geqslant k_1 M_2 M_3 t^{n-j} \int_t^\infty p(s) \frac{h^{j-1}(s)}{s} \left(\frac{f(x(s), x(h(s)))}{x(h(s))} \right) ds, \qquad \text{for} \quad t \geqslant t_9, \tag{13}$$

where $t_9 = \max\{t_7, t_8\}$. Let $t_{10} \geqslant t_9$ be such that $h(t) \geqslant 1$ and for a fixed, but arbitrary $k_2 \in (0, 1)$, $M_2 M_3 t^{n-j} \geqslant k_2 t$, for all $t \geqslant t_{10}$. Then from (13) it follows that

$$c^{-1} \geqslant k_1 k_2 \limsup_{t \to \infty} t \int_t^\infty p(s) \frac{h(s)}{s} \, ds, \tag{14}$$

which is identical to (5). The rest now follows as in the case of $j = 1$.

If $j = n - 1 \geqslant 3$, then by Lemma 4, for $k_2 \in (0, 1)$, there exists t_{11} such that

$$x^{(n-2)}(t) \geqslant k_2 t x^{(n-1)}(t), \qquad \text{for} \quad t \geqslant t_{11}. \tag{15}$$

From (3) and (15), we have

$$x^{(n-2)}(t) \geqslant k_2 t \int_t^\infty p(s) \left(\frac{f(x(s), x(h(s)))}{x^{(n-2)}(h(s))} \right) x^{(n-2)}(h(s)) \, ds, \qquad \text{for} \quad t \geqslant t_{11}. \tag{16}$$

By Lemma 3, for $k_1 \in (0, 1)$, there exists $t_{12} \geqslant t_{11}$ such that

$$x^{(n-2)}(h(t)) \geqslant k_1 \frac{h(t)}{t} x^{(n-2)}(t), \qquad \text{for} \quad t \geqslant t_{12}. \tag{17}$$

The argument used in obtaining (10) holds in this setting, hence, for some $M_3 > 0$ and $h(t) \geqslant t_7$

$$x(h(t)) \geqslant M_3 h^{n-3}(t) \, x^{(n-2)}(h(t)).$$

By (C_7), w.l.o.g., we may assume $M_3 h^{n-3}(t) \geqslant 1$ for all $t \geqslant t_{13} = \max\{t_7, t_{12}\}$. Thus,

$$x(h(t)) \geqslant x^{(n-2)}(h(t)), \qquad \text{for} \quad t \geqslant t_{13}. \tag{18}$$

Combining (16), (17), and (18) yields

$$\begin{aligned} x^{(n-2)}(t) &\geqslant k_1 k_2 t \int_t^\infty p(s) \frac{h(s)}{s} x^{(n-2)}(s) \frac{f(x(s), x(h(s)))}{x(h(s))} \, ds \\ &\geqslant x^{(n-2)}(t) \inf_{s \geqslant t} \frac{f(x(s), x(h(s)))}{x(h(s))} k_1 k_2 t \int_t^\infty p(s) \frac{h(s)}{s} \, ds \qquad \text{for} \quad t \geqslant t_{13}. \end{aligned}$$

From this we obtain (5) and the argument proceeds as in the case of $j = 1$.

The case where (C_9) holds instead of (C_8) follows in a similar manner.

To give another set of sufficient conditions for the oscillation of (1), we need the following lemma.

LEMMA 5. *Assume* (C_1), (C_3), (C_4), (C_7), and (C_{10}) *hold. Let* $x(t)$ *be a nonoscillatory solution of* (1) *and its corresponding* j (*as in Lemma* 1) *be given. Then,*

$$w(t)=\begin{cases} x^{(n-1)}(t)/x^{(j-1)}(\lambda t) & \text{if } 1\leqslant j<n-1, \text{ and } \lambda\in(0,1) \\ x^{(n-1)}(t)/x^{(n-2)}(t) & \text{if } j=n-1, \end{cases}$$

satisfies

$$\int_t^\infty w^2(s)\,ds<\infty, \tag{19}$$

and

$$\int_t^\infty w^2(s)\,ds+kd\int_t^\infty p(s)\frac{h(s)}{s}\,ds\leqslant w(t), \tag{20}$$

for t *sufficiently large, where* $d>0$ *may depend on* $x(t)$, $0<k<1$ *is an arbitrary constant which is independent of* $x(t)$.

Proof. We will give the proof for the case $1<j<n-1$ (the proof for $j=1$ or $j=n-1$ is similar). W.l.o.g., we may assume $x(t)>0$ for all t sufficiently large. In view of the argument given in the proof of Theorem 1, for any k_1, $0<k_1<1$, there exists $t_8\geqslant t_0$ such that

$$\begin{aligned} 0&=x^{(n)}(t)+p(t)\,f(x(t),x(h(t)))\\ &=x^{(n)}(t)+p(t)\,x(h(t))\left(\frac{f(x(t),x(h(t)))}{x(h(t))}\right)\\ &\geqslant x^{(n)}(t)+k_1M_3\,p(t)\frac{h^{j-1}(t)}{t}x^{(j-1)}(t)\left(\frac{f(x(t),x(h(t)))}{x(h(t))}\right), \qquad \text{for}\quad t\geqslant t_8 \end{aligned} \tag{21}$$

in which we have used (10) and (11), respectively.

Let $w(t)=x^{(n-1)}(t)/x^{(j-1)}(\lambda t)$. From (21), we have

$$w'(t)+\lambda\frac{x^{(n-1)}(t)\,x^{(j)}(\lambda t)}{(x^{(j-1)}(\lambda t))^2}+k_1M_3\,p(t)\frac{h^{j-1}(t)}{t}\left(\frac{f(x(t),x(h(t)))}{x(h(t))}\right)\leqslant 0,$$

and, applying Lemma 2 to $x^{(j)}(\lambda t)$, we obtain

$$w'(t)+\lambda M_4t^{n-j-1}w^2(t)+k_1M_3\,p(t)\frac{h^{j-1}(t)}{t}\left(\frac{f(x(t),x(h(t)))}{x(h(t))}\right)\leqslant 0.$$

Now, from (C_7) and $x'(t)>0$, $\lim_{t\to\infty} x(h(t))$ tends monotonically upward to either a positive finite value or $+\infty$. In either case, there exists a $d>0$ such that

$$\frac{f(x(t), x(h(t)))}{x(h(t))} \geqslant d,$$

for all t sufficiently large. Moreover, for t sufficiently large, $\lambda M_4 t^{n-j-1} \geqslant 1$ and $M_3 h^{j-2}(t) \geqslant 1$. Thus for all t sufficiently large,

$$w'(t) + w^2(t) + k_1\, dp(t)\frac{h(t)}{t} \leqslant 0. \tag{22}$$

The rest follows as in Lemma 2 from [7, p. 381].

Following [7], we consider the sequence of functions

$$\{\alpha_m(t)\}, \qquad m = 0, 1, 2, ..., t \in [t_0, \infty), \tag{23}$$

where

$$\alpha_0(t) = \delta c \int_t^\infty p(s)\frac{h(s)}{s}\, ds, \qquad \alpha_m(t) = \int_t^\infty \alpha_{m-1}^2(s)\, ds + \alpha_0(t), \qquad m = 1, 2, ...,$$

and δ is a constant, $0<\delta<1$. We can now state the following results:

THEOREM 2. *Let* (C_1), (C_3), (C_4), (C_7), *and* (C_{10}) *hold, and let there exist a constant* $\delta, 0<\delta<1$, *such that one of the conditions* (C_{11}) *or* (C_{12}) *is satisfied, then Eq.* (1) *is oscillatory.*

Proof. It is easy to see that (C_{11}) implies

$$\int^\infty \int_t^\infty p(s)\frac{h(s)}{s}\, ds\, dt = \infty. \tag{24}$$

To show that (C_{12}) also implies (24), we assume that $\int^\infty \int_t^\infty p(s)(h(s)/s)\, ds\, dt < \infty$ and obtain a contradiction.

It can be seen that, for any $\eta>0$,

$$t\int_t^\infty p(s)\frac{h(s)}{s}\, ds \leqslant \eta,$$

for sufficiently large t, and hence

$$\int_t^\infty p(s)\frac{h(s)}{s}\, ds \leqslant \frac{\eta}{t}.$$

Thus,

$$\alpha_0(t) = \delta c \int_t^\infty p(s) \frac{h(s)}{s} ds \leqslant \frac{\delta c \eta}{t},$$

for all large t.

Choose $\eta > 0$ so that $c_0 = \delta c \eta \leqslant \frac{1}{4}$, then we have

$$\alpha_n(t) \leqslant c_n/t, \qquad n = 1, 2, ...,$$

where $c_n = c_{n-1}^2 + c_0$. The sequence $\{c_n\}$ is bounded by $(1 - \sqrt{1 - 4c_0})/2$ and hence

$$\alpha_n(t) \leqslant \frac{1 - \sqrt{1 - 4c_0}}{2t}, \qquad n = 1, 2, ...,$$

which contradicts (C_{12}).

In view of the above facts, it suffices to prove that (1) does not have unbounded nonoscillatory solutions. Noting that in Lemma 5, since $x(t)$ is unbounded, we may take $d = c$, the rest of the proof follows from Lemma 5 and the argument given in [7, p. 382].

COROLLARY. *Let* (C_1), (C_7), *and* (C_{10}) *hold. If* $F \in C(R)$, $yF(y) > 0$ *for* $y \neq 0$, $f(x, y) \equiv F(y)$ *satisfies* (C_4), *and if there exists a constant* δ, $0 < \delta < 1$, *such that one of the conditions* (C_{11}) *or* (C_{12}) *is satisfied, then the equation*

$$x^{(n)}(t) + p(t) F(x(h(t))) = 0, \qquad n \text{ even}, \tag{25}$$

is oscillatory.

Remarks 2. (a) If $n = 2$, then Theorems 1 and 2 (including the corollary) become Theorems 1–3 in [4].

(b) Corollary 4, Theorem 5, and Corollary 6 of [7] can be easily formulated for Eqs. (1) and (25) extending some well-known oscillation criteria. (See Remark 3 of [4].)

(c) The nth-order equations given in Examples 1 and 2 of [2] satisfy the hypotheses of our Theorem 1. Hence they are oscillatory by our criterion as well. However, unlike [2, 6], we do not require a minimum linear growth rate for the delay $h(t)$. Thus, a delay such as $h(t) = \ln(t)$ is allowed in our theorems.

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