Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 1105–1111
Applied Mathematics Letters
www.elsevier.com/locate/aml

A note on ideal Nash equilibrium in multicriteria games☆

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Received 11 April 2007; received in revised form 6 November 2007; accepted 11 December 2007

Abstract

This paper is a contribution to the problem of existence of ideal Nash equilibrium in noncooperative multicriteria games in strategic form. We give an existence theorem by using the maximal element theorem due to Deguire et al. [P. Deguire, K.K. Tan, G.X.-Z. Yuan, The study of maximal elements, fixed points for L_S-majorized mappings and their applications to minimax and variational inequalities in product topological spaces, Nonlinear Anal. TMA 37 (1999) 933–951] and the characterization provided by Voorneveld et al. [M. Voorneveld, S. Grahn, M. Dufwenberg, Ideal equilibria in non cooperative multicriteria games, Math. Methods Oper. Res. 52 (2000) 65–77].

Keywords: Multicriteria games; Ideal equilibria; Nash equilibria; Efficient equilibria; Maximal elements

1. Introduction

In recent years, many authors have studied the game problems with vector payoffs, for example, see [1,4,6,11,12] and the references therein. Although many concepts have been suggested to solve multicriteria games [3,7,8], the notion of the Pareto–Nash equilibrium, introduced by Shapley (1959), is the most studied concept in game theory. In [10], the authors introduced the new concept of ideal Nash equilibrium for finite multicriteria games which has the best properties. In fact, the ideal Nash strategy of each player enables him to maximize all his criteria when the other players choose their ideal equilibrium strategies. In consequence, when its existence is guaranteed it is preferable to solve the game using the ideal Nash equilibrium. In [10], the authors gave a characterization and axiomatization of the ideal Nash equilibrium and showed its existence in a particular class of bicriteria games constructed from ordinal potential games.

The aim of this paper is to point out a more general class with a nonempty set of ideal Nash equilibria. By using the maximal element theorem due to Deguire et al. [5] and the characterization given in [10], we provide an existence theorem for a more general class of multicriteria games.

☆ The authors are grateful to Denis Bouyssou, Guillaume Hollard and Tarik Tazdaït for useful comments and suggestions. The usual disclaimer applies.
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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.12.009

2. Preliminaries and definitions

Consider a multicriteria game

$$G = \langle I, (X_i)_{i\in I}, (f_i)_{i\in I}\rangle, \tag{1}$$

where $I = \{1, 2, \ldots, n\}$ is a finite set of players; for all $i \in I$, $X_i \subset R^{l_i}$ is the set of pure strategies of player $i \in I$; $X = \prod_{i=1}^{n} X_i$ is the set of outcomes of the game and for all $i \in I$, $f_i : X \longrightarrow \mathbb{R}^{r(i)}$ is the vector payoff function of player $i \in I$, where $r(i) \in \mathbb{N}$ is the number of criteria of player i. Throughout this paper, we assume that each player is a maximizer.

We can interpret the multicriteria game (1) as follows: each player $i \in I$ is considered as an organization which includes $r(i)$ members. Each member $k \in \{1, \ldots, r(i)\}$ of the ith organization has a payoff function $f_{ik} : X_i \longrightarrow \mathbb{R}$. A choice of strategy $x_i \in X_i$ by the organization i is supposed to be taken by common agreement of all the members with the objective to maximize the payoff of each of them, taking into account the fact that the payoff of each member depends also on the strategy choices $x_j \in X_j$ of the other organizations $j \in I \setminus \{i\}$. The idea is from that of Voorneveld [10].

We denote by $\Delta_m = \{\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m_+, \sum_{i=1}^{m} \mu_i = 1\}$, the unit simplex in $\mathbb{R}^m$ and $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m, x_i \geqq 0, \forall i = 1, \ldots, m\}$.

For vectors $x, y \in \mathbb{R}^m$, we denote by $\langle x, y\rangle = \sum_{k=1}^{m} x_k y_k$ the inner product of x *and* y; e_k the kth standard basis vector of $\mathbb{R}^m$.

For $a = (a_1, a_2, \ldots, a_m), b = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$; we define vector inequalities as follows:

$$a \geqq b \Leftrightarrow a_j \geqq b_j \quad \text{for all } j \in \{1, \ldots, m\};$$
$$a \geq b \Leftrightarrow a \geqq b \quad \text{and} \quad a \neq b;$$
$$a > b \Leftrightarrow a_j > b_j \quad \text{for all } j \in \{1, \ldots, m\}.$$

We design by $a \ngeqq b$ (*respectively* $a \ngeq b, a \not> b$) the negation of the relations $a \geqq b$ (*respectively* $a \geq b, a > b$). Let be $\lambda = (\lambda_1, \ldots, \lambda_n) \in \prod_{i=1}^{n} \Delta_{r(i)}$ and consider the λ-weighted game

$$G_\lambda = \langle I, (X_i)_{i\in I}, (g_i)_{i\in I}\rangle \tag{2}$$

associated to the multicriteria game (1), where each player has the same set of strategies as in G; for all $i \in I$, the payoff function $g_i : X \longrightarrow \mathbb{R}$ is defined by $g_i(x) = \langle \lambda_i, f_i(x)\rangle = \sum_{k=1}^{k=r(i)} \lambda_{ik} f_{ik}(x)$, where $\lambda_i \in \Delta_{r(i)}$ is the vector of weights for the vector criteria $f_i(\cdot)$ of player $i \in I$.

Definition 1. Let $\{F_i\}_{i\in I}$ be a family of multivalued maps $F_i : W \longrightarrow 2^{Z_i}$, where I is any index set and for all $i \in I$, W, Z_i are topological vector spaces and 2^{Z_i} is the set of all subsets of Z_i.

The point $\bar{x} \in W$ is said to be a maximal element for the family $\{F_i\}_{i\in I}$, if $F_i(\bar{x}) = \phi$, for all $i \in I$.

Definition 2 (*[2]*). Let W, Z be topological vector spaces and 2^Z the family of all subsets of Z, M a nonempty convex subset of W and let $P : M \longrightarrow 2^Z$ be a multivalued map such that for each $x \in M$, $P(x)$ is a closed, convex cone with nonempty interior.

A multivalued function $H : M \times M \longrightarrow 2^Z$ is called $P(x)$-quasi-convex-like, if for each $x, y_1, y_2 \in M$ *and* $t \in [0, 1]$, we have either

$$H(x, ty_1 + (1-t)y_2) \subseteq H(x, y_1) - P(x),$$

or

$$H(x, ty_1 + (1-t)y_2) \subseteq H(x, y_2) - P(x).$$

H is called $P(x)$-quasi-concave-like if $(-H)$ is $P(x)$-quasi-convex-like.

Remark 1. If the function H is defined on $M \subset W$ into $\mathbb{R}$ and $P(x) = \mathbb{R}_+$ for all $x \in M$, then the $P(x)$-quasi-concavity-like is equivalent to the quasi-concavity.

3. Ideal Nash equilibrium

In 1959, Loyd Shapley generalized the Nash equilibrium concept for strategic multicriteria games, hereby called efficient and weakly efficient Nash equilibrium. Their existence is established by scalarization of the vector payoff functions of the players, see for example [9,12]. Recently, Ansari [1] studied the existence of weak efficient Nash equilibrium by the Maximal element approach.

Definition 3 (*[9]*). A situation $x^* \in X$ is said to be

- a weakly efficient Nash equilibrium for the multicriteria game (1), if for each $i \in I$ and $y_i \in X_i$,

$$f_i(x^*) \not< f_i(y_i, x^*_{-i});$$

- an efficient Nash equilibrium for the game (1), if for each $i \in I$ and $y_i \in X_i$,

$$f_i(x^*) \not\leq f_i(y_i, x^*_{-i}).$$

We will denote by $X^{EN}(G)$ the set of efficient Nash equilibria and by $X^{WEN}(G)$ the set of weakly efficient Nash equilibria for the multicriteria game G.

In 2000, Voorneveld et al. [10] proposed the concept of ideal Nash equilibrium, in which each player maximizes all his criteria simultaneously. Thus, it is important to look for this equilibrium in a multicriteria game before all other concepts.

Definition 4 (*[10]*). A situation $x \in X$ is said to be an ideal Nash equilibrium for the multicriteria game (1), if for each player $i \in I$ and for all $y_i \in X_i$,

$$f_i(x) = f_i(x_i, x_{-i}) \geqq f_i(y_i, x_{-i}).$$

We will denote by $X^{IN}(G)$ the set of ideal Nash equilibria of the game G.

Remark 2 (*[10]*).

1. It is easy to see that $X^{IN}(G) \subseteq X^{EN}(G) \subseteq X^{WEN}(G)$.
2. If each player has only one criterion, then the ideal Nash equilibria, efficient Nash equilibria and Weakly efficient Nash equilibria coincide with the Nash equilibria.

Definition 5 (*[10]*). A collection $\Lambda \subset \prod_{i \in I} \Delta_{r(i)}$ of weight vectors is called representative for the game G, if for each organization $i \in I$ and each of its members $k \in \{1, \ldots, r(i)\}$, there exists a weight vector in Λ assigning weight one to this organization member:

$$\forall i \in I, \forall k \in \{1, 2, \ldots, r(i)\}, \exists \lambda = (\lambda_j)_{j \in I} \in \Lambda, \quad \text{with } \lambda_i = e_k \in \Delta_{r(i)}.$$

Let $\psi_i : X_{-i} \longrightarrow \mathbb{R}^{r(i)}$ be a vector function defined by

$$\psi_{ik}(x_{-i}) = \sup_{y_i \in X_i} f_{ik}(y_i, x_{-i}), \quad \forall k = \overline{1, r(i)}, \forall i \in I.$$

The following theorem of characterization of ideal Nash equilibrium is given in [10] for finite multicriteria games. It stays valid for the strategic multicriteria games defined by (1).

Theorem 3.1. *Let G be the multicriteria game defined by (1) and Λ a representative collection for G. The following statements are equivalent:*

(a) $x \in X^{IN}(G)$,
(b) $\forall i \in I$, $f_i(x) = \psi_i(x_{-i})$;
(c) $x \in \bigcap_{\lambda \in \prod_{i \in I} \Delta_{r(i)}} X^I(G_\lambda)$;
(d) $x \in \bigcap_{\lambda \in \Lambda} X^I(G_\lambda)$,

where $X^I(G_\lambda)$ is the set of Nash equilibria for the game (G_λ).

Proposition 3.1 (*[10]*). *The smallest representative collection for the multicriteria game G has* $\max_{i \in I} r(i)$ *elements. Hence,* $\max_{i \in I} r(i)$ *scalarizations suffice to determine $X^{IN}(G)$.*

4. Existence results

In this section, we give our main results about the existence of ideal Nash equilibrium in strategic form multicriteria games. We recall that Voorneveld et al. [10] gave a class of bicriteria games that they constructed from ordinal potential game in which this equilibrium exists. Our study is based on the maximal element for a particular family of multivalued maps and the characterization of this equilibrium given in Theorem 3.1.

Let Λ be a representative collection for the game (1). For each $i \in I$, define a multivalued map $A_i : X \longrightarrow 2^{X_i}$ by

$$A_i(x) = \{y_i \in X_i / \exists \lambda \in \Lambda \text{ such that } \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle < 0\}. \tag{3}$$

The following proposition shows that the existence of ideal Nash equilibrium for the game (1) is equivalent with the existence of a maximal element for the family of maps defined by (3).

Proposition 4.1. *A situation $x^* \in X$ is an ideal Nash equilibrium for the game (1) if and only if $x^* \in X$ is a maximal element for the family of multivalued maps defined by (3).*

Proof. Consecutively using Theorem 3.1, the definition of a Nash equilibrium, and (3), it follows that

$$\begin{aligned} x^* \in X^{IN}(G) &\Leftrightarrow x^* \in \bigcap_{\lambda \in \Lambda} X^I(G_\lambda) \\ &\Leftrightarrow \forall \lambda \in \Lambda, \forall i \in I, \forall y_i \in X_i : \langle \lambda_i, f_i(x*)\rangle - \langle \lambda_i, f_i(y_i, x^*_{-i})\rangle \geqq 0 \\ &\Leftrightarrow \forall i \in I : A_i(x^*) = \phi, \end{aligned}$$

finishing the proof. □

By using Theorem 7 in [5] and the Proposition 4.1, we derive an existence result of ideal Nash equilibria.

Theorem 4.1. *For the game (1), we suppose that for each $i \in I$,*

1. *X_i is a nonempty convex set;*
2. *the function f_i is continuous on X;*
3. *for each $x \in X$, the function $y_i \longmapsto f_i(y_i, x_{-i})$ is $\mathbb{R}_+^{r(i)}$-quasi-concave-like;*
4. *there exists a nonempty compact subset K of X and a nonempty, compact convex subset D_i of X_i for each $i \in I$, such that for all $x \in X \setminus K$, there exists $i \in I$ and $\tilde{y}_i \in D_i$ such that $f_i(x) - f_i(\tilde{y}_i, x_{-i}) < 0$.*

Then, there exists an ideal Nash equilibrium for the multicriteria game (1).

Proof. Let Λ be a smallest representative collection for the game (1) and $A_i : X \longrightarrow 2^{X_i}$ the correspondence defined by

$$A_i(x) = \{y_i \in X_i \quad \exists \lambda \in \Lambda \text{ such that } \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle < 0\}.$$

We shall prove that the family of multivalued maps $\{A_i\}_{i \in I}$ satisfies all assumptions of Theorem 7 in [5].

- For all $x \in X$, $i \in I$ and $\lambda \in \Lambda$, we have $\langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(x_i, x_{-i})\rangle = 0$, thus $x_i \notin A_i(x)$.
- For all $x \in X$ and $i \in I$, $A_i(x)$ is convex. Indeed, let $y_i^1, y_i^2 \in A_i(x)$. Then, there exist $\lambda^1, \lambda^2 \in \Lambda$ such that

$$\langle \lambda_i^1, f_i(x)\rangle - \langle \lambda_i^1, f_i(y_i^1, x_{-i})\rangle < 0;$$
$$\langle \lambda_i^2, f_i(x)\rangle - \langle \lambda_i^2, f_i(y_i^2, x_{-i})\rangle < 0.$$

Let $\alpha \in [0, 1]$. By hypothesis 3, f_i is $\mathbb{R}_+^{r(i)}$-quasi-concave-like on X_i, then $(-f)$ is $\mathbb{R}_+^{r(i)}$-quasi-convex-like, and we have either

$$-f_i(\alpha y_i^1 + (1-\alpha) y_i^2, x_{-i}) = -f_i(y_i^1, x_{-i}) - \beta^1, \quad \beta^1 \in \mathbb{R}_+^{r(i)},$$

or

$$-f_i(\alpha y_i^1 + (1-\alpha) y_i^2, x_{-i}) = -f_i(y_i^2, x_{-i}) - \beta^2, \quad \beta^2 \in \mathbb{R}_+^{r(i)}.$$

It follows that either

$$\langle \lambda_i^1, f_i(x)\rangle - \langle \lambda_i^1, f_i(\alpha y_i^1 + (1-\alpha) y_i^2, x_{-i})\rangle = \langle \lambda_i^1, f_i(x)\rangle - \langle \lambda_i^1, f_i(y_i^1, x_{-i})\rangle - \langle \lambda_i^1, \beta^1\rangle < 0$$

or

$$\langle \lambda_i^2, f_i(x)\rangle - \langle \lambda_i^2, f_i(\alpha y_i^1 + (1-\alpha) y_i^2, x_{-i})\rangle = \langle \lambda_i^2, f_i(x)\rangle - \langle \lambda_i^2, f_i(y_i^2, x_{-i})\rangle - \langle \lambda_i^2, \beta^2\rangle < 0$$

which implies in both cases that

$$\alpha y_i^1 + (1-\alpha) y_i^2 \in A_i(x),$$

hence $A_i(x)$ is convex on X_i.

• We prove that for all $y_i \in X_i$ and $i \in I$, the set $A_i^{-1}(y_i)$ is open in X.

By hypothesis 2,

$$A_i^{-1}(y_i) = \bigcup_{\lambda \in \Lambda} \{x \in X, \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle < 0\}$$

is a finite union of pre-images of the open set $(-\infty, 0)$ under the continuous functions $x \mapsto \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle$ and hence open.

• We prove that the set $A_i = \{x \in X : A_i(x) \neq \phi\}$ is open for all $i \in I$. We have

$$\begin{aligned} A_i &= \{x \in X, \exists y_i \in X_i, \exists \lambda \in \Lambda, \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle < 0\} \\ &= \bigcup_{y_i \in X_i} \bigcup_{\lambda \in \Lambda} \{x \in X, \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(y_i, x_{-i})\rangle < 0\} \\ &= \bigcup_{y_i \in X_i} A_i^{-1}(y_i). \end{aligned}$$

Hence, the set A_i is open as the union of open sets.

• From assumption 4 of Theorem 4.1, for all $x \in X \setminus K$, there exist $i \in I$ and $\tilde{y}_i \in D_i \subset X_i$ such that $f_i(x) - f_i(\tilde{y}, x_{-i}) < 0$.

Since Λ is a representative collection, then for all $i \in I$ and $k \in \{1, \ldots, r(i)\}$ there exists $\lambda = (\lambda_j)_{j \in I}$ with $\lambda_i = e_k$. It follows that

$$f_{ik}(x) - f_{ik}(\tilde{y}_i, x_{-i}) = \langle \lambda_i, f_i(x)\rangle - \langle \lambda_i, f_i(\tilde{y}_i, x_{-i})\rangle < 0,$$

thus $\tilde{y}_i \in A_i(x)$ and $A_i(x) \cap D_i \neq \phi$.

All conditions of theorem 7 in [5] are satisfied, then there exists $\bar{x} \in X$ such that $A_i(\bar{x}) = \phi$ for all $i \in I$ and, by Proposition 4.1, $\bar{x}$ is an ideal Nash equilibrium. □

We illustrate our result by the following example.

Example 1. Consider the game

$$G = \langle I, (X_i)_{i \in I}, (f_i)_{i \in I}\rangle, \tag{4}$$

where $I = \{1, 2\}$, $r(1) = r(2) = 2$, $X_1 =]-1, 1[$, $X_2 = [0, 1]$, $f_1(x_1, x_2) = (-x_1^2 + x_2^2, x_2 \cos \frac{\pi}{2} x_1)$ and $f_2(x_1, x_2) = (f_{21}(x_1, x_2), f_{22}(x_1, x_2))$ with

$$f_{21}(x_1, x_2) = \begin{cases} 2x_1^2 x_2, & \text{if } x_2 \in \left[0, \frac{1}{2}\right], x_1 \in X_1 \\ -2(x_2 - 1)x_1^2, & \text{if } x_2 \in \left[\frac{1}{2}, 1\right], x_1 \in X_1, \end{cases} \qquad f_{22}(x_1, x_2) = (x_1 + 1)\sin \pi x_2.$$

∗ For all $i \in I = \{1, 2\}$, the set X_i is convex.

∗ To show that the function $y_1 \longrightarrow f_1(y_1, x_2)$ is $\mathbb{R}_+^2$-quasi-concave-like, we must prove that for all $\lambda \in [0, 1]$ and $y_1, z_1 \in X_1$ we have

$$\begin{aligned} &\text{either} \quad f_1(\lambda y_1 + (1-\lambda) z_1, x_2) = f_1(y_1, x_2) + \epsilon \\ &\text{or} \quad f_1(\lambda y_1 + (1-\lambda) z_1, x_2) = f_1(z_1, x_2) + \epsilon, \quad \text{where } \epsilon \in \mathbb{R}_+^2. \end{aligned}$$

If $|y_1| > |z_1|$, then

$$\begin{cases} f_{11}(\lambda y_1 + (1-\lambda) z_1, x_2) = f_{11}(y_1, x_2) + \epsilon_1, & \epsilon_1 \in \mathbb{R}_+ \\ f_{12}(\lambda y_1 + (1-\lambda) z_1, x_2) = f_{12}(y_1, x_2) + \epsilon_2, & \epsilon_2 \in \mathbb{R}_+. \end{cases}$$

It follows that

$$f_1(\lambda y_1 + (1-\lambda)z_1, x_2) = f_1(y_1, x_2) + \epsilon, \quad \epsilon = (\epsilon_1, \epsilon_2) \in \mathbb{R}^2_+.$$

If $|y_1| \leq |z_1|$, then

$$\begin{cases} f_{11}(\lambda y_1 + (1-\lambda)z_1, x_2) = f_{11}(z_1, x_2) + \epsilon_1, \\ f_{12}(\lambda y_1 + (1-\lambda)z_1, x_2) = f_{12}(z_1, x_2) + \epsilon_2. \end{cases}$$

It follows that

$$f_1(\lambda y_1 + (1-\lambda)z_1, x_2) = f_1(z_1, x_2) + \epsilon, \quad \epsilon \in (\epsilon_1, \epsilon_2) \in \mathbb{R}^2_+.$$

In the same way, we show that the function $y_2 \longrightarrow f_2(x_1, y_2)$ is $\mathbb{R}^2_+$-quasi-concave-like.

To verify the last assumption of Theorem 4.1, we set $K = [-\frac{1}{2}, \frac{1}{2}] \times [0, 1]$, $D_1 = [-\frac{1}{2}, \frac{1}{2}]$, $D_2 = [\frac{1}{2}, 1]$.

(a) $\forall \tilde{x} \in X \setminus K = (]-1, -\frac{1}{2}[\cup]\frac{1}{2}, 1[) \times [0, 1]$, with $\tilde{x}_2 \neq 0$, $\exists i = 1 \in I$ and $\tilde{y}_1 \in D_1 = [-\frac{1}{2}, \frac{1}{2}]$ such that

$$f_{11}(\tilde{x}) - f_{11}(\tilde{y}_1, \tilde{x}_2) = -\tilde{x}_1^2 + \tilde{y}_1^2 = \tilde{y}_1^2 - \tilde{x}_1^2.$$

Since $\tilde{y}_1 \in [-\frac{1}{2}, \frac{1}{2}]$ and $\tilde{x}_1 \in]-1, -\frac{1}{2}[\cup]\frac{1}{2}, 1[$, then

$$f_{11}(\tilde{x}) - f_{11}(\tilde{y}_1, \tilde{x}_2) < 0,$$

and

$$f_{12}(\tilde{x}) - f_{12}(\tilde{y}_1, \tilde{x}_2) = \tilde{x}_2 \left(\cos\frac{\pi}{2}\tilde{x}_1 - \cos\frac{\pi}{2}\tilde{y}_1\right) < 0.$$

(b) $\forall \tilde{x} \in X \setminus K$ such that $\tilde{x} = (\tilde{x}_1, 0)$, $\exists i = 2$ and $\tilde{y}_2 = \frac{1}{2} \in D_2 = [\frac{1}{2}, 1]$ such that

$$f_{21}(\tilde{x}) - f_{21}(\tilde{x}_1, \tilde{y}_2) = -\tilde{x}_1^2 < 0$$

and

$$f_{22}(\tilde{x}) - f_{22}(\tilde{x}_1, \tilde{y}_2) = -(\tilde{x}_1 + 1) < 0.$$

All assumptions of Theorem 4.1 are verified, then there exists an ideal Nash equilibrium $\bar{x} \in X$.

To find $\bar{x}$, we follow the proof of Theorem 4.1. Let $\Lambda = \{(e_1, e_1), (e_2, e_2)\}$ be the smallest representative collection and $A_1(\bar{x}) = \{y_1 \in X_1, \exists \lambda \in \Lambda / \langle \lambda_1, f_1(\bar{x})\rangle - \langle \lambda_1, f_1(y_1, \bar{x}_2)\rangle < 0\}$. We have

$$\begin{aligned} A_1(\bar{x}) = \emptyset &\iff \forall y_1 \in X_1, \forall \lambda \in \Lambda\ \langle \lambda_1, f_1(\bar{x})\rangle - \langle \lambda_1, f_1(y_1, \bar{x}_2)\rangle \geqq 0 \\ &\iff \begin{cases} (-\bar{x}_1^2 + \bar{x}_2^2) - (-y_1^2 + \bar{x}_2^2) \geqq 0 & \forall y_1 \in X_1; \\ \bar{x}_2 \cos\frac{\pi}{2}\bar{x}_1 - \bar{x}_2 \cos\frac{\pi}{2} y_1 \geqq 0, & \forall y_1 \in X_1. \end{cases} \\ &\iff \begin{cases} (y_1^2 - \bar{x}_1^2) \geqq 0, \quad \forall y_1 \in X_1; & \text{(a)} \\ \bar{x}_2 \left(\cos\frac{\pi}{2}\bar{x}_1 - \cos\frac{\pi}{2} y_1\right) \geqq 0, \quad \forall y_1 \in X_1. & \text{(b)} \end{cases} \end{aligned} \tag{1}$$

$$1\text{(a)} \implies \bar{x}_1 = 0 \tag{2}$$

$$1\text{(b)} \implies \bar{x}_2 \left(1 - \cos\frac{\pi}{2} y_1\right) \geqq 0, \quad \forall y_1 \in X_1.$$

With $A_2(\bar{x}) = \emptyset$ and $\bar{x}_1 = 0$, we find $\bar{x}_2 = \frac{1}{2}$.

Thus $\bar{x} = (0, \frac{1}{2})$ is the ideal Nash equilibrium for the considered game.

Following Lemma 4 and the proof of Theorem 7 in [5], we deduce:

Remark 3. If for each $i \in I$, X_i is a nonempty compact convex subset, then the conclusion of Theorem 4.1 holds without assumption 4.

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