Christoffel-type functions for $m$-orthogonal polynomials for Freud weights

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Abstract

This paper gives upper and lower bounds of the Christoffel-type functions $\lambda_{jn}(W^m, m; x)$, $j = m - 2, m - 4, \ldots, m - 2[m/2]$, for the $m$-orthogonal polynomials for a Freud weight $W = e^{-Q}$, which are given as follows. Let $a_n = a_n(Q)$ be the $n$th Mhaskar–Rahmanov–Saff number, $\phi_n(x) = \max\{n^{-2/3}, 1 - |x|/a_n\}$, and $d > 0$. Assume that $Q \in C(\mathbb{R})$ is even, $Q'' \in C[0, \infty)$, $Q'(x) > 0$, $x \in (0, \infty)$, $Q(0) = 0$, and for some $A, B > 1$

$$A \leq \left(\frac{x Q'(x)}{Q(x)}\right)' \leq B, \quad x \in (0, \infty).$$

Then for $x \in \mathbb{R}$

$$\lambda_{jn}(W^m, m; x) \geq \begin{cases} c \left(\frac{a_n}{n}\right)^{j+1} W(x)^m \phi_n(x)^{-1/2}, & m \text{ is even, } j = 0, \\ c \left(\frac{a_n}{n}\right)^{j+1} W(x)^m, & \text{otherwise,} \end{cases}$$

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and for $|x| \leq a_n(1 + d n^{-2/3})$
\[ \lambda_{jn}(W^m, m; x) \leq c \left( \frac{a_n}{n} \right)^{j+1} W(x)^m \phi_n(x)^{(1-m)/2}. \]

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1. Introduction and main results

We denote by $N, N_1$, or $N_2$ the set of positive, odd, and even integers, respectively. We also denote by $R$ the set of real numbers.

Let $\mu$ be a nondecreasing function on $R$ with infinitely many points of increase such that all moments of $d \mu$ are finite. We call $d \mu$ a measure. If $\mu$ happens to be absolutely continuous then we will usually write $w$ instead of $d \mu$ and will call $w$ a weight. The symbol $P_n$ stands for the set of algebraic polynomials of degree at most $n$. The symbol $\hat{P}$ denotes the exact degree of the polynomial $P \neq 0$, i.e., $P \in \hat{P} \setminus \hat{P}_{P-1}$.

We denote by $c, c_1, \ldots$ positive constants independent of variables and indices, unless otherwise indicated; their value may be different at different occurrences, even in subsequent formulas. We write $a_n \sim b_n$ if $c_1 \leq a_n/b_n \leq c_2$ holds for every $n$. The notations $a(x) \sim b(x)$ and $a_n(x) \sim b_n(x)$ have similar meaning.

Throughout this paper let $m \in N$ ($m \geq 2$), $M_1 = \{j \leq m - 3 : m - j \in N_1\}$, and $M_2 = \{j \leq m - 2 : m - j \in N_2\}$. Put $P^*_n = \{P(x) = c(x - y_1) \cdots (x - y_r) : c, y_1, \ldots, y_r \in R, r \leq n\}$ and $P^*_n(x) = \{P \in P^*_n : P(x) = 1\}$ for $x \in R$. We agree $P^*_0 = P_0$.

We define the $m$-monic orthogonal polynomials
\[ P_n(d \mu, m; x) = x^n + \cdots, \quad n = 0, 1, \ldots, \]
for which
\[ \int_R |P_n(d \mu, m; x)|^m d \mu(x) = \min_{P(x) = x^n + \cdots} \int_R |P(x)|^m d \mu(x). \]  \hfill (1.1)

According to Theorem 4 in [1], if $x_k = x_{kn}(d \mu, m)$ with
\[ x_1 < x_2 < \cdots < x_n \]  \hfill (1.2)
are the zeros of $P_n(d \mu, m; x)$ then the Gaussian quadrature formula
\[ \int_R f(x) \text{sgn} P_n(d \mu, m; x)^m d \mu(x) = \sum_{k=1}^n \sum_{j=0}^{m-2} \hat{\lambda}_{kj} f^{(j)}(x_k) \]  \hfill (1.3)
is exact for all $f \in P_{mn-1}$, where the Christoffel numbers $\hat{\lambda}_{kj} = \hat{\lambda}_{kn}(d \mu, m)$ are given by
\[ \hat{\lambda}_{kj} = \int_R A_{kj}(x) \text{sgn} P_n(d \mu, m; x)^m d \mu(x) \]  \hfill (1.4)
and $A_{kj} \in \mathbf{P}_{mn-1}$ are the fundamental polynomials of Hermite interpolation, which satisfy

$$A_{kj}^{(p)}(x_q) = \delta_{kq}\delta_{jp}, \quad j, p = 0, 1, \ldots, m - 1, \ k, q = 1, 2, \ldots, n.$$  

As we know, orthogonal polynomials ($m = 2$) have a long history of study and a classical theory. One of the important contents of this theory are the Christoffel functions

$$\lambda_n(d\mu; x) = \min_{P \in \mathbf{P}_{n-1}, P(x) = 1} \int_{\mathbb{R}} P(t)^2 d\mu(t),$$  

which are closely related to the Christoffel numbers

$$\lambda_{kn}(d\mu) = \lambda_n(d\mu; x_{kn}(d\mu)), \quad k = 1, 2, \ldots, n.$$  

Here we accept the notation $P_{n}(d\mu) = P_{n}(d\mu, 2)$, $x_{kn}(d\mu) = x_{kn}(d\mu, 2)$, etc. The study and applications of the Christoffel functions can be found in [7].

The author in [8,11] defines the Christoffel-type functions $\lambda_{jn}(d\mu, m; x)$, which are the extension of $\lambda_n(d\mu; x)$ to the $m$-orthogonal polynomials and are given as follows.

Given a fixed point $x \in \mathbb{R}$, an index $j$, $0 \leq j \leq m - 2$, and $n \in \mathbb{N}$, for $P \in \mathbf{P}_{n-1}$ with $P(x) = 1$ let the polynomial

$$A_j(P, x; t) = A_{jnm}(P, x; t) = \frac{1}{j!} (t - x)^j B_j(P, x; t) P(t)^m$$  

with $B_j(P, x; \cdot) \in \mathbf{P}_{m-j-2}$ satisfy the conditions

$$A_j^{(i)}(P, x; x) = \delta_{ij}, \quad i = 0, 1, \ldots, m - 2.$$  

It is easy to see that $A_j(P, x; t)$ must exist and be unique.

**Definition 1.1 (Shi [11, Definition 1.1]).** The Christoffel-type function $\lambda_{jn}(d\mu, m; x)$ with respect to $d\mu$ is defined by

$$\lambda_{jn}(d\mu, m; x) = \inf_{P \in \mathbf{P}_{n-1}(x)} \int_{\mathbb{R}} A_j(P, x; t)\text{sgn}[(t - x)P(t)]^m d\mu(t)$$  

for $j \in \mathbf{M}_2$ and by

$$\lambda_{jn}(d\mu, m; x) = \int_{\mathbb{R}} A_j(P, x; t)\text{sgn}[(t - x)P(t)]^m d\mu(t)$$  

for $j \in \mathbf{M}_1$, where the polynomial $P$ in (1.9) is the solution of (1.8) in the case when $j \in \mathbf{M}_2$.

According to Theorem 2.1 in [11] there is a unique polynomial $P \in \mathbf{P}_{n-1}^*(x)$ such that Eq. (1.8) holds for every $j \in \mathbf{M}_2$. So the definition of $\lambda_{jn}(d\mu, m; x)$ for $j \in \mathbf{M}_1$ is reasonable. Meanwhile we have

$$\lambda_{0n}(d\mu, 2; x) = \lambda_n(d\mu; x)$$  

by Corollary 2.2 in [11] and

$$\lambda_{kjn}(d\mu, m) = [\text{sgn} P_{n}^{(1)}(d\mu, m; x_{kn}(d\mu, m))]^m] \lambda_{jn}(d\mu, m; x_{kn}(d\mu, m)), \quad k = 1, 2, \ldots, n$$  

by Theorem 2.3 in [11].
In [9,11] the author gives estimations of \( \lambda_{jn}(u, m; x) \) for a weight \( u \) satisfying

\[
u \sim w \quad \text{a.e.,} \tag{1.10}\]

where \( w \) is a \textit{generalized Jacobi weight}:

\[
w(x) = \prod_{i=1}^{r} |x - t_i|^p_i, \quad |x| < 1, \quad w(x) = 0, \quad |x| \geq 1,
-1 = t_1 < t_2 < \cdots < t_r = 1 \quad (r \geq 2), \quad p_i > -1, \quad i = 1, 2, \ldots, r. \tag{1.11}\]

**Theorem 1.1** (Shi [11, Theorem 3.3]). Let relation (1.10) prevail. Then with the constants associated with the symbol \( \sim \) depending on \( u \) and \( m \),

\[
\lambda_{jn}(u, m; x) \sim \lambda_n(u; x)A_n(x)^j \sim \frac{1}{n} w_n(x)A_n(x)^j, \quad x \in [-1, 1], \quad j \in M_2. \tag{1.12}\]

Here

\[
w_n(x) = \left[ (1 + x)^{1/2} + \frac{1}{n} \right]^{2p_1+1} \left[ (1 - x)^{1/2} + \frac{1}{n} \right]^{2p_r+1} \prod_{i=2}^{r-1} \left[ |x - t_i| + \frac{1}{n} \right]^{p_i} \tag{1.13}\]

and

\[
A_n(x) = \left( \frac{1 - x^2}{n} \right)^{1/2} + \frac{1}{n^2}. \]

\textit{Freud weights} on an infinite interval are as significant as generalized Jacobi weights on a finite interval.

**Definition 1.2** (Lubinsky and Mastroianni [5, Definition 1.1]). Let \( W = e^{-Q} \) where \( Q \in C(\mathbb{R}) \) is even, \( Q' \in C(0, \infty) \), \( Q'(x) > 0 \), \( x \in (0, \infty) \), and for some \( A, B > 1 \)

\[
A \leq \frac{(xQ'(x))^r}{Q'(x)} \leq B, \quad x \in (0, \infty). \tag{1.14}\]

Then we write \( W \in \mathcal{F} \).

Assume, further, that \( Q(0) = 0 \) and \( Q' \in C[0, \infty) \). In this case we write \( W \in \mathcal{F}^* \).

For \( W \in \mathcal{F} \) the \( q \)th \textit{Mhaskar–Rahmanov–Saff number} \( a_q = a_q(Q) \) is defined by the positive root of the equation

\[
q = \frac{2}{\pi} \int_{0}^{1} a_q t Q(a_q t)(1 - t^2)^{-1/2} dt, \quad q > 0. \tag{1.15}\]

Levin and Lubinsky give the following important results, in which

\[
\phi_n(x) = \max \left\{ n^{-2/3}, 1 - \frac{|x|}{a_n} \right\}. \]
Theorem 1.2 (Levin and Lubinsky [4, Theorem 1.1]). Let $W \in F$, $n \in \mathbb{N}$, and $d > 0$. Then

$$
\begin{cases}
\lambda_n(W^2; x) \sim \frac{a_n}{n} W(x)^2 \phi_n(x)^{-1/2}, & |x| \leq a_n(1 + dn^{-2/3}), \\
\lambda_n(W^2; x) \geq c \frac{a_n}{n} W(x)^2 \phi_n(x)^{-1/2}, & x \in \mathbb{R}.
\end{cases}
$$

In this paper we shall give upper and lower bounds of the Christoffel-type functions $\lambda_{jn}(W^m, m; x)$ with $j \in \mathbb{M}_2$ for a Freud weight $W$.

Theorem 1.3. Let $W \in F^*, n \in \mathbb{N}$, $d > 0$, and $j \in \mathbb{M}_2$. Then for $x \in \mathbb{R}$

$$
\lambda_{jn}(W^m, m; x) \geq \begin{cases}
c \left( \frac{a_n}{n} \right)^{j+1} W(x)^m \phi_n(x)^{-1/2}, & m \in \mathbb{N}_2, j = 0, \\
c \left( \frac{a_n}{n} \right)^{j+1} W(x)^m & \text{otherwise},
\end{cases}
$$

and for $|x| \leq a_n(1 + dn^{-2/3})$

$$
\lambda_{jn}(W^m, m; x) \leq c \left( \frac{a_n}{n} \right)^{j+1} W(x)^m \phi_n(x)^{(1-m)/2}.
$$

We shall give some auxiliary lemmas in Section 2 and the proof of Theorem 1.3 in Section 3.

2. Auxiliary lemmas

We need some known results. Here $\|f\| = \sup_{x \in \mathbb{R}} |f(x)|$, $f \in C(\mathbb{R})$.

Lemma 2.1 (Shi [11, Lemma 2.1]). We have

$$
B_j(P, x; t) = \sum_{i=0}^{m-j-2} b_i(t - x)^i,
$$

where

$$
b_i = b_i(P, x) = \frac{1}{i!} \left[ P(t)^{-m} \right]^{(i)}_{t=x}, \quad i = 0, 1, \ldots.
$$

Moreover, for $P \in \mathbb{P}_n^*(x)$ and $j \in \mathbb{M}_2$

$$
b_{m-j-2} > 0, \quad B_j(P, x; t) > 0, \quad t \in \mathbb{R}.
$$

Lemma 2.2 (Shi [8, Theorem 3]). Let $m \in \mathbb{N}_2$. We have

$$
\lambda_0(d\mu, m; x) \geq \lambda_m/2(d\mu; x).
$$

Lemma 2.3 (Nevai [6, Lemma 6.3.8, p. 108]). Let $\nu(x) = (1 - x^2)^{-1/2}$ and

$$
K_n(\nu; x, t) = \frac{T_n(x)T_{n-1}(t) - T_n(t)T_{n-1}(x)}{\pi(x-t)}, \quad n \geq 2,
$$
where \( T_n \) stands for the \( n \)th Chebyshev polynomial of the first kind. Then
\[
|K_n(v; x, t)| \leq c \min \left\{ n, \frac{(1-x^2)^{1/2} + (1-t^2)^{1/2}}{|x-t|} \right\}, \quad x, t \in [-1, 1],
\] (2.5)
where \( c \) is an absolute constant.

**Lemma 2.4** (Freud [2, (3.7), p. 102; p. 104]). Let
\[
\pi_{n-1}(x, t) = \frac{K_n(v; x, t)}{K_n(v; x, x)}.
\]
Then
\[
|\pi_{n-1}(x, t)| \leq 4, \quad n \geq 3, \quad x, t \in [-1, 1]
\] (2.6)
and
\[
K_n(v; x, x) \sim n, \quad |x| \leq 1.
\] (2.7)

By [2, Theorem 3.1, p. 19] the polynomial \( \pi_{n-1}(x, t) \) in \( t \) has simple real zeros only and hence
\[
\pi_{n-1}(x, \cdot) \in P_{n-1}^*(x).
\] (2.8)

**Lemma 2.5** (Hardy [3, Theorem 27, pp. 71–72]). Let \( A, B, p \geq 0 \) and \( AB + p > 0 \). Then
\[
(A + B)^p \leq c(p)(A^p + B^p).
\] (2.9)

**Lemma 2.6.** Let \( \ell \in \mathbb{N}, s = 1 + \lfloor n/\ell \rfloor, a > 0, \) and  
\[
S_{n,a,x}(t) = \pi_{s-1}(x/a, t/a)^\ell.
\] (2.10)
Then
\[
|b_i(S_{n,a,x}, x)| \leq c(\ell, i, m) [a \Delta_n(x/a)]^{-i}, \quad |x| \leq a.
\] (2.11)

**Proof.** By Bernstein inequality and (2.6) for \( |x|, |t| \leq a \)
\[
|([S_{n,a,x}(t)^m]^{(i)} |t=x| = \left| \frac{d^i}{dt^i} \pi_{s-1}(x/a, t/a)^\ell |_{t=x} \right| 
\leq c[a \Delta_n(x/a)]^{-i} \max_{|x|,|t| \leq a} |\pi_{s-1}(x/a, t/a)^\ell| 
\leq c[a \Delta_n(x/a)]^{-i}.
\]
Since
\[
[S_{n,a,x}(t)^m S_{n,a,x}(t)^{-m}]^{(i)} = 0, \quad i \geq 1,
\]
we have
\[ b_t(S_{n,a,x}, x) = - \sum_{v=0}^{i-1} b_v(S_{n,a,x}, x) \left\{ \frac{1}{(i-v)!} [S_{n,a,x}(t)^m]^{(i-v)}_{t=x} \right\} \]
and hence
\[ |b_t(S_{n,a,x}, x)| \leq \sum_{v=0}^{i-1} |b_v(S_{n,a,x}, x)| \left[ |S_{n,a,x}(t)^m|^{(i-v)}_{t=x} \right] \]
\[ \leq c \sum_{v=0}^{i-1} |a A_n(x/a)|^{v-i} |b_v(S_{n,a,x}, x)|. \]

By induction we obtain (2.11). □

Lemma 2.7 (Shi [10, Lemma 1]). If \( u(t) \leq C w(t) \) and
\[ N \geq 8 + 2 \left[ p_1 + p_r + \frac{1}{2} \sum_{i=2}^{r-1} |p_i| \right], \tag{2.12} \]
where \( w \) is given in (1.11) and the symbol \([y]\) denotes the integral part of the real number \( y \), then the inequality
\[ S = \int_{-1}^{1} |\pi_{n-1}(x, t)|^N u(t) dt \leq c(C, N, w) n^{-1} w_n(x) \tag{2.13} \]
holds for all \( x \in [-1, 1] \) and \( n \in \mathbb{N} \).

Lemma 2.8. Let \( u(t) \leq C w(t) \), \( 0 \leq p \leq m \),
\[ N = 4m + m \left[ p_1 + p_r + \frac{1}{2} \sum_{i=2}^{r-1} |p_i| \right], \quad s = 1 + \left[ \frac{m(n-1)}{N+m} \right], \tag{2.14} \]
and
\[ P_{n-1}(t) = \pi_{s-1}(x, t)^{(N+m)}/m, \tag{2.15} \]
where \( w \) is given in (1.11). Then the inequality
\[ S = \int_{-1}^{1} |P_{n-1}(t)|^m |t - x|^p u(t) dt \leq c(C, m, w) \lambda_n(w; x) A_n(x)^p \tag{2.16} \]
holds for all \( x \in [-1, 1] \) and \( n \in \mathbb{N} \).

Proof. By Lemma 2.7 for \( 0 \leq q \leq m \)
\[ \int_{-1}^{1} |\pi_{s-1}(x, t)|^N (1 - t^2)^q u(t) dt \leq c s^{-1} \delta_s(x)^q w_s(x), \quad |x| \leq 1, \quad s \in \mathbb{N}, \tag{2.17} \]
where \( \delta_s(x) = (1 - x^2)^{1/2} + s^{-1} \). Since \( \bar{c}P_{n-1} = (s - 1)(N + m)/m \leq n - 1 \), by (2.5)–(2.7), (2.9), and (2.17) we have

\[
S = \int_{-1}^{1} |\pi_{s-1}(x, t)|^{N+m} |t - x|^p u(t) dt \\
= \int_{-1}^{1} |\pi_{s-1}(x, t)|^{N+m-p} |(t - x)K_s(v; x, t)|^p K_s(v; x, x)^{-p} u(t) dt \\
\leq cs^{-p} \int_{-1}^{1} |\pi_{s-1}(x, t)|^N [(1 - x^2)^{1/2} + (1 - t^2)^{1/2}]^p u(t) dt \\
\leq cs^{-p} \left[ (1 - x^2)^{p/2} \int_{-1}^{1} |\pi_{s-1}(x, t)|^N u(t) dt + \int_{-1}^{1} |\pi_{s-1}(x, t)|^N (1 - t^2)^{p/2} u(t) dt \right] \\
\leq cs^{-p-1} w_s(x) [(1 - x^2)^{p/2} + \delta_s(x)^p] \\
\leq cs^{-p-1} \delta_s(x)^p w_s(x) \\
= cs^{-1} w_s(x) \Delta_s(x)^p \\
\leq cn^{-1} w_n(x) \Delta_n(x)^p. \quad \Box
\]

**Lemma 2.9** (Levin and Lubinsky [4, Lemma 5.2]). Let \( W \in \mathcal{F} \) and \( d > 1 \). Then the relation

\[
\left| \frac{a_p}{a_q} - 1 \right| \sim \left| \frac{p}{q} - 1 \right| \tag{2.18}
\]

uniformly holds for \( q \in (0, \infty) \) and \( p \in [q/d, dq] \).

**Lemma 2.10** (Levin and Lubinsky [4, Theorem 1.8]). Let \( W \in \mathcal{F} \), \( 0 < p \leq \infty \), and \( d > 0 \). Then for \( P \in \mathcal{P}_n \)

\[
\| PW \|_{L_p(\mathbb{R})} \leq c \| PW \|_{L_p(|x| \leq an(1-dn^{-2/3}))}. \tag{2.19}
\]

**Lemma 2.11** (Levin and Lubinsky [4, Theorem 1.9]). Let \( W \in \mathcal{F}^* \). Then for \( P \in \mathcal{P}_n \)

\[
\| P'W \| \leq c \frac{n}{a_n} \| PW \|. \tag{2.20}
\]

**Lemma 2.12.** Let \( W \in \mathcal{F} \) and \( \lambda > 0 \). Then

\[
a_q(\lambda Q) = a_{q/\lambda}. \tag{2.21}
\]

**Proof.** Replacing \( q \) by \( q/\lambda \) or \( Q \) by \( \lambda Q \) in (1.15), we have

\[
q = \frac{2}{\pi} \int_0^1 a_{q/\lambda}(Q)t(\lambda Q)'(a_{q/\lambda}(Q)t)(1 - t^2)^{-1/2} dt
\]

or

\[
q = \frac{2}{\pi} \int_0^1 a_q(\lambda Q)t(\lambda Q)'(a_q(\lambda Q)t)(1 - t^2)^{-1/2} dt,
\]

respectively. Comparing these two equations and observing \( a_{q/\lambda} = a_q(\lambda Q) \), we obtain (2.21). \( \Box \)
Following Levin and Lubinsky [4, pp. 485–488] define for \( x \in [-1, 1] \setminus \{0\} \)
\[
\mu_n(x) = \frac{2}{\pi^2} \int_0^1 \frac{(1 - x^2)^{1/2} a_n t Q'(a_n t) - a_n x Q'(a_n x)}{(1 - t^2)^{1/2} n(t^2 - x^2)} \, dt.
\]
Then by Lemma 7.1 in [4]
\[
\mu_n(x) > 0, \quad x \in (-1, 1) \setminus \{0\}
\] (2.22)
and
\[
\int_{-1}^1 \mu_n(x) \, dx = 1.
\] (2.23)

Now first we choose
\[-1 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1\]
so that
\[
\int_{t_k}^{t_{k+1}} \mu_n(t) \, dt = \begin{cases} 
\frac{1}{2n}, & k = 0, n, \\
\frac{1}{n}, & 1 \leq k \leq n - 1.
\end{cases}
\] (2.24)

Next, for a fixed \( x \in \mathbb{R} \) we choose \( \sigma_k = \sigma_{kn}(x), \ 1 \leq k \leq n \), as follows. For \( t_i \leq x < t_{i+1} \) with \( 2 \leq i \leq n - 2 \), put
\[
\sigma_k = \begin{cases} 
0, & k = i, i + 1, \\
2, & k = i - 1, i + 2, \\
1 & \text{otherwise},
\end{cases}
\]
for \( x \geq t_{n-1} \), put
\[
\sigma_k = \begin{cases} 
0, & k = n - 1, n, \\
3, & k = n - 2, \\
1 & \text{otherwise},
\end{cases}
\]
for \( x < t_2 \), put
\[
\sigma_k = \begin{cases} 
0, & k = 1, 2, \\
3, & k = 3, \\
1 & \text{otherwise}.
\end{cases}
\]
Then we write
\[
P_{n,x}(t) = \prod_{k=1}^n (t - t_k)^{\sigma_k}
\] (2.25)
and
\[
R_{n,x}(t) = \frac{P_{n,x}(t/a_n)}{P_{n,x}(x/a_n)}.
\] (2.26)
Lemma 2.13. Let \( d > 0 \). If \( W \in \mathcal{F} \) then
\[
\lambda_{n+1,\infty}(W; x) := \inf_{P \in P_{n, P(x)=1}} \| P W \| \leq \| R_{n,x} W \| \leq c W(x), \quad |x| \leq a_n(1 + dn^{-2/3}),
\] (2.27)
and if \( W \in \mathcal{F}^* \) then
\[
|b_i(R_{n,x}, t)| \leq c(i, m, W) \left( \frac{n}{a_n} \right)^i, \quad |x| \leq a_n(1 + dn^{-2/3}).
\] (2.28)

**Proof.** Inequality (2.27) is given in [4, pp. 514–515]. To prove (2.28) applying (2.20) repeatedly we obtain
\[
\| [R^m_{n,x}]^{(i)} W^m \| \leq c \left( \frac{mn}{amn} \right)^i \| R^m_{n,x} W^m \| \leq c \left( \frac{n}{a_n} \right)^i \| R_{n,x} W \|^m.
\] (2.29)
Thus, by (2.29), (2.27), and (2.21) for \( |x| \leq a_n(1 + dn^{-2/3}) \)
\[
\| [R^m_{n,x}]^{(i)} W^m \| \leq c \left( \frac{n}{a_n} \right)^i \| R_{n,x} W \|^m \leq c \left( \frac{n}{a_n} \right)^i W(x)^m
\]
which implies
\[
\left[ [R_{n,x}(t)^m]^{(i)} W(x)^m \right] \leq c \left( \frac{n}{a_n} \right)^i W(x)^m
\]
and hence
\[
\left[ [R_{n,x}(t)^m]^{(i)} \right] \leq c \left( \frac{n}{a_n} \right)^i.
\]
By the same argument as that of Lemma 2.6 we obtain (2.28).  \( \square \)

3. Proof of theorem 1.3

The proof follows and properly modifies the ideas of Levin and Lubinsky in [4].

The first line of (1.17) follows from (2.4) and (1.16). Let us prove the second. Let \( P \in P_{n-1}^* \) satisfy (1.8) with \( d\mu(t) = W(t)^m dt \), that is,
\[
\lambda_{jn}(W^m, m; x) = \int_{R} |A_j(P, x; s)| W(s)^m ds.
\]
Then by (1.5)
\[
A_j(P, x; t)^2 \leq \lambda_{mn}(W^{2m}; t)^{-1} \int_{R} A_j(P, x; s)^2 W(s)^{2m} ds \leq \lambda_{mn}(W^{2m}; t)^{-1} \| A_j(P, x; \cdot) W^m \| \int_{R} |A_j(P, x; s)| W(s)^m ds = \lambda_{mn}(W^{2m}; t)^{-1} \lambda_{jn}(W^m, m; x) \| A_j(P, x; \cdot) W^m \|.
\]
By (1.16) and (2.18)
\[
[A_j(P, x; t) W(t)^m]^2 \leq c \frac{n}{a_n} \lambda_{jn}(W^m, m; x) \| A_j(P, x; \cdot) W^m \|
which yields
\[ \|A_j(P, x; \cdot) W^m\| \leq c \frac{n}{a_n} \lambda_{jn}(W^m, m; x) \|A_j(P, x; \cdot) W^m\| \]
and hence
\[ \|A_j(P, x; \cdot) W^m\| \leq c \frac{n}{a_n} \lambda_{jn}(W^m, m; x). \] (3.1)

Applying (2.20) \( j \) times and using (3.1), we obtain
\[ W(x)^m = A^{(j)}(P, x; x) W(x)^m \leq c \left( \frac{n}{a_n} \right)^{j+1} \lambda_{jn}(W^m, m; x) \]
which gives the second line of (1.17).

Let us prove (1.18). We use (2.19) to get
\[ \lambda_{jn}(W^m, m; x) \leq \frac{1}{j!} \int_{\mathbb{R}} |(t - x)^j B_j(P, x; t) P(t)^m| W(t)^m dt \]
\[ \leq \frac{1}{j!} \sum_{i=0}^{m-2-j} |b_i(P, x)| \int_{\mathbb{R}} |t - x|^{i+j} |P(t)|^m W(t)^m dt \]
\[ \leq c \sum_{i=0}^{m-2-j} |b_i(P, x)| \int_{-a}^{a} |t - x|^{i+j} |P(t)|^m W(t)^m dt, \] (3.2)
where by (2.21) \( a = a_{mn}(mQ)(1 + dn^{-2/3}) = a_n(1 + dn^{-2/3}) \). Choose \( P = P_{q-1} P_{n-q} \), where
\[ P_{q-1}(t) = R_{q-1,x}(t) \] (3.3)
and
\[ P_{n-q}(t) = S_{n-q,a,x}(t) = \pi_{s-1}(x/a, t/a)^{(N+m)/m}, \quad s = 1 + \left[ \frac{m(n-q)}{N+m} \right]. \] (3.4)

Then by (3.2)–(3.4)
\[ \lambda_{jn}(W^m, m; x) \]
\[ \leq c \sum_{i=0}^{m-2-j} |b_i(P_{q-1} P_{n-q}, x)| \|P_{q-1} W^m\| \int_{-a}^{a} |t - x|^{i+j} |P_{n-q}(t)|^m dt \]
\[ = c \sum_{i=0}^{m-2-j} |b_i(P_{q-1} P_{n-q}, x)| \|P_{q-1} W^m\| a^{i+1} \int_{-1}^{1} |t - x/a|^{i+j} |P_{n-q}(at)|^m dt \]
\[ := c \sum_{i=0}^{m-2-j} S_i. \] (3.5)

Let us estimate \( S_i \) for \( 0 \leq i \leq m - 2 - j \). By (2.27) and (2.21)
\[ \|P_{q-1}^m W^m\| = \|P_{q-1} W\|^m \leq c W(x)^m, \quad |x| \leq a_{q-1}(1 + d(q - 1)^{-2/3}), \] (3.6)
and applying Lemma 2.8 and Theorem 1.1
\[
\int_{-1}^{1} |t - x/a|^{i+j} |P_{n-q}(at)|^m dt \leq c \Delta_{n-q}(x/a)^{i+j+1}, \quad |x| \leq a.
\] (3.7)

Also, by (2.28), (2.11), and (2.18)
\[
|b_i(P_{q-1}P_{n-q}, x)| = \left| \frac{1}{i!} [P_{q-1}(t)^{-m} P_{n-q}(t)^{-m}]^{(i)}_{t=x} \right|
\leq c \sum_{v=0}^{i} |b_v(P_{q-1}, x)| \cdot |b_{i-v}(P_{n-q}, x)|
\leq c \sum_{v=0}^{i} \left( \frac{q-1}{aq-1} \right)^v [a \Delta_{n-q} \left( \frac{x}{a} \right)]^{v-i}
\leq c \left[ a_n \Delta_{n-q} \left( \frac{x}{a} \right) \right]^{-i} \sum_{v=0}^{i} \left[ n \Delta_{n-q} \left( \frac{x}{a} \right) \right]^v,
\quad q \geq [n/2], \; |x| \leq a(q-1)(1 + d(q - 1)^{-2/3}).
\] (3.8)

Then by (3.6)–(3.8) we have
\[
S_i \leq c W(x)^m \left[ a_n \Delta_{n-q} \left( \frac{x}{a} \right) \right]^{j+1} \sum_{v=0}^{i} \left[ n \Delta_{n-q} \left( \frac{x}{a} \right) \right]^v,
\quad q \geq [n/2], \; |x| \leq a(q-1)(1 + d(q - 1)^{-2/3}).
\] (3.9)

To estimate \( \Delta_{n-q}(x/a) \) we separate three cases according to range of \( x \).

**Case 1:** \( |x| \leq a_{n/2} \). In this case let \( q = [n/2] \). Then by (2.18)
\[
\Delta_{n-q} \left( \frac{x}{a} \right) \leq \frac{c}{n} \phi_n(x)^{-1/2}.
\] (3.10)

**Case 2:** \( a_{n/2} < |x| \leq a_n(1 - n^{-2/3}) \). In this case let \( q \) satisfy
\[
a_{q-1} < |x| \leq a_q.
\]
Clearly, for \( n \) large enough we have
\[
n/3 \leq q \leq n(1 - n^{-2/3}).
\] (3.11)

By (2.18)
\[
1 - \frac{|x|}{a} \sim 1 - \frac{a_q}{a_n} \sim 1 - \frac{q}{n}
\]
and hence by (3.11)
\[
\frac{1}{(n-q)^2} \leq n^{-2/3} \leq 1 - \frac{q}{n} \sim 1 - \frac{|x|}{a}.
\]
Thus
\[
\Delta_{n-q} \left( \frac{x}{a} \right) \leq \frac{c}{n-q} \left( 1 - \frac{|x|}{a} \right)^{1/2}
\]
\[
= \frac{c}{n(1-q/n)} \left( 1 - \frac{|x|}{a} \right)^{1/2} \leq \frac{c}{n} \left( 1 - \frac{|x|}{a} \right)^{-1/2}
\]
which implies (3.10).

Case 3: \(a_n(1-2n^{-2/3}) \leq |x| \leq a_n(1+dn^{-2/3})\). In this case let \(q = n - \lfloor n^{1/3} \rfloor\). By (2.18)
\[
1 - \frac{|x|}{a} \leq cn^{-2/3} \sim \frac{1}{(n-q)^2}
\]
and hence
\[
\Delta_{n-q} \left( \frac{x}{a} \right) \leq \frac{c}{(n-q)^2} \sim n^{-2/3} \sim \frac{1}{n} \phi_n(x)^{-1/2}
\]
which again implies (3.10). Also by (2.18)
\[
\frac{|x|}{a_{q-1}} \leq \frac{a_n}{a_{q-1}} \leq 1 + c(q - 1)^{-2/3}.
\]
Using all these estimations inequality (3.9) with (3.10) yields
\[
S_i \leq c \left( \frac{a_n}{n} \right)^{j+1} W(x)^m \phi_n(x)^{-i(j+1)/2}.
\]
Then (1.18) follows from (3.5). □

References