ELSEVIER Journal of Pure and Applied Algebra 111 (1996) 205–211

JOURNAL OF PURE AND APPLIED ALGEBRA

Semistability of Artin and Coxeter Groups

Michael L. Mihalik *

Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA

Communicated by J.D. Stasheff; received 28 March 1995; revised 26 June 1995

Abstract

Semistability at infinity is a geometric invariant for finitely presented groups. If G is a finitely presented group then the semistability of G can give information about the fundamental group at infinity for G, the cohomology of G, and the shape of the boundary of G. In [6], M. Davis exhibits Coxeter groups with somewhat pathological behavior at infinity. Our main theorems are that all Artin and all Coxeter groups are semistable at infinity.

1991 Math. Subj. Class.: 20F32

1. Introduction

Semistability at infinity is a geometric invariant of finitely presented groups. If a finitely presented group G is semistable at ∞, then one can define $\pi_1^\infty(G)$, the fundamental group at infinity for G (see for example [10] or [11]). If G acts properly discontinuously, cocompactly and by isometries on a CAT(0) space X, then the boundary of X, ∂X, is well defined. Geoghegan [8] has pointed out that a one-ended CAT(0) group G is semistable at infinity iff ∂X has the shape of a locally connected continuum [1]. All Coxeter groups are CAT(0) groups and while (by Theorem 1.1) a boundary of a Coxeter group must have the shape of a locally connected continuum, there are Coxeter groups with non-locally connected boundaries (see Remark 4.2 for more on this).

It is conjectured that if G is a finitely presented group then $H^2(G;\mathbf{Z}G)$ is free abelian, but this is currently unknown, even for 2-dimensional duality groups (where one is discussing the dualizing module (see [2]). If G is semistable at infinity then $H^2(G;\mathbf{Z}G)$ is free abelian (see [9]).

* E-mail: mihalikm@ctrvax.vanderbilt.edu.

[1] A continuum is a compact connected metric space.

SSDI 0022-4049(95)00117-4

It is unknown if all finitely presented groups are semistable at infinity.

In [6] Davis constructs, in every dimension ≥ 4, closed aspherical manifolds not covered by Euclidean space. The fundamental group of each of these manifolds is a subgroup of finite index of a Coxeter group, and in Section. 16 of [6], these particular Coxeter groups are shown to be semistable at infinity, but not simply connected at infinity. The non-simple connectivity at infinity implies that Davis' manifolds are not covered by Euclidean space. These Coxeter groups are the first observed examples for which the fundamental group at infinity is not either trivial, $\mathbf{Z}$ or an infinite rank free group. In fact, Davis shows that the fundamental groups at infinity of his constructions are inverse limits of free products of more and more copies of $\pi_1(H)$, where H is a homology 3-sphere (the bonding maps kill the last $\pi_1(H)$ factor). While Coxeter groups indeed provide exotic behavior at infinity they are semistable at infinity.

Theorem 1.1. *All Coxeter groups are semistable at infinity.*

Corollary 1.2. *If G is a Coxeter group then $H^2(G;\mathbf{Z}G)$ is free abelian.*

Corollary 1.3. *If G is a one-ended Coxeter group, acting by isometries, properly discontinuously and cocompactly on the* CAT(0) *space X, then the boundary of X has the shape of a locally connected continuum.*

We observe that our main theorem and results in [4] imply that if M is a closed irreducible 3-manifold and $\pi_1(M)$ is a subgroup of finite index of an infinite Coxeter group, then the universal cover of M is Euclidean 3-space. This can also be seen using [7, 17].

We also consider a related type of group – Artin groups. We show

Theorem 1.4. *All Artin groups are semistable at infinity.*

Remark 1.5. In Section 2 we define Coxeter groups and Artin groups through certain presentations. A third type of group, with similar presentation, can be studied. In the case of 3 generators these groups have presentation of the form $\langle x, y, z : (xy)^m = (xz)^n = (yz)^\ell = 1,\ m, n, \ell \geq 2\rangle$. The Tietze moves $u = xy$ giving $\langle u, y, z : u^m = (uy^{-1}z)^n = (yz)^\ell = 1\rangle$, $v = uy^{-1}z$ giving $\langle u, v, y : u^m = v^n = (y^2u^{-1}v)^\ell = 1\rangle$ and $w = y^2u^{-1}v$ giving $\langle u, v, w, y : u^m = v^n = w^\ell = y^2u^{-1}vw^{-1} = 1\rangle$ show these groups are Fuchsian (see Section 4.6) of [19]). Hence they contain a surface group of finite index. This implies that they are semistable at ∞ and π_1^∞ is $\mathbf{Z}$, for such a group.

2. Groups

A *Coxeter group* is a group with a presentation of the following form: $\langle s_1, \ldots, s_n : s_i^2 = 1$ for $i \in \{1, \ldots, n\}, (s_is_j)^{m_{ij}} = 1$ where $i < j$ ranges over some subset of $\{1, \ldots, n\} \times \{1, \ldots, n\}$ and $m_{ij} \geq 2\rangle$.

Suppose G is a finitely generated group with finite generating set A. The *Cayley graph* of G with respect to A, $\Gamma(G,A)$, has as vertex set G. There is a directed edge labeled a, from g to ga for each $g \in G$ and $a \in A$. Hence if w is a word in A, then for any vertex $v \in G$, there is a unique edge path beginning at v with edge labels defined by w. Conversely, any edge path in Γ defines a unique element in $G \times F$ where F is the free group on A.

Define a metric d on Γ by defining each edge to be isometric to a unit interval and $d(g,h)$ to be the length of a shortest edge path from g to h. Define the *length* of g, $\ell(g)$, to be $d(g,1)$.

We need the following basic facts:

Lemma 2.1. *If $[v,w]$ is an edge of Γ, then $\ell(v) \neq \ell(w)$.*

Proof. There is a homomorphism $f : G \longrightarrow \{-1,1\}$ where $f(s_i) = -1$ for all i. Hence if α is any edge path from 1 to v then $f(v) = (-1)^{|\alpha|}$ where $|\alpha|$ is the number of edges in α. Let e be the edge of Γ from v to w. Then $f(w) = (-1)^{|\alpha|+1}$. Now $(-1)^{\ell(v)} = (-1)^{|\alpha|} \neq (-1)^{|\alpha|+1} = (-1)^{\ell(w)}$.

Lemma 2.2. *If v is a vertex of Γ, s_i and s_j are generators of G, $\ell(vs_i) = \ell(vs_j) > \ell(v)$ and s_is_j has order $m_{ij} \neq \infty$, then consider the edge loop $(s_is_j)^{m_{ij}}$ at v. If α is a shortest path from 1 to v, then $\langle \alpha, h \rangle$ is geodesic when h is either of the two edge paths of length m_{ij}, beginning at v and alternating between edges with labels s_i and s_j.*

Proof. This follows easily from Lemma 3 of [7].

Theorem 2.3 (Brown [5]). *If G is a Coxeter group with presentation as above, then the subgroup of G generated by the subset $S \subset \{s_1, \ldots, s_n\}$ has presentation $\langle S : s_i^2 = 1$ for all $s_i \in S$, $(s_is_j)^{m_{ij}} = 1$ if $s_i, s_j \in S$ and $(s_is_j)^{m_{ij}}$ is a relation of the above presentation of $G \rangle$.*

In the free group on $\{x,y\}$, let $(x,y)_m$ be the word of length m that begins with x and the letters alternate between x and y.

An *Artin group* is a group with presentation $\langle s_1, \ldots, s_n : (s_i,s_j)_{m_{ij}} = (s_j,s_i)_{m_{ij}}$ where $i < j$ ranges over some subset of $\{1,\ldots,n\} \times \{1,\ldots,n\}$ and $m_{ij} \geq 2$ for all such $i,j \rangle$.

In [12] (Theorem 4.13) van der Lek shows

Theorem 2.4. *If G is the Artin group with presentation as above, then the subgroup of G generated by the subset $S \subset \{s_1, \ldots, s_n\}$ has presentation $\langle S : (s_i,s_j)_{m_{ij}} = (s_j,s_i)_{m_{ij}}$ if $s_i, s_j \in S$ and $(s_i,s_j)_{m_{ij}} = (s_j,s_i)_{m_{ij}}$ is a relator of the above presentation of $G \rangle$.*

3. Semistability

We use [14] as a basic reference on semistability. In this paper all spaces are locally finite CW-complexes. If A is subset of a space then $St(A)$ is the union of all cells that intersect A. Define $St^n(A) \equiv St(St^{n-1}(A))$ for $n \geq 1$ (here $St^0(A) \equiv A$). A continuous function f mapping the space X to the space Y is *proper* if for each compact set $C \subset Y$, $f^{-1}(C)$ is compact in X. A *ray* in X is a proper map $r : ([0,\infty)) \longrightarrow X$.

Two rays, r and s, *converge to the same end* of X if for any compact set $C \subset X$ there is an $N \geq 0$ such that $r([N,\infty))$ and $s([N,\infty))$ lie in the same component of $X - C$. "Converges to the same end" is an equivalence relation on the set of all proper maps $r : [0,\infty) \longrightarrow X$ and the set of all such equivalence classes is called the *set of ends* of X.

An end of X is *semistable* if any two rays of X converging to that end are properly homotopic. This is equivalent to the following: An end $[r]$ of X is semistable if for some (equivalently any) ray $s \in [r]$ and subset $St^N(*)$ of X ($*$ a vertex of X) there is an integer $M > N$ such that for any $T > M$ and loop α, based on r with image in $X - St^M(*)$, α is homotopic rel. r, to a loop in $X - St^T(*)$ by a homotopy in $X - St^N(*)$. A space X is *semistable at* ∞ if each end of X is semistable.

By the local finiteness of X we have:

Lemma 3.1. *If* $[r]$ *is an end of* X *and* $*$ *a vertex of* X *then there is a ray* $s \in [r]$ *such that* $s(0) = *$, *for any integer* $N \geq 0$ s *maps the interval* $[N, N+1]$ *isometrically to an edge of* X *and in the 1-skeleton of* X, s *is a geodesic.*

A finitely presented group G is *semistable at infinity* if for some (equivalently any) finite, connected complex Y with $\pi_1(Y) = G$, the universal cover of Y is semistable at infinity. A generalization of this definition to finitely generated groups is given in [14].

The following three results will be needed in the next section.

Theorem 3.2 (Mihalik and Tschantz [15]). *If the finitely presented group* G *is the amalgamated product* $A *_C B$ *where* A *and* B *are finitely presented and semistable at infinity, and* C *is finitely generated, then* G *is semistable at infinity.*

Theorem 3.3 (Mihalik and Tschantz [16]). *All* 1*-relator groups are semistable at infinity.*

Theorem 3.4 (Mihalik [14]). *If* G *is finitely presented,* A *and* B *are finitely generated subgroups of* G *such that* $A \cup B$ *generates* G, A *and* B *are one ended and semistable at infinity or two ended and if* $A \cap B$ *contains an element of infinite order, then* G *is semistable at infinity.*

Example. If $i,j \in \{1,2,3\}$ let e_{ij} be the 3×3 matrix whose entries are zero with the exception of a 1 in the ith row and jth column. Set $z_0 = 1 + e_{12}$, $z_1 = 1 + e_{13}$, $z_2 = 1 + e_{23}$, $z_3 = 1 + e_{21}$, $z_4 = 1 + e_{31}$, $z_5 = 1 + e_{32}$. Then $SL_3(\mathbf{Z})$ is generated by $\{z_0, \ldots, z_5\}$. The subgroup generated by $\{z_i, z_{i+1}\}$ for $i \in \{0, \cdots, 4\}$ is isomorphic to $\mathbf{Z} \times \mathbf{Z}$. Hence four applications of Theorem 3.4 show $SL_3(\mathbf{Z})$ is semistable at infinity. In fact, $SL_3(\mathbf{Z})$ is known to be simply connected at infinity.

4. The main results

Proof of Theorem 1.1 (*All Coxeter groups are semistable at infinity*). Let $G \equiv \langle s_1,\ldots,s_n : s_i^2 = 1$ for all $i \in \{1,\ldots,n\}$, $(s_is_j)^{m_{ij}} = 1$ where $i < j$ ranges over some subset of $\{1,\ldots,n\} \times \{1,\ldots,n\}$ and $m_{ij} \geq 2\rangle$.

The proof will be presented in two parts. First we show that the problem can be reduced to the case where the pairs i,j in the above presentation of G range over all pairs $i < j$. This reduction is achieved through an application of Theorem 3.2.

The second part is a geometric argument to show that groups with a presentation as in the above reduction are semistable at infinity.

If in the above presentation of G there is a pair $i < j$ such that m_{ij} is not defined (i.e. there is no relator $(s_is_j)^{m_{ij}}$ for this choice of i,j) then consider the subgroups A, B and C of G with respective generating sets $\{s_1,\ldots,\hat{s}_i,\ldots,s_n\}$, $\{s_1,\ldots,\hat{s}_j,\ldots,s_n\}$ and $\{s_1,\ldots,\hat{s}_i,\ldots,\hat{s}_j,\ldots s_n\}$. By Theorem 2.3 a presentation for A is $\langle s_1,\ldots,\hat{s}_i,\ldots s_n : s_k^2 = 1$ for all $k \in \{1,\ldots,\hat{i},\ldots,n\}$, $(s_ks_\ell)^{m_{k\ell}} = 1$ where $k < \ell$ ranges over all pairs listed in the above presentation of G, except when $k = i$ or $\ell = i\rangle$. Similarly one can obtain a presentation for B and C.

If H is the amalgamated product $U *_W V$ where $U \equiv \langle u_1,\ldots,u_n,w_1,\ldots,w_m : R\rangle, V \equiv \langle v_1,\ldots,v_p,w_1,\ldots,w_m : S\rangle$ and $\langle w_1,\ldots,w_m\rangle$ generates the common copy of W in both presentations, then $\langle u_1,\ldots,u_n,\ v_1,\ldots,v_p,\ w_1,\ldots,w_m : R \cup S\rangle$ is a presentation for H. This implies that $G = A *_C B$, and by Theorem 3.1 it suffices to show A and B are semistable at infinity. Continued reductions of this fashion reduce the theorem to showing the following:

Lemma 4.1. *If $G = \langle s_1,\ldots,s_n : s_i^2 = 1$ for all $i, (s_is_j)^{m_{ij}} = 1$ where $i < j$ range over all pairs in $\{1,\ldots,n\} \times \{1,\ldots,n\}\rangle$ then G is semistable at infinity.*

Proof. If G is finite (e.g. $G \equiv \langle s_1 : s_1^2 = 1\rangle$) then G is semistable at infinity. Otherwise let X be the standard 2-complex associated with the above presentation and $\tilde{X}$ the universal cover of X. The 1-skeleton Γ of $\tilde{X}$ is the Cayley graph of G with respect to $\{s_1,\ldots,s_n\}$. Let 1 be the vertex corresponding to the identity of G. For each pair $i < j$ in $\{1,\ldots,n\} \times \{1\ldots,n\}$ let λ_{ij} be the edge loop $(s_is_j)^{m_{ij}}$ (with initial vertex 1). Observe that λ_{ij} is homotopically trivial in $St(1) \subset \tilde{X}$. Given $St^N(1)$, let α be an edge loop in $\tilde{X} - St^{N+1}(1)$. Let v be a vertex of α closest to 1. Say s_j and s_i are the edge labels leading into and out of v respectively, along α. We will show the edge path $\langle s_j, s_i\rangle$ (with middle vertex v) is homotopic rel(0,1) to an edge path, each of whose vertices are further from 1 than $d(v,1)$, by a homotopy in $\tilde{X} - St^N(1)$.

The loop $(s_is_j)^{m_{ij}}$ is homotopically trivial in $St(v)$ and so in $\tilde{X} - St^N(1)$. Hence the edge s_j leading into v followed by the edge s_i leading out of v can be replaced by the edge path $(s_is_j)^{m_{ij}-1}$, each of whose vertices are further from 1 than $d(v,1)$ (by Lemma 2.2). Continuing we can move the loop α as far as we like from 1.

This process can be done relative to base ray by an elementary adjustment.

Assume r is a geodesic edge path base ray with initial vertex 1 (see Lemma 3.1) and α is an edge loop based at the vertex $w = r(n)$ with $\text{im}(\alpha) \subset \tilde{X} - St^{N+1}(1)$. If e is the edge $r[n, n+1]$ then apply the above process to the loop $\langle e^{-1}, \alpha, e \rangle$. Repeat this as e^{-1} and e are "replaced" in the above process, thus always obtaining a loop "further out" and still based on r. □

Remark 4.2. F. Paulin observed that the proof of Lemma 4.1 should show that Coxeter groups satisfying the hypothesis of Lemma 4.1 have locally connected boundaries. To see this (as suggested by K. Ruane) one can easily check that a certain geometric condition $(\ddagger_M)$, found in [1] and implying local connectivity of the boundary of a space, is satisfied in the space $\tilde{X}$ (of Lemma 4.1). This implies that if G is word hyperbolic and as in Lemma 4.1, then the boundary of G is locally connected. It is an open problem if all word hyperbolic groups have locally connected boundaries.

If G acts by isometries, properly discontinuously and cocompactly on CAT(0) spaces X and Y, it is unknown if ∂X and ∂Y are homeomorphic.

Let F^n be the free group on n-generators. In [3], Bowers and Ruane show that if the group $F^n \times Z$ acts by isometries, properly discontinuously and cocompactly on a CAT(0) space X, then the boundary of X is homeomorphic to the suspension of the Cantor set, a non-locally connected space. If A is the group with two elements then $(A * A) \times (A * A * A)$ is a Coxeter group containing a subgroup of finite index of this form and so is a Coxeter group with non-locally connected boundary.

Proof of Theorem 1.4 (*All Artin groups are semistable at infinity*). Let $A = \langle S : (s_i, s_j)_{m_{ij}} = (s_j, s_i)_{m_{ij}}$ for some distinct pairs, $i < j \rangle$. Let Λ be the graph with vertex set S and an edge between s_i and s_j if $(s_i, s_j)_{m_{ij}} = (s_j, s_i)_{m_{ij}}$ for some $i \neq j$. If $A_1, \ldots, A_k$ are the vertex sets of the components of Λ then A is the free product of the subgroups of A generated by the A_i. By Theorem 3.2 it suffices to show that the subgroups of A generated by such A_i are semistable at infinity. By Theorem 2.4 each such subgroup is an Artin group with the induced presentation.

If s and t are generators with relation $(s,t)_m = (t,s)_m$, then the 1-relator group $G = \langle s, t : (s,t)_m = (t,s)_m \rangle$ is non-trivial and torsion free. By Stallings' end theorem [18], if G had more than 1 end, it would decompose into a free product $U * V$. As G is torsion free with 2 generators, U and V would be infinite cyclic. (The minimal number of generators of a free product is the sum of the minimal number of generators of the factors – see [13].) But G is not free, and so must be 1-ended. By Theorem 3.3, G is semistable at infinity, and by Theorem 3.4, A is semistable at infinity. □

Acknowledgements

I would like to thank Mike Davis, John Meier, Efstratios Prassidis, John Ratcliffe, Barry Spieler and Steve Tschantz for helpful conversations.

References

[1] M. Bestvina and G. Mess, The boundary of a negatively curved group, J. Amer. Math. Soc. 4 (1991) 469–481.

[2] R. Bieri, Homological dimension of discrete groups, Queen Mary College Math. Notes (London, 1976).

[3] P. Bowers and K. Ruane, Boundaries of nonpositively curved groups of the form $G \times Z^n$, Glasgow Math. J., to appear.

[4] M. Brin and T. Thickstun, Open irreducible 3-manifolds that are end 1-movable, Topology 26 (1987) 211–222.

[5] K.S. Brown, Buildıngs (Springer, New York, 1989).

[6] M.W. Davis, Groups generated by reflections and aspherical manifolds not covered by Euclidean space, Ann. of Math. (2) 117 (1983) 293–325.

[7] M. Davis and M. Shapiro, Coxeter groups are almost convex, Geom. Dedicata 39 (1991) 55–57.

[8] R. Geoghegan, The shape of a group, in: H. Torunczyk, Ed., Geometric and Algebraic Topology, Banach Center Publ., Vol. 18 (PWN, Warsaw, 1986) 271–280.

[9] R. Geoghegan and M.L. Mihalik, Free abelian cohomology of groups and ends of universal covers, J. Pure Appl. Algebra 36 (1985) 123–137.

[10] R. Geoghegan and M.L. Mihalik, The fundamental group at infinity, Topology, to appear.

[11] B. Jackson, End invariants of group extensions, Topology 21 (1982) 71–81.

[12] H. van der Lek, The homotopy type of complex hyperplane complements, Ph.D. Thesis, Univ. of Nijmegen, 1983.

[13] W. Magnus, A. Karrass and D. Solitar, Combinatorial Group Theory (Dover, New York, 1976).

[14] M. Mihalik, Semistability at ∞ of finitely generated groups, and solvable groups, Topology Appl. 24 (1986) 254–269.

[15] M. Mihalik and S. Tschantz, Semistability of amalgamated products and HNN-extensions, Mem. AMS, No. 471, Vol. 98, 1992.

[16] M. Mihalik and S. Tschantz, One relator groups are semistable at infinity, Topology 31 (4) (1992) 801–804.

[17] V. Poenaru, Almost convex groups, Lipschitz combings and π_1^∞ for universal covering spaces of closed 3-manifolds, J. Diff. Geom. 35 (1992) 103–130.

[18] J. Stallings, Group theory and three dimensional manifolds, Yale Math. Monographs, Vol. 4 (Yale Univ. Press, New Haven, CT, 1972).

[19] H. Zieschang, E. Vogt and H.D. Coldwey, Surfaces and planar discontinuous groups, in: A. Dold and B. Eckmann, Eds., Lecture Notes in Mathematics, Vol. 835 (Springer, New York, 1980).